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Some q-representations of the q-analogue of the Hurwitz zeta function

Algunas $q-$representaciones del $q-$análogo de la función zeta de Hurwitz

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Abstract. This work is an attempt to motivate the study of, and research on, the theory of special functions. As main result, some new representations of the q–analogue of the Hurwitz zeta function are shown.

Key words and phrases. Hurwitz zeta function, q–hypergeometric series, q–shifted factorial, special function.

Resumen. Este trabajo es un intento por motivar el estudio e investigación en la teoría de funciones especiales. Como resultado fundamental, se deducen nuevas formas de representar el q–análogo de la función zeta de Hurwitz.

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1. Introduction

It is well known that the development of the q-calculus has been steadily pushed forward by the discovery of new applications of the q-hypergeometric series to areas as diverse as combinatorics, quantum theory, number theory, and statistical mechanics [2, 4]. At present, due to the great diversity of special functions of interest, it becomes attractive to identify some classifying criteria that may allow us to group special functions sharing some characteristic into a single class. In many instances, the classifying criteria to be considered may be stated in terms of the functions having some particular character (i.e. way of being represented) which might naturally facilitate the study of certain properties of special functions. As an example, we could take the generalized hypergeometric character of a function, understanding for it the fact that the function admits a representation of the form

$$ {}_r\phi_s \left[\begin{array}{c} a_1, \ldots, a_r \\ \\ b_1, \ldots, b_s \end{array} \middle| \, q; z \right] \equiv \sum_{n\geq 0} \frac{\langle a_1; q\rangle_n \cdots \langle a_r; q\rangle_n}{\langle 1; q\rangle_n \langle b_1; q\rangle_n \cdots \langle b_s; q\rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n $$

(see [2, 4, 5, 6] for more details). Here, $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{i=0}^{s}$ are complex numbers subject to the condition that $b_j \neq 0 \wedge -k$ where $k \in \mathbb{N}$ for $j = 1, 2, \ldots, s$, and $\langle \cdot; q\rangle_k$ denotes the q–shifted factorial [2, 4, 5, 6, 7], defined by

$$ \langle a; q\rangle_n \equiv \begin{cases} 1, & n = 0, \\ \prod\limits_{0\leq j\leq n-1} \left(1 - q^{a+j}\right), & n = 1, 2, \ldots. \end{cases} $$

It is important to mention that the theory of hypergeometric series is fundamental for mathematical physics, since almost all elementary functions can be expressed as either hypergeometric or ratios of hypergeometric series, and many non–elementary functions in this field can be expressed as hypergeometric functions [3].

The purpose of this paper is to present some q-hypergeometric representations of the q-analogue of the Hurwitz zeta function, which is defined by

$$ \zeta_q(s, \alpha) \equiv \sum_{n\geq 0} \frac{q^{(n+\alpha)(s-1)}}{\{n+\alpha\}_q^s}, \quad 0 < q < 1, \quad 0 < \alpha \leq 1, \tag{1.1} $$

where $\{z\}_q$ denotes a q-analogue of a complex number defined by

$$ \{z\}_q \equiv \frac{1 - q^z}{1 - q}, \quad q \in \mathbb{C} \setminus \{1\}. $$

The series (1.1) is convergent for Re $s > 1$, and it is proved in [1] that q-analogue of the Hurwitz zeta function can be rewritten as

$$ \zeta_q(s, \alpha) = q^{\alpha(s-1)} \{\alpha\}_q^{-s} \sum_{n\geq 0} q^{-n} \left[\frac{\langle \alpha; q\rangle_n}{\langle \alpha+1; q\rangle_n} q^n \right]^s. \tag{1.2} $$

We take this property as the starting point for what follows.

2. Main results

Lemma 2.1. *The following two relations are valid:*

$$ \langle \alpha; q\rangle_n = \langle \alpha; q\rangle_{n+1} + q^{n+\alpha} \langle \alpha; q\rangle_n, \tag{2.1} $$

$$ \langle \alpha; q\rangle_n = \langle 1; q\rangle_n \sum_{0\leq k\leq n} q^{(\alpha+1)k} \frac{\langle -1; q\rangle_k \langle \alpha+1; q\rangle_{n-k}}{\langle 1; q\rangle_k \langle 1; q\rangle_{n-k}}. \tag{2.2} $$

Proof. Taking into account the following identity [2, p. 14, exp. 79]

$$\langle\alpha;q\rangle_{n+k}=\langle\alpha;q\rangle_k\,\langle\alpha+k;q\rangle_n\,, \tag{2.3}$$

we easily obtain the result (2.1).

Now, in order to prove (2.2), let us use the following identity [8, p. 25]

$$\langle\alpha+\beta;q\rangle_n=\sum_{0\leq k\leq n}\binom{n}{k}_q q^{\beta k}\,\langle\alpha;q\rangle_k\,\langle\beta;q\rangle_{n-k}\,, \tag{2.4}$$

where [6, p. 3, exp. 8]

$$\binom{n}{k}_q\equiv\frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\,\langle 1;q\rangle_{n-k}},\quad k=0,1,\ldots,n.$$

Taking into account the following relation [9, p. 12, exp. 0.3.3]

$$\binom{n}{k}_q=\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\,(-1)^k\,q^{nk-\binom{k}{2}},$$

and [2, p. 14, exp. 74]

$$\langle -n;q\rangle_k=\frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{n-k}}\,(-1)^k\,q^{\binom{k}{2}-nk},\quad 0\leq k<n, \tag{2.5}$$

we deduce easily from (2.4) the result (2.1). This completes the proof. ☑

In the rest of the paper we will use the following abbreviation:

$${}_r\phi_s^n\left[\begin{array}{c|c} a_1,\ldots,a_r & \\ & q;z \\ b_1,\ldots,b_s & \end{array}\right]:=\left({}_r\phi_s\left[\begin{array}{c|c} a_1,\ldots,a_r & \\ & q;z \\ b_1,\ldots,b_s & \end{array}\right]\right)^n,\quad n=1,2,\ldots.$$

Theorem 2.2. *Let s be an integer number, with $s>1$, $|q|<1$ and $0<\alpha\leq 1$. Then the q-analogue of the Hurwitz zeta function (1.1) admits the following representations.*

i.)

$$\begin{aligned}\zeta_q(s,\alpha)=q^{\alpha(s-1)}\,\{\alpha\}_q^{-s}\sum_{k\geq 0}(-1)^k\,q^{\binom{k}{2}+k\alpha}&\\ \times\,{}_2\phi_1\left[\begin{array}{c|c} -k,2\alpha & \\ & q;q^{1-\alpha+k} \\ \alpha+1 & \end{array}\right]{}_2\phi_1^{s-1}\left[\begin{array}{c|c} -k,1 & \\ & q;q \\ \alpha+1 & \end{array}\right]&\\ \times\,{}_{s+1}\phi_s\left[\begin{array}{c|c} k+1,k+1,k+\alpha,\ldots,k+\alpha & \\ & q;q^{s-1} \\ k+\alpha+1,\ldots,k+\alpha+1 & \end{array}\right].&\end{aligned} \tag{2.6}$$

ii.)

$$\begin{aligned}\zeta_q(s,\alpha) = q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}(-1)^k q^{\binom{k+1}{2}} \\ \times {}_2\phi_1\left[\begin{array}{c} -k,1 \\ \\ \alpha+1 \end{array}\middle| q;q^{k+\alpha}\right] {}_2\phi_1\left[\begin{array}{c} -k,1-\alpha \\ \\ 1 \end{array}\middle| q;q^{k+\alpha}\right] \\ \times {}_2\phi_1^{s-1}\left[\begin{array}{c} -k,1 \\ \\ \alpha+1 \end{array}\middle| q;q\right] \\ \times {}_s\phi_{s-1}\left[\begin{array}{c} k+\alpha,\ldots,k+\alpha \\ \\ k+\alpha+1,\ldots,k+\alpha+1 \end{array}\middle| q;q^{s-1}\right]. \end{aligned} \tag{2.7}$$

iii.)

$$\begin{aligned}\zeta_q(s,\alpha) = q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}q^{(\alpha+1)k}{}_2\phi_1\left[\begin{array}{c} -k,\alpha+2 \\ \\ \alpha+1 \end{array}\middle| q;q^{k-1}\right] \\ \times {}_2\phi_1^{s-1}\left[\begin{array}{c} -k,1 \\ \\ \alpha+1 \end{array}\middle| q;q\right] \\ \times {}_{s+1}\phi_s\left[\begin{array}{c} k,\alpha+1,k+\alpha,\ldots,k+\alpha \\ \\ k+\alpha+1,\ldots,k+\alpha+1 \end{array}\middle| q;q\right]. \end{aligned} \tag{2.8}$$

iv.)

$$\begin{aligned}\zeta_q(s,\alpha) = (1-q^{\alpha})q^{\alpha(s-1)}\{\alpha\}_q^{-s}{}_s\phi_{s-1}\left[\begin{array}{c} 1,\alpha,\ldots,\alpha \\ \\ \alpha+1,\ldots,\alpha+1 \end{array}\middle| q;q^{s-1}\right] \\ + q^{\alpha s}\{\alpha\}_q^{-s}{}_{s+1}\phi_s\left[\begin{array}{c} 1,\alpha,\ldots,\alpha \\ \\ \alpha+1,\ldots,\alpha+1 \end{array}\middle| q;q^{s}\right]. \end{aligned} \tag{2.9}$$

Proof. Taking into account the equalities [2, p. 19, exp. 109]

$${}_2\phi_1\left[\begin{array}{c} -n,b \\ \\ c \end{array}\middle| q;q^{n+c-b}\right] = \frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n}, \quad n=0,1,\ldots \tag{2.10}$$

and

$$\langle -n;q\rangle_k = 0, \quad \text{whenever } n<k,$$

we have

$$\begin{aligned}\frac{\langle\alpha;q\rangle_n}{\langle 1;q\rangle_n} &= {}_2\phi_1\left[\begin{array}{c|c} -n,1-\alpha & \\ & q;q^{n+\alpha} \\ 1 & \end{array}\right] \\ &= \sum_{k\geq 0}\frac{\langle -n;q\rangle_k\langle 1-\alpha;q\rangle_k}{\langle 1;q\rangle_k\langle 1;q\rangle_k}q^{k(n+\alpha)} = \sum_{0\leq k\leq n}\frac{\langle -n;q\rangle_k\langle 1-\alpha;q\rangle_k}{\langle 1;q\rangle_k\langle 1;q\rangle_k}q^{k(n+\alpha)}\end{aligned}$$

and

$$\begin{aligned}\frac{\langle 1;q\rangle_n}{\langle\alpha+1;q\rangle_n} &= {}_2\phi_1\left[\begin{array}{c|c} -n,\alpha & \\ & q;q^{n+1} \\ \alpha+1 & \end{array}\right] \\ &= \sum_{k\geq 0}\frac{\langle -n;q\rangle_k\langle \alpha;q\rangle_k}{\langle 1;q\rangle_k\langle \alpha+1;q\rangle_k}q^{k(n+1)} = \sum_{0\leq k\leq n}\frac{\langle -n;q\rangle_k\langle \alpha;q\rangle_k}{\langle 1;q\rangle_k\langle \alpha+1;q\rangle_k}q^{k(n+1)}.\end{aligned}$$

Thus, from the above and (1.2) we get

$$\begin{aligned}\zeta_q(s,\alpha) = q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s-1}\langle 1;q\rangle_n}{\langle\alpha+1;q\rangle_n^{s}}q^{n(s-1)} \\ \times\sum_{0\leq k\leq n}\frac{\langle -n;q\rangle_k\langle 1-\alpha;q\rangle_k}{\langle 1;q\rangle_k\langle 1;q\rangle_k}q^{k(n+\alpha)}\end{aligned}$$

and

$$\begin{aligned}\zeta_q(s,\alpha) = q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s}}{\langle\alpha+1;q\rangle_n^{s-1}\langle 1;q\rangle_n}q^{n(s-1)} \\ \times\sum_{0\leq k\leq n}\frac{\langle -n;q\rangle_k\langle \alpha;q\rangle_k}{\langle 1;q\rangle_k\langle \alpha+1;q\rangle_k}q^{k(n+1)},\end{aligned}$$

respectively. Then, using (2.5) we obtain

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s} \\ &\times\sum_{k\geq 0}\sum_{n\geq k}\frac{\langle\alpha;q\rangle_n^{s-1}\langle 1-\alpha;q\rangle_k\langle 1;q\rangle_n\langle 1;q\rangle_n}{\langle\alpha+1;q\rangle_n^{s}\langle 1;q\rangle_k\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}\left(-q^{\alpha}\right)^k q^{\binom{k}{2}+n(s-1)} \\ &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_{n+k}^{s-1}\langle 1-\alpha;q\rangle_k\langle 1;q\rangle_{n+k}\langle 1;q\rangle_{n+k}}{\langle\alpha+1;q\rangle_{n+k}^{s}\langle 1;q\rangle_k\langle 1;q\rangle_k\langle 1;q\rangle_n} \\ &\qquad\times\left(-q^{s-1+\alpha}\right)^k q^{\binom{k}{2}+n(s-1)}\end{aligned}$$

and

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s} \\ &\quad\times\sum_{k\geq 0}\sum_{n\geq k}\frac{\langle\alpha;q\rangle_n^s\langle\alpha;q\rangle_k}{\langle\alpha+1;q\rangle_n^{s-1}\langle 1;q\rangle_k\langle\alpha+1;q\rangle_k\langle 1;q\rangle_{n-k}}(-q)^k q^{\binom{k}{2}+n(s-1)} \\ &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_{n+k}^s\langle\alpha;q\rangle_k}{\langle\alpha+1;q\rangle_{n+k}^{s-1}\langle 1;q\rangle_k\langle\alpha+1;q\rangle_k\langle 1;q\rangle_n} \\ &\qquad\times(-q^s)^k q^{\binom{k}{2}+n(s-1)},\end{aligned}$$

respectively. Finally, from (2.3) we infer

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}(-1)^k q^{\binom{k}{2}+k\alpha}\frac{\langle 1-\alpha;q\rangle_k}{\langle\alpha+1;q\rangle_k}\frac{\langle\alpha;q\rangle_k^{s-1}}{\langle\alpha+1;q\rangle_k^{s-1}}q^{k(s-1)} \\ &\qquad\times\sum_{n\geq 0}\frac{\langle k+1;q\rangle_n\langle k+1;q\rangle_n\langle k+\alpha;q\rangle_n^{s-1}}{\langle 1;q\rangle_n\langle k+\alpha+1;q\rangle_n^s}q^{n(s-1)}\end{aligned}$$

and

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}(-1)^k q^{\binom{k+1}{2}} \\ &\qquad\times\frac{\langle\alpha;q\rangle_k}{\langle\alpha+1;q\rangle_k}\frac{\langle\alpha;q\rangle_k}{\langle 1;q\rangle_k}\frac{\langle\alpha;q\rangle_k^{s-1}}{\langle\alpha+1;q\rangle_k^{s-1}}q^{k(s-1)} \\ &\qquad\qquad\times\sum_{n\geq 0}\frac{\langle k+\alpha;q\rangle_n^s}{\langle 1;q\rangle_n\langle k+\alpha+1;q\rangle_n^{s-1}}q^{n(s-1)},\end{aligned}$$

which, taking into account (2.10) and [2, p. 20, exp. 113]

$$ {}_2\phi_1\left[\begin{array}{c|c}-n,b & \\ & q;q\\ c & \end{array}\right]=\frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n}q^{bn},\quad n=0,1,\ldots.$$

coincides with (2.6) and (2.7) respectively.

Now, by Lemma 2.1, expression (2.2), and expression (2.3) we get

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{k\geq 0}q^{(\alpha+1)k}\frac{\langle -1;q\rangle_k}{\langle\alpha+1;q\rangle_k}\frac{\langle\alpha;q\rangle_k^{s-1}}{\langle\alpha+1;q\rangle_k^{s-1}}q^{k(s-1)} \\ &\qquad\times\sum_{n\geq 0}\frac{\langle k;q\rangle_n\langle\alpha+1;q\rangle_n\langle k+\alpha;q\rangle_n^{s-1}}{\langle 1;q\rangle_n\langle k+\alpha+1;q\rangle_n^s}q^{n(s-1)},\end{aligned}$$

which coincides with (2.8).

Finally, according to Lemma 2.1 and expression (2.1) , we obtain

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s-1}\langle\alpha;q\rangle_n}{\langle\alpha+1;q\rangle_n^{s}}q^{n(s-1)}\\ &= q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s-1}\langle\alpha;q\rangle_{n+1}}{\langle\alpha+1;q\rangle_n^{s-1}\langle\alpha+1;q\rangle_n}q^{n(s-1)}\\ &\qquad + q^{\alpha s}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s}}{\langle\alpha+1;q\rangle_n^{s}}q^{ns}.\end{aligned}$$

Since

$$\langle\alpha+1;q\rangle_n = \frac{1-q^{n+\alpha}}{1-q^{\alpha}}\langle\alpha;q\rangle_n$$

and

$$\langle\alpha;q\rangle_{n+1} = \left(1-q^{n+\alpha}\right)\langle\alpha;q\rangle_n ,$$

we deduce

$$\begin{aligned}\zeta_q(s,\alpha) = (1-q^{\alpha})\,q^{\alpha(s-1)}\{\alpha\}_q^{-s}\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s-1}}{\langle\alpha+1;q\rangle_n^{s-1}}q^{n(s-1)} + q^{\alpha s}\{\alpha\}_q^{-s}\\ \times\sum_{n\geq 0}\frac{\langle\alpha;q\rangle_n^{s}}{\langle\alpha+1;q\rangle_n^{s}}q^{ns},\end{aligned}$$

which corresponds to (2.9). This completes the proof. ☑

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