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A mathematical model for mosquito infestation .

Un modelo matemático de la infestación de mosquitos.

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Abstract

We propose and analyze a mathematical model in ordinary differential equations to describe the dynamics of mosquitoes infested by bacteria. The introduction of some bacteria in mosquitoes population aims to diminish gradually the transmission of vector host-diseases. This is a good strategy of biological control.

Keywords. Ordinary differential equation, bacterial infestation , basic reproduction number, local stability, Simulation.

Resumen

Proponemos y analizamos un modelo matemático que describe la dinámica de los mosquitos infestados por bacterias. La introducción de ciertas bacterias en la población de mosquitos apunta a disminuir gradualmente la transmisión de enfermedades de huesped-vector. Esta es una buena estrategia de control biológico.

Palabras clave. Ecuación diferenciales ordinarias, infestación bacterial, número reproductivo básico, Simulación.

1. Introduction. The strategy of introduce bacteria in the mosquitoes population is very important for the biological control of vector in host-vector diseases. The introduction of bacteria in mosquitoes is called bacterial infestation, many researchers propose mathematical predictions for biological control of different host-vector diseases.

These kind of differential equation models are described in ([1], [2], [4]) . We refer to the model of *infested population only* models, type S-I epidemic models. It is important to think that the development of models of intermediate complexity works as a conceptual bridge between simple and complex model as those which implement demographic and epidemiological structure.

First, in *section 2*, we propose and analyze simpler S-I infestation model that differ from the others in the fact that an infested mosquito could produce a susceptible newborn and maintain sufficient simplicity to the model to be used in practical empirical work. we take into account the density dependence due competition for resources.

2. S-I infestation model. The mathematical model is considered through the infestation of a bacterium that inhibits the transmission of the virus that characterizes the diseases by indirect transmission. We divide the population of mosquitoes in aquatic and adult age and with epidemiological subdivision of infested and not infested will be taken into account, those populations are described in Table 1.1. The flow diagram of the infestation dynamics is given in Fig. 2 and the variables are detailed in Table 1.2.

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Variables	Meaning
J_U	mosquitoes of aquatic age not infested by the inhibitory bacteria
S_U	mosquitoes in adulthood not infested by the inhibitory bacteria
J_W	mosquitoes of aquatic age infested by the inhibitory bacteria
A_W	mosquitoes in adulthood infested by the inhibitory bacteria

TABLE 2.1
Meaning of the variables of the system (3.1)

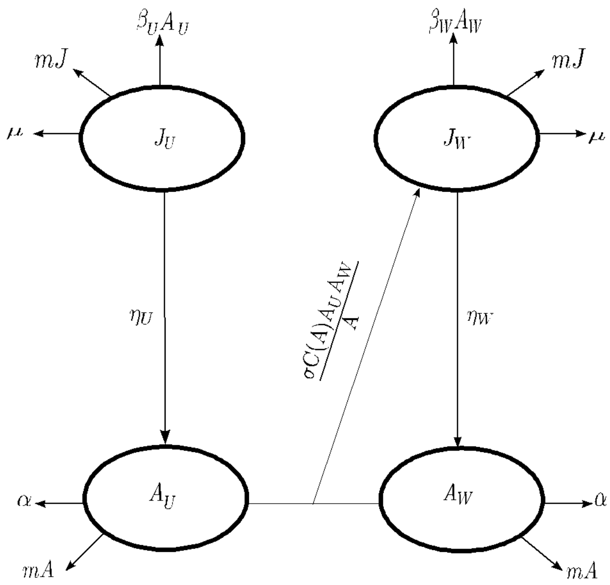

FIGURE 2.1. *Flow diagram of the bacterial infestation of mosquitoes*

The dynamical system is given by

$$(3.1)\qquad \begin{cases} \dfrac{dJ_U}{dt} &= \beta_U A_U + (1-\xi)\beta_W A_W - \eta_U J_U - \mu J_U - mJ_U J \ , \\[2ex] \dfrac{dJ_W}{dt} &= \xi\beta_W A_W + \dfrac{\sigma C(A) A_U A_W}{A} - \eta_W J_W - \mu J_W - mJ J_W \ , \\[2ex] \dfrac{dA_U}{dt} &= \eta_U J_U - \alpha A_U - mAA_U \ , \\[2ex] \dfrac{dA_W}{dt} &= \eta_W J_W - \alpha A_W - mAA_W \ . \end{cases} ,$$

The parameters of the model (3.1) are constant and are detailed in Table 5.

Considerations. It is assumed that aquatic stage mosquitoes reproduce with dinamycs $\beta_i A_i$ where β_i $(i = U, W)$ is the number of newborns produced by an adult mosquito per unit of time, considering that (ξ) is the probability that a infested mosquito in aquatic stage born from an adult infested mosquito . By natural death aquatic mosquitoes and adults leave their aquatic states at a rate μJ (or αA) respectively where μ is the per capita mortality rate of aquatic mosquitoes and α is the per capita death rate of adult mosquitoes.
Aquatic mosquitoes migrate to the adult stage at a rate of ηJ where η is the rate of per capita emigration of aquatic mosquitoes to the adult stage.

3. Basic Results. In this section, we study some basic results of the solutions of the system (3.1) which will be very useful to use into the proof of stability and persistence results.

Theorem 3.1. *For all* $J_U^0, A_U^0, J_W^0, A_W^0 > 0$, *there exists* $J_U, A_U, J_W, A_W : (0,\infty) \to (0,\infty)$ *which solve the system (3.1) with initial conditions* $J_U = J_U^0, A_U = A_U^0, J_W = J_W^0, A_W = A_W^0$.

Parameters	Meaning
β_U	adult mosquito maternal transmission rate without bacteria
β_W	maternal transmission rate of adult mosquito with bacteria
η_U	average time for a mosquito in aquatic stage without bacteria it becomes an adult mosquito without bacteria
η_W	average time for a mosquito in aquatic stage with bacteria to become an adult mosquito with bacteria
$1/\mu$	average life time of the mosquito in aquatic stage
$1/\alpha$	average life time of the adult mosquito
m	interspecific competition rate among mosquitoes
ξ	probability that a mosquito in aquatic stage with bacteria born from an adult mosquito with bacteria

TABLE 2.2
Biological meaning of System Parameters (3.1)

Theorem 3.2. *All solutions of this system (3.1) are bounded.*

The following reproduction rates are defined

$$\begin{aligned} R_U &= \left(\frac{\beta_U}{\alpha}\right)\left(\frac{\eta_U}{\eta_U+\mu}\right), \\ R_W &= \left(\frac{\beta_W}{\alpha}\right)\left(\frac{\eta_W}{\eta_W+\mu}\right). \end{aligned}$$

with biological meaning

$$\begin{aligned} R_U &= \text{average number of births of non infested mosquitoes} \\ &\quad\ \text{by a typical not infested mosquito during its life}, \\ R_W &= \text{average number of births of mosquitoes per each} \\ &\quad\ \text{typical infested mosquito infested during its life} \end{aligned}$$

Proposition 3.3. *If $R_U > 1$, then the system (3.1) has a single equilibrium free of infestation, $(J_U^*, A_U^*, 0, 0)$.*

Proposition 3.4. *If $R_W > 1$ and $\xi = 1$, then the system (3.1) has a single point of total infestation equilibrium (TI), $(0, 0, J_W^*, A_W^*)$,*

4. Global Stability of the Susceptible Extinction Equilibrium. We derive a criteria in the case that the susceptible subpopulation eventually win the competition with the infected subpopulation (the most desirable result).

Theorem 4.1. *If $R_U \leq 1$, $R_W > 1$ and $\xi = 1$, then the total infestation equilibrium (TI), $(0, 0, J_W^*, A_W^*)$, is global attractor in $\mathbb{R}_+^4$ for the system (3.1).*

Proof. To show that the TI is a global attractor in $\mathbb{R}_+^4$, consider the following Lyapunov function :

$$V(X) = J_U + \varrho_U A_U\,, \qquad X = (J_U, A_U, J_W, A_W)\,,$$

with $\varrho_U > 0$ will be chosen later.
The derivative of V along a solution of the system (3.1) is

$$\dot{V}(X) = (\eta_U + \mu)\left(\varrho_U \frac{\eta_U}{\eta_U+\mu} - 1\right) J_U + (\beta_U - \varrho_U \alpha) A_U - \frac{C J_U J_W}{J} - \frac{C A_U A_W}{A} - mVN\,.$$

We choose $\varrho_U = \dfrac{\beta_U}{\alpha}$. Then

$$\dot{V}(X) = (\eta_U + \mu)(R_U - 1) J_U - \frac{C J_U J_W}{J} - \frac{C A_U A_W}{A} - mVN\,.$$

Since $R_U \leq 1$, then $\dot{V}(X) \leq 0$ for $X \in \mathbb{R}^4_+$.

So as to apply *Lyapunov-LaSalle's Invariance Principle*, we see that :

By the boundedness of the solutions (*Theorem 3.2) of the system (3.1), we obtain that*

$$\dot{V}(X) \leq 0 \qquad \textit{for} \quad X \in \Omega_c = \mathbb{R}^4_+.$$

In the other hand,

$$E = \{X \in \mathbb{R}^4_+ : \dot{V}(X) = 0\} = \{(0, 0, J_W, A_W)\}$$

is an invariant set, as a consequence of the Principle, the trajectory enters $M = E$ and all solutions tend to E as $t \to \infty$.

To prove the asymptotic behavior of the TI equilibrium of the system (3.1), we need to consider the convergence theory developed by Thieme for limit equations, *described in* Appendix B. *We have that under the given conditions, any solution of system (3.1) is attracted to the $J_U A_U$-plane, then we can decouple the system into the second two equations, it means*

$$\text{(4.1)} \qquad \begin{cases} J'_W &= \beta_W A_W + \dfrac{\sigma C(J) J_U J_W}{J} - (\eta_W + \mu) J_W - m J_W J \ , \\ A'_W &= \eta_W J_W + \dfrac{\upsilon C(A) A_U A_W}{A} - \alpha A_W - m A_W A \ , \end{cases}$$

for $t \to \infty$, the corresponding limit system of the above system 4.1 is given by

$$\text{(4.2)} \qquad \begin{cases} J'_W &= \beta_W A_W - (\eta_W + \mu) J_W - m J_W (J_W + A_W) \equiv F(J_W, A_W) \ , \\ A'_W &= \eta_W J_W - \alpha A_W - m A_W (J_W + A_W) \equiv G(J_W, A_W) \ . \end{cases}$$

We define the function $\rho : (0, \infty)^2 \to \mathbb{R}^2$ by $\rho(J_W, A_W) = \dfrac{1}{J_W A_W}$, it follows

$$\begin{aligned} \frac{\partial(\rho F)}{\partial J_W} + \frac{\partial(\rho G)}{\partial A_W} &= \frac{\partial \left(\dfrac{\beta_W}{J_W} - \dfrac{(\eta_W + \mu)}{A_2} - \dfrac{m(J_W + A_W)}{A_W} \right)}{\partial J_W} \\ &\quad + \frac{\partial \left(-\dfrac{\alpha}{J_W} + \dfrac{\eta_W}{A_W} - \dfrac{m(J_W + A_W)}{J_W} \right)}{\partial A_W} \\ &= \left(-\frac{\beta_W}{J_W^2} - \frac{m}{A_W} \right) + \left(-\frac{\eta_W}{A_W^2} - \frac{m}{J_W} \right) \\ &< 0 \end{aligned}$$

for all $(J_W, A_W) \in (0, \infty)^2$. Then, by the Bendixson-Dulac criterium , there are no periodic orbits in $(0, \infty)^2$. The solutions of system ((3.1)) are bounded and there is at least one equilibrium point, so all equilibria is isolated. Since its ω-limit is contained in $(0, \infty)^2$ and $(0, \infty)^2$ is simply connected, we conclude that the ω-limit set is an equilibrium. Since (J^*_W, A^*_W) is the only equilibrium in $(0, \infty)^2$, we have that the $\omega(J_W, A_W) = (J^*_W, A^*_W)$ of the system ((3.1)). As the ω-limit set of a bounded solution of system ((3.1)) is contained in $(0, \infty)^2$, we satisfied all the conditions of Theorem 0.2 *(see Appendix B), in consequence, the ω-limit set of system consists of the equilibrium (J^*_W, A^*_W).*

In consequence, the $SE = (0, 0, J^*_W, A^*_W)$ is a global attractor of the system (3.1). □

5. Sensitivity Analysis of R_W.

We used LHS and PRCC methods to estimated the sensitivity of R_U.

$$\text{(5.1)} \qquad R_W = \left(\frac{\beta_W}{\alpha} \right) \left(\frac{\eta_W}{\eta_W + \mu_W} \right)$$

We considered the following values for the parameters

β_W	$(0.43, 0.57)$
$1/\mu_W$	$(0.01, 0.04)$
$1/\alpha$	$(0.04, 0.07)$
η_W	$(0.108, 0.123)$

- We made 11 simulations to obtain more sensitivity parameter (5.1)
- For the graphic, the colors are different tonalities of blue and red. When the tonality of red is more intense the parameter R_W will be more sensitive, is the same with the blue tonality color.

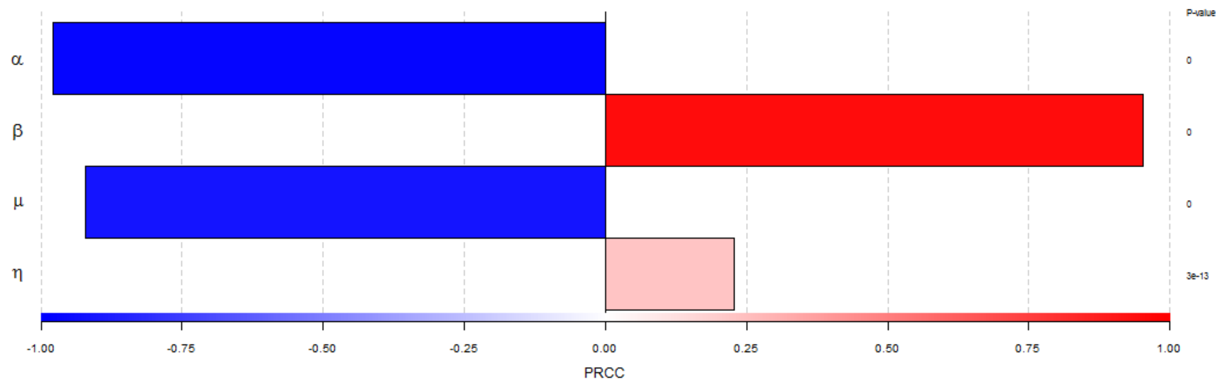

FIGURE 5.1. *Final Simulation*

5.1. Simulations.

Interpretation. The more sensitivity parameters are α and β

1) There exists a direct proportional correlation (positive) between R_W and β ($PRCC \approx 0.952$). It means when we increase the value of β the value of R_W will increase.
2) There exists a indirect proportional correlation (negative) between R_W and α ($PRCC \approx -0.978$). It means when we increase the value of α the value of R_W will decrease

Parameter	PRCC	lower bound	upper bound	p-value
β	0.952	0.946	0.958	0.00
α	-0.978	-0.980	-0.975	0.00
η	0.228	0.168	0.286	0.00
μ	-0.920	-0.929	-0.909	0.00

6. Acknowledgements. We gratefully thank MOSTICAW Project and the FONDECYT financial aid.

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