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The fundamental theorem of calculus for the Riemann–Stieltjes integral

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Abstract. We consider an extension of the ordinary derivative called the Ω–derivative, and develop some of its properties. Our main result is a generalization of the fundamental Theorem of Calculus that applies to Riemann–Stieltjes integrals in which the integrator is continuous and strictly increasing.

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Resumen. Consideramos una extensión de la derivada usual, llamada la Ω–derivada y desarrollamos algunas de sus propiedades. Nuestro resultado principal es una generalización del teorema fundamental del cálculo que es aplicable a integrales de Riemann–Stieltjes cuyos integradores son continuos y estrictamente crecientes.

1. Introduction

In this article we consider an extension of the ordinary derivative, which we call the Ω–derivative, and develop some of its properties. The Ω–derivative (the derivative with respect to Ω) is defined in [3]. We give a somewhat simpler definition, and our approach is more elementary than the approach taken in that paper. In particular,we don't use any measure theoretic concepts, and we consider the Riemann–Stieltjes integral rather that the Lebesgue–Stieltjes integral. The relationship between the Ω–derivative and the ordinary derivative is analogous to the relationship between the Riemann–Stieltjes integral and the Riemann integral. Our main result is a generalization of the Fundamental

Theorem of Calculus that applies to Riemann–Stieltjes integrals in which the integrator is continuous and strictly increasing.

2. The Ω–derivative

We begin with the definition of the Ω–derivative, cf. [3, p. 619–620].

Definition 1. Suppose f and Ω are real-valued functions defined on the same open interval (bounded or unbounded) and that Ω is continuous and strictly increasing. Suppose x_0 is a point in this interval. We say that f is Ω–differentiable at x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{\Omega(x)-\Omega(x_0)} \quad \text{exists.}$$

If this limit exists we denote its value by $D_\Omega f(x_0)$,which we call the Ω–derivative of f at x_0.

Of course, if $\Omega(x) = x$, then the Ω–derivative of f is the usual ordinary derivative of f. Notice that if $f'(x_0)$ and $\Omega'(x_0)$ both exist and $\Omega'(x_0) \neq 0$, then

$$\begin{aligned} D_\Omega f(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{\Omega(x)-\Omega(x_0)} \\ &= \lim_{x\to x_0} \frac{[f(x)-f(x_0)]\,/\,(x-x_0)}{[\Omega(x)-\Omega(x_0)]\,/\,(x-x_0)} = \frac{f'(x_0)}{\Omega'(x_0)}. \end{aligned}$$

Examples

1. If $f(x) = c$ (c a constant), then

$$D_\Omega f(x_0) = \lim_{x\to x_0} \frac{c-c}{\Omega(x)-\Omega(x_0)} = 0$$

2. If $f(x) = \Omega(x)$, then

$$D_\Omega f(x_0) = \lim_{x\to x_0} \frac{\Omega(x)-\Omega(x_0)}{\Omega(x)-\Omega(x_0)} = 1.$$

   Thus, Ω is Ω–differentiable.

3. Let $f(x) = x^{2/3}$ and $\Omega(x) = x^{1/3}$. Then

$$D_\Omega f(x_0) = \frac{f'(x_0)}{\Omega'(x_0)} = 2x_0^{1/3}, \quad x_0 \neq 0.$$

   Also, by direct calculation, $D_\Omega f(0) = 0$, so

$$D_\Omega f(x_0) = 2x_0^{1/3}, \quad \text{for all} x_0.$$

4. In general, if $f(x) = x^p$ and $\Omega(x) = x^q$, $p \in \mathbf{R}$ and $q > 0$, then

$$D_\Omega f(x_0) = \frac{p}{q} x_0^{p-q}, \quad x > 0.$$

Theorem 1 *If f is Ω–differentiable at x_0, then f is continuous at x_0.*

Proof.

$$\begin{aligned}\lim_{x\to x_0} f(x)-f(x_0) &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{\Omega(x)-\Omega(x_0)}\cdot[\Omega(x)-\Omega(x_0)]\\ &= D_\Omega f(x_0)\cdot 0 = 0,\end{aligned}$$

since Ω is continuous at x_0. ☑

3. Ω–derivative rules

Theorem 2. *Suppose f and g are both Ω–differentiable at x_0.Then $f+g$, fg, and cf (c a constant) are each Ω–differentiable at x_0. Their Ω–derivatives are as follows:*

1. $D_\Omega(f+g)(x_0) = D_\Omega f(x_0) + D_\Omega g(x_0)$
2. $D_\Omega(fg)(x_0) = D_\Omega f(x_0)g(x_0) + f(x_0)D_\Omega g(x_0)$
3. $D_\Omega(cf)(x_0) = cD_\Omega f(x_0)$

The proof can be obtained simply by mimicking the proof for the corresponding results for ordinary derivatives, so we omit it.

Theorem 3. *If $f(x) = (\Omega(x)+c)^n$ (c a constant), $n\in\mathbf{N}$, then*

$$D_\Omega f(x_0) = n\,(\Omega(x_0)+c)^{n-1}\,.$$

(As usual, here we interpret 0^0 as 1.)

Proof. We give a proof by induction. For $n=1$, the result follows easily from previous results. Next, assume that the formula is true for $n\in\mathbf{N}$. If we let

$$g(x) = (\Omega(x)+c)^{n+1} = f(x)\,(\Omega(x)+c)\,,$$

then, using the rule for the Ω–derivative of the product of two functions, we have

$$\begin{aligned}D_\Omega g(x_0) &= n\,(\Omega(x_0)+c)^{n-1}\,(\Omega(x_0)+c) + (\Omega(x_0)+c)^n\cdot 1\\ &= (n+1)\,(\Omega(x_0)+c)^n\,,\end{aligned}$$

which is what we needed to show.

4. Maximum and minimum values

Definition 2. We say that f is Ω–differentiable on an open interval if it is Ω–differentiable at every point in the interval.

Theorem 4. *Suppose f is Ω–differentiable on the interval (a,b). If f has a relative maximum or a relative minimum at $x_0\in(a,b)$, then $D_\Omega f(x_0) = 0$.*

Proof. Suppose f has a relative maximum at $x_0 \in (a,b)$. Then there exists a number $\delta > 0$ such that

$$f(x_0) \geq f(x) \quad \text{whenever} \quad a < x_0 - \delta < x_0 < x_0 + \delta < b.$$

Therefore, if $x_0 - \delta < x < x_0$ we have

$$\frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} \geq 0,$$

since Ω is strictly increasing. Letting $x \to x_0^-$, we see that we must have

$$D_\Omega f(x_0) \geq 0.$$

Similarly, if $x_0 < x < x_0 + \delta$, then

$$\frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} \leq 0\,.$$

Letting $x \to x_0^+$, we see that we must have

$$D_\Omega f(x_0) \leq 0.$$

It follows that

$$D_\Omega f(x_0) = 0.$$

The proof for a relative minimum is similar. ☑

If f and Ω are differentiable at x_0 and $\Omega'(x_0) \neq 0$, then this result follows from the corresponding standard calculus result, since

$$D_\Omega f(x_0) = \frac{f'(x_0)}{\Omega'(x_0)}.$$

However, as the following example illustrates, our result doesn't require either f or Ω to be differentiable at x_0.

Example. Let $f(x) = x^{2/3}$ and $\Omega(x) = x^{1/3}$. Then f has a relative minimum at $x = 0$ and $D_\Omega f(0) = 0$.

Proposition 1. *Suppose f is Ω–differentiable at x_0 and that $D_\Omega f(x_0) > 0$. Then there exists a number $\delta > 0$ such that*

$$f(x) < f(x_0) < f(y) \quad \textit{whenever} \quad x_0 - \delta < x < x_0 < y < x_0 + \delta.$$

A similar result holds if $D_\Omega f(x_0) < 0$.

Proof. Since f is Ω–differentiable, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} - D_\Omega f(x_0) \right| < \epsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta.$$

Choosing $\epsilon = \frac{1}{2} D_\Omega f(x_0)$ we obtain

$$\frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} > \frac{1}{2} D_\Omega f(x_0)$$

for $0 < |x - x_0| < \delta$. The result now follows easily, since Ω is strictly increasing.

5. Mean value type theorems

Theorem 5. *Suppose f is continuous on the closed interval $[a, b]$ and Ω–differentiable on the open interval (a, b). If $f(a) = f(b)$, then there exists a number $x_0 \in (a, b)$ such that $D_\Omega f(x_0) = 0$.*

Proof. If the minimum and maximum points of f both occur at the endpoints a and b, then f is a constant function and $D_\Omega f(x_0) = 0$, for all $x_0 \in (a, b)$. If one of the minimum or maximum points of f occurs at $x_0 \in (a, b)$, then$D_\Omega f(x_0) = 0$, by Theorem 4. ☑

Example. Let $f(x) = x^{2/3}$ and $\Omega(x) = x^{1/3}$. Then $f(-1) = f(1)$ and$D_\Omega f(0) = 0$.

Theorem 6. *Assume f and g are both continuous on the closed interval $[a, b]$ and Ω–differentiable on the open interval (a, b). Then there exists a number $x_0 \in (a, b)$ such that*

$$[f(b) - f(a)]\, D_\Omega g(x_0) = [g(b) - g(a)]\, D_\Omega f(x_0).$$

Proof. Apply the previous theorem to the function

$$h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x).$$ ☑

Theorem 7. *Assume f is continuous on the closed interval $[a, b]$ and Ω–differentiable on the open interval (a, b). Then there exists a number $x_0 \in (a, b)$ such that*

$$D_\Omega f(x_0) = \frac{f(b) - f(a)}{\Omega(b) - \Omega(a)}.$$

Proof. Let $g = \Omega$ in the previous theorem. ☑

If Ω is differentiable on (a, b), this result follows from the Generalized Mean Value Theorem, but our result doesn't require this assumption.

Example. Let $f(x) = x^{2/3}$ and $\Omega(x) = x^{1/3}$. Then

$$\frac{f(8) - f(-1)}{\Omega(8) - \Omega(-1)} = 1$$

On the other hand,

$$D_\Omega f(x_0) = 2x_0^{1/3} = 1$$

for $x_0 = \frac{1}{8} \in (-1, 8)$.

Proposition 2.

1. *If $D_\Omega f(x) \geq 0$, for all x in an open interval, then f is increasing on that interval.*

2. *If* $D_\Omega f(x) \leq 0$, *for all* x *in an open interval, then* f *is decreasing on that interval.*
3. *If* $D_\Omega f(x) = 0$, *for all* x *in an open interval, then* f *is constant on that interval.*

Proof. Consider the following equation

$$f(x_2) - f(x_1) = [\Omega(x_2) - \Omega(x_1)]\, D_\Omega f(x_0)$$

which holds for any $x_1 < x_2$ in the interval and some $x_0 \in (x_1, x_2)$. The proposition follows immediately, since $\Omega$is strictly increasing. ☑

Example. Let $f(x) = x^{3/5}$ and $\Omega(x) = x^{1/3}$. Then

$$D_\Omega f(x) = \frac{9}{5} x^{4/15} \geq 0$$

Thus, $f(x) = x^{3/5}$ is increasing on $(-\infty, \infty)$.

6. The fundamental theorem of calculus for the Riemann–Stieltjes integral

Definition 3. Let f be a function defined on an open interval I. We say that $F$is an Ω-antiderivative of f on I, if

$$D_\Omega F(x) = f(x), \quad \text{for all } x \in I.$$

We will denote by $\mathcal{R}(\Omega)$ the set all Riemann–Stieltjes integrable functions with respect to Ω, where Ω is a continuous, strictly increasing function on a closed, bounded interval $[a, b]$.

Theorem 8. *Let* $f \in \mathcal{R}(\Omega)$.*The function*

$$F(x) = \int_a^x f(t)\, d\Omega(t), \quad x \in [a, b],$$

is continuous on $[a, b]$. *Moreover, if* f *is continuous on* $[a, b]$, *then* F *is an* Ω-*antiderivative of* f *on* (a, b). *Proof.* By the Mean Value Theorem for Riemann–Stieltjes integrals, for any$x \neq y \in [a, b]$, we have

$$F(y) - F(x) = c\,[\Omega(y) - \Omega(x)]\,, \quad \text{for some } c \in [m, M],$$

where $m = \text{ínf}\,\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$.Since Ω is continuous on $[a, b]$, this implies F is continuous on $[a, b]$. If f is continuous on $[a, b]$, then we can replace c, in the equation above, by$f(z)$, for some z between x and y. The remainder of the theorem follows if we divide both sides of the equation above by$\Omega(y) - \Omega(x)$ and let $y \to x$. (Note that since f is continuous $f(z) \to f(x)$ as $y \to x$.) ☑

Theorem 9. *Let $f \in \mathcal{R}(\Omega)$.Suppose F is continuous on $[a, b]$ and an Ω-antiderivative of f on (a, b). Then*

$$\int_a^b f(x)\, d\Omega(x) = F(b) - F(a).$$

Proof. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. By Theorem 5, there exists $t_k \in [x_{k-1}, x_k]$ such that

$$F(x_k) - F(x_{k-1}) = f(t_k)\,[\Omega(x_k) - \Omega(x_{k-1})]$$

It follows that,

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} f(t_k)\,[\Omega(x_k) - \Omega(x_{k-1})]$$

Therefore,

$$F(b) - F(a) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(t_k)\,[\Omega(x_k) - \Omega(x_{k-1})] = \int_a^b f(x)\, d\Omega(x) \quad ☑$$

Example. Let $f(x) = x^{2/3}$ and $\Omega(x) = x^{1/3}$. Then

$$D_\Omega f(x) = 2x^{1/3}, \quad \text{for all } x.$$

Therefore,

$$\int_a^b 2x^{1/3}\, d\Omega = b^{2/3} - a^{2/3}$$

Note that if $a = 0$ or $b = 0$ or $a < 0 < b$, then this is equal to the *improper* Riemann integral

$$\int_a^b 2x^{1/3}\Omega'(x)\, dx = \int_a^b \frac{2}{3} x^{-1/3}\, dx.$$

Remark. The authors have discovered many other applications of the Ω–derivative to elementary calculus and differential equations. For example, we are able to prove a version of Taylor's theorem for the Ω–derivative as well as several results related to ordinary differential equations. We hope to be able to publish some of these results in the near future.

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