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Epsilon-delta proofs and uniform continuity

Demostraciones de límites y continuidad usando sus definiciones con epsilon y delta y continuidad uniforme

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Abstract. We present two heuristic methods to get epsilon-delta proofs. From these methods, a new approach to study uniform continuity of real functions comes up. In addition, some results on uniform continuity of homeomorphisms in the real line are established.

Key words and phrases. Limits, continuity, uniform continuity.

Resumen. Se presentan dos métodos heurísticos para obtener demostraciones sobre límites y continuidad usando sus deficiones en términos de epsilon y delta. De estos métodos surge un nuevo abordaje para estudiar la continuidad uniforme de funciones reales. Además se establecen algunos resultados sobre la continuidad uniforme de homeomorfismos de la recta real.

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1. Introduction

Despite its importance, the construction of proofs of limits using only the definition of limit is not a topic discussed much on basic calculus courses. As we know, in most cases those proofs are difficult and involve hard estimates and several algebraic manipulations, which make this type of proofs fundamental for those who want to understand some other concepts in mathematical analysis. For these reasons, this paper aims to study two issues:

1. Epsilon-delta proofs: the task of giving a proof of the existence of the limit of a function based on the epsilon-delta definition.
2. The role of delta-epsilon functions (see Definition 2.2) in the study of the uniform continuity of a continuous function.

We begin by recalling the definition of limit of a function.

Definition 1.1. Let I be a non empty open interval, $f : I \to \mathbb{R}$ a function defined on I. We say that f tends to $L \in \mathbb{R}$ when x tends to $a \in I$, and write

$$\lim_{x \to a} f(x) = L,$$

if for every $\epsilon > 0$, there exists $\delta > 0$ so that

$$\text{if } x \in I \text{ and } 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon. \tag{1}$$

If $L = f(a)$, then we say that f is continuous at a.

1.1. How to construct a epsilon-delta proof? In this section, we present two heuristic methods to obtain epsilon-delta proofs. That is, we want to prove that

$$\lim_{x \to a} f(x) = L,$$

by applying Definition 1.1. The question is: given a positive number ϵ, how to find out a positive number δ, such that (1) is satisfied? As we know, for most functions f, this is a complicated task which involves not only a good knowledge of the properties of f, but also tricky algebraic calculations and hard estimates. Namely,

Lemma 1.1. ***The first method.*** *Let I be a nonempty interval, $f : I \to \mathbb{R}$ a function defined on I, and $a \in I$. If there exists $b > 0$ and $g : [0, b) \to [0, g(b))$ an increasing, bijective function satisfying*

$$|f(x) - f(a)| \leq g(|x - a|), \tag{2}$$

then, for $\epsilon > 0$, there exists

$$\delta = g^{-1}(\epsilon_0), \tag{3}$$

with $0 < \epsilon_0 < g(b)$, such that

$$\text{if } x \in I \text{ and } |x - a| < \delta, \quad \text{then} \quad |f(x) - f(a)| < \epsilon. \tag{4}$$

In particular, f is a continuous function at a.

Proof. Without loss of generality, we can consider $\epsilon > 0$ such that $0 < \epsilon < g(b)$. Taking $0 < \delta = g^{-1}(\epsilon)$, we obtain that if $0 \leq y < \delta$ then $0 \leq g(y) < \epsilon$. Therefore, if $|x - a| < \delta = g^{-1}(\epsilon)$, since g is increasing we have $g(|x - a|) < \epsilon$, and from (2), we conclude that $|f(x) - f(a)| < \epsilon$. This finishes the proof of the lemma. ☑

Example 1.1. Using epsilon-delta proofs, we want to show that

$$\lim_{x \to 2} x^2 = 4. \tag{5}$$

According to Lemma 1.1, it is enough to get an estimate of the form

$$|x^2 - 4| \leq g_1(|x - 2|). \tag{6}$$

From the triangular inequality, we obtain that

$$|x^2 - 4| = |x - 2||x + 2| \leq |x - 2|(|x - 2| + 4), \tag{7}$$

Therefore, the function $g_1 : [0, \infty) \to [0, \infty)$, $g_1(y) = y(y + 4)$ satisfies the condition of Lemma 1.1. Thus, for $\epsilon > 0$ there exists

$$\delta = g_1^{-1}(\epsilon) = \sqrt{4 + \epsilon} - 2 \tag{8}$$

such that (4) is satisfied. Specifically, we have that for every $\epsilon > 0$ there exists $\delta \in (0, \sqrt{\epsilon + 4} - 2]$ such that

$$if \quad |x - 2| < \delta, \quad \text{then} \quad |x^2 - 4| < \epsilon. \tag{9}$$

As we wanted to prove.

Example 1.2. Another way to prove (5) proceeds as follows: If $|x - 2| < 1$, then from (7), we obtain that

$$|x^2 - 4| < 5|x - 2|. \tag{10}$$

Hence, the function $g_2 : [0, 1) \to [0, 5)$, $g_2(y) = 5y$ satisfies the condition of Lemma 1.1. Thus, for $\epsilon > 0$ there exists

$$\delta = g_2^{-1}(\epsilon) = \frac{\epsilon}{5} \tag{11}$$

such that (9) is satisfied.

For monotonic and bijective functions, another way to find out the delta number can be done as follows,

Lemma 1.2. ***The second method.*** *Let I, J be open intervals such that $a \in I$. Considering $f : I \to J$, an increasing bijective function (actually continuous), then, for $\epsilon > 0$ with $f(a) - \epsilon, f(a) + \epsilon \in J$, there exists*

$$\delta = \text{mín}\{f^{-1}(f(a) + \epsilon) - a, a - f^{-1}(f(a) - \epsilon)\} \tag{12}$$

such that,

$$\textit{if } x \in I \textit{ and } |x - a| < \delta, \quad \textit{then} \quad |f(x) - f(a)| < \epsilon. \tag{13}$$

In particular, f is a continuous function at a.

Proof. Since $f(a) - \epsilon, f(a) + \epsilon \in J$, the number δ given in (12) is well defined. Now, if $|x - a| < \delta$ then

$$f^{-1}(f(a) - \epsilon) \leq a - \delta < x < a + \delta \leq f^{-1}(f(a) + \epsilon). \tag{14}$$

Since f is increasing, applying f to the previous inequality we get that

$$f(a) - \epsilon < f(x) < f(a) + \epsilon. \tag{15}$$

In other words, $|f(x) - f(a)| < \epsilon$, as we wanted to prove. ☑

Remark 1.1. In Lemma 1.2, if the function f is taken to be decreasing instead of increasing, the conclusion of the lemma remains valid if we change the value of δ in (12) by

$$\delta' = \text{mín}\{a - f^{-1}(f(a) + \epsilon), f^{-1}(f(a) - \epsilon) - a\}. \tag{16}$$

Example 1.3. Since $f : (0, \infty) \to (0, \infty)$, $f(x) = x^2$ is an increasing bijective function. Then we can apply Lemma 1.2 to get a epsilon-delta proof of (5). Thus, for $\epsilon > 0$, there exists

$$\delta = \text{mín}\{2 - \sqrt{4 - \epsilon}, \sqrt{\epsilon + 4} - 2\} = \sqrt{\epsilon + 4} - 2,$$

such that the condition (9) is satisfied.

2. Uniform continuity

In this section, from epsilon-delta proofs we move to the study of the relationship between continuity and uniform continuity. For this purpose, we introduce the concept of *delta-epsilon function*, which is essential in our discussion. Using this concept, we also give a characterization of uniform continuity in Theorem 2.1. In addition, as an application of this theorem, we give a sufficient condition for an unbounded homeomorphism not to be uniformly continuous at infinite. Now, we recall the definition of uniform continuity.

Definition 2.1. Let I be a nonempty interval. A function $f : I \to \mathbb{R}$ is called uniformly continuous on I, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, we have that $|f(x) - f(y)| < \epsilon$.

Example 2.1. The function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & if \quad x < 0, \\ \ \ 1 & if \quad x > 0, \end{cases} \tag{17}$$

is continuous but not uniformly continuous.

Example 2.2. It is well known that continuous functions defined on compact sets are uniformly continuous (see [1]).

Definition 2.2. Let I be a nonempty interval, $f : I \to \mathbb{R}$ a continuous function. For a fixed $\epsilon > 0$, we say that a function $\delta_\epsilon : I \to (0, \infty)$ is a **delta-epsilon function** for f, if $\delta_\epsilon(a)$ satisfies the continuity definition for f at a. Namely

$$\text{if } \ x \in I \ \text{ and } \ |x - a| < \delta_\epsilon(a), \quad \text{then} \quad |f(x) - f(a)| < \epsilon.$$

Example 2.3. For $\epsilon > 0$, the function

$$\delta_\epsilon(a) = \begin{cases} -a & if \quad a < 0, \\ \ \ a & if \quad a > 0, \end{cases} \tag{18}$$

is a delta-epsilon function for the function of the Example 2.1.

Theorem 2.1. *Let I an interval, $f : I \to \mathbb{R}$ a continuous function. Then f is uniformly continuous on I if and only if there exists a family $\{\delta_\epsilon\}_{\epsilon>0}$ of delta-epsilon functions for f such that,*

$$\eta_\epsilon := \inf_{a\in I} \delta_\epsilon(a) > 0, \tag{19}$$

for every $\epsilon > 0$.

Proof. If $f : I \to \mathbb{R}$ is uniformly continuous on I, then for $\epsilon > 0$ there is $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, we have that $|f(x) - f(y)| < \epsilon$. Thus, the constant function $\delta_\epsilon : I \to (0, \infty)$, $\delta_\epsilon(a) = \delta$, is a delta-epsilon function for f that clearly satisfies the condition (19). Conversely, let $\{\delta_\epsilon\}_{\epsilon>0}$ be a family of delta-epsilon functions for the continuous function f that satisfies the condition (19). Hence, for $\epsilon > 0$ and $x, y \in I$, if $|x - y| < \eta_\epsilon \leq \delta_\epsilon(x)$, since f is continuous at x and $\delta_\epsilon(x)$ is so that the continuity definition 1.1 is verified at x, we can conclude that $|f(x) - f(y)| < \epsilon$. ☑

Remark 2.1. Roughly speaking, the previous theorem tells us that continuous functions f that admit a family of delta-epsilon constant functions $\{\eta_\epsilon\}_{\epsilon>0}$ are uniformly continuous.

Example 2.4. For every positive M , the function $f : [0, M] \to \mathbb{R}$, $f(x) = x^2$ is uniformly continuous. In fact, for $\epsilon > 0$ the function $\delta_\epsilon : [0, M] \to (0, \infty)$ defined as

$$\delta_\epsilon(a) = \sqrt{a^2 + \epsilon} - a,$$

is a delta-epsilon function for f that is continuous, decreasing and

$$\inf_{a\in[0,M]} \delta_\epsilon(a) = \sqrt{M^2 + \epsilon} - M > 0.$$

Hence, from Theorem 2.1, we get that f is uniformly continuous. We notice that in this example the function $\rho_\epsilon(a) = \sqrt{M^2 + \epsilon} - M$ is a delta-epsilon function for f that is constant and less than δ_ϵ.

Remark 2.2. In terms of Theorem 2.1, in order to show that a continuous function f is not uniformly continuous, we must verify that any family of delta-epsilon functions has an element δ_{ϵ_0} such that

$$\inf_{a\in I} \delta_{\epsilon_0}(a) = 0.$$

This seems to be a difficult task. The proof of the following theorem gives us a good idea of how to deal with this type of problem.

Theorem 2.2. *Let $b \in \mathbb{R}$ and let $f : [b, \infty) \to [f(b), \infty)$ be an increasing homeomorphism. If there exists $\epsilon_0 > 0$ with*

$$\lim_{x\to\infty} f^{-1}(f(x) + \epsilon_0) - x = 0, \tag{20}$$

then f is not uniformly continuous.

Proof. We give two proofs of this theorem. We first observe that, since the image of the function f is an unbounded interval, the expression $f^{-1}(f(x)+\epsilon_0)$ is always well defined, in other words, it is not necessary to impose any restrictions on the variable ϵ_0.

- First proof. It will be divided in four steps:
1- For $\epsilon > 0$, the function given by

$$\delta_\epsilon(x) = \text{mín}\left\{f^{-1}(f(x)+\epsilon) - x, x - f^{-1}(f(x)-\epsilon)\right\} \tag{21}$$

is a delta–epsilon function for f that is maximum. In fact, from Lemma 1.2, for every $\epsilon > 0$, δ_ϵ is a delta-epsilon function. Now, we proceed to prove that δ_ϵ is **maximum**, that is, δ_ϵ is greater than any other delta-epsilon function for f. We will prove this by contradiction. If ρ_ϵ is another delta-epsilon function for f such that there exists x_0 in (b,∞) with $\delta_\epsilon(x_0) < \rho_\epsilon(x_0)$, then, we can consider p such that

$$x_0 + \delta_\epsilon(x_0) < p < x_0 + \rho_\epsilon(x_0), \tag{22}$$

and then $\delta_\epsilon(x_0) < p - x_0 < \rho_\epsilon(x_0)$. Now, if we assume that

$$\delta_\epsilon(x_0) = f^{-1}(f(x_0)+\epsilon) - x_0,$$

from (22) we get that

$$f^{-1}(f(x_0)+\epsilon) < p.$$

Since f is an increasing function, we obtain that $\epsilon < f(p) - f(x_0)$. This contradicts the fact that ρ_ϵ is a delta-epsilon function for f. The proof that δ_ϵ is maximum when $\delta_\epsilon(x_0) = x_0 - f^{-1}(f(x_0)-\epsilon)$ proceeds similarly.
2- Let $\epsilon > 0$ and ρ_ϵ a delta-epsilon function for f. If

$$\text{ínf}_{x\in[b,\infty)} \delta_\epsilon(x) = 0, \quad \text{then,} \quad \text{ínf}_{x\in[b,\infty)} \rho_\epsilon(x) = 0. \tag{23}$$

The proof of this assertion follows from step one, since

$$0 < \rho_\epsilon(x) \le \delta_\epsilon(x), \quad \text{for all} \quad x \in [b,\infty).$$

3- Let $0 < \epsilon \le \epsilon_0$, then, $\text{ínf}_{x\in[b,\infty)} \delta_\epsilon(x) = 0$. In fact, by definition of δ_ϵ in (21) and since f is increasing, we have that

$$\delta_\epsilon(x) \le f^{-1}(f(x)+\epsilon) - x \le f^{-1}(f(x)+\epsilon_0) - x, \quad \text{fol all} \quad x \in [b,\infty).$$

Hence, from condition (20), we obtain that $\text{lím}_{x\to\infty} \delta_\epsilon(x) = 0$ and thus

$$\eta_\epsilon = \text{ínf}_{a\in I} \delta_\epsilon(a) > 0.$$

4- End of proof. Let $\{\rho_\epsilon\}_\epsilon > 0$ be a family of delta-epsilon functions for f. From step 3, for every ϵ with $0 < \epsilon < \epsilon_0$ we have $\text{ínf}_{x\in[b,\infty)} \delta_\epsilon(x) = 0$. By step 2, $\text{ínf}_{x\in[b,\infty)} \rho_\epsilon(x) = 0$, so finally, from Theorem 2.1, we can conclude that f is not uniformly continuous.

- Second proof. Since $\lim_{x\to\infty} f^{-1}(f(x)+\epsilon_0)-x=0$ for some ϵ_0, then for every $\eta>0$ there exists $R>0$ such that, for all $x>R$, we have that
$$f^{-1}(f(x)+\epsilon_0)-x<\eta, \tag{24}$$
or equivalently, since f is increasing, $\epsilon_0<f(x+\eta)-f(x)$. Hence, for any $\delta>0$, taking η such that $0<\eta<\delta$ and $a>R$, we have that $|(a+\eta)-a|=\eta<\delta$, but $\epsilon_0<f(a+\eta)-f(a)$. This shows that f is not uniformly continuous. ☑

Corollary 2.1. *If $f:I\to\mathbb{R}$ admits a maximum delta–epsilon function δ_{ϵ_0} such that*
$$\inf_{x\in I}\delta_{\epsilon_0}(x)=0,$$
then f is not uniformly continuous.

Example 2.5. From examples 2.1 and 2.3 we have that, for $0<\epsilon<2$, δ_ϵ is a maximum delta-epsilon function. In addition,
$$\inf_{x\in\mathbb{R}-\{0\}}\delta_\epsilon(x)=0.$$
Then, from Corollary 2.1, we obtain that f (the function given in Example 2.1) is not uniformly continuous on $\mathbb{R}-\{0\}$.

Example 2.6. For M positive, the function $f:[M,\infty)\to\mathbb{R}$, $f(x)=x^2$ is not uniformly continuous. In fact, for every $\epsilon>0$ we have that
$$\lim_{x\to\infty}\sqrt{x^2+\epsilon}-x=0,$$
hence, from Theorem 2.2, we get that f is not uniformly continuous.

Corollary 2.2. ***(A well-known result)*** *Let b,c be real numbers with $b<c$. Then every increasing homeomorphism $f:[b,c)\to[f(b),\infty)$ is not uniformly continuous.*

Proof. It is possible to prove that
$$\lim_{x\to c^-}f(x)=\infty, \tag{25}$$
and that for every $\epsilon>0$,
$$\lim_{x\to c^-}f^{-1}(f(x)+\epsilon)-x=0. \tag{26}$$
Now, the proof of the corollary follows from Theorem 2.2 by making convenient alterations. ☑

3. **Examples**

In this section, we give several examples of how to use the heuristic methods introduced above to get epsilon-delta proofs. We also give examples of how to use the technique developed in Section 2 to study uniform continuity.

1. *Linear functions.* For $m > 0$, the linear function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx$ is a continuous, bijective and increasing function. Therefore, to find out a family of delta-epsilon functions for f, it is enough to apply the formula (12) of Lemma 1.2. Thus, for $\epsilon > 0$ and $a \in \mathbb{R}$, there exists
$$\delta_\epsilon(a) = \frac{\epsilon}{m}, \tag{27}$$
so that, if
$$|x - a| < \delta, \quad \text{then} \quad |mx - ma| < \epsilon. \tag{28}$$
Furthermore, since for every $\epsilon > 0$
$$\inf_{a\in\mathbb{R}} \delta_\epsilon(a) = \frac{\epsilon}{m} > 0,$$
then from Theorem 2.1 we conclude that f is uniformly continuous.
2. *The power functions with natural power.* Since for $n \in \mathbb{N}$ the power function $f : (0, \infty) \to (0, \infty)$, $f(x) = x^n$, is continuous, bijective and increasing, then from lemma 1.2, for $\epsilon > 0$, there exists
$$\delta_\epsilon(a) = \sqrt[n]{a^n + \epsilon} - a, \tag{29}$$
so that, if
$$|x - a| < \delta, \quad \text{then} \quad |x^n - a^n| < \epsilon. \tag{30}$$
Now, since for $n > 1$
$$\lim_{a\to\infty} \sqrt[n]{a^n + \epsilon} - a = 0,$$
then from Theorem 2.2, for $n > 1$ the function $f(x) = x^n$ is not uniformly continuous on $[M, \infty)$, for any M positive. On the other hand, the power function $f(x) = x^n$ is uniformly continuous on $[0, M]$ for every $n \in \mathbb{N}$. In fact, for every $\epsilon > 0$, the constant functions
$$\inf_{a\in[0,M]} \delta_\epsilon(a) = \sqrt[n]{M^n + \epsilon} - M,$$
form a family of delta-epsilon functions for f on $[0, M]$.
3. *The function* $f(x) = \frac{1}{x}$. Since $f : (0, \infty) \to (0, \infty)$, $f(x) = \frac{1}{x}$ is a continuous, bijective and decreasing function, the formula (16) of the Lemma 1.2 can be applied. Hence, for $\epsilon > 0$, there exists
$$\delta_\epsilon(a) = \frac{a^2\epsilon}{1 + a\epsilon} \tag{31}$$
so that, if
$$|x - a| < \delta, \quad \text{then} \quad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon. \tag{32}$$

Additionally, for $M > 0$ and $\epsilon > 0$, the function $\delta_\epsilon : [M, \infty) \to (0, \infty)$ given in (31) is increasing, so we obtain that

$$\operatorname*{ínf}_{a \in [M,\infty)} \delta_\epsilon(a) = \frac{M^2\epsilon}{1 + M\epsilon} > 0.$$

Hence, we conclude from Theorem 2.1 that f is uniformly continuous on $[M, \infty)$. In contrast, since

$$\operatorname*{ínf}_{a \in (0,M]} \delta_\epsilon(a) = 0,$$

then by making slight changes to Corollary 2.2, we can conclude that f is not uniformly continuous on $(0, M]$.

4. *The square root function.* Function $f : (0, \infty) \to (0, \infty)$, $f(x) = \sqrt{x}$ is continuous, bijective, and increasing. Hence, for $\sqrt{a} > \epsilon > 0$, from Lemma 1.2 there exists

$$\delta_\epsilon(a) = 2\sqrt{a}\epsilon - \epsilon^2 \tag{33}$$

such that, if

$$|x - a| < \delta, \quad \text{then} \quad |\sqrt{x} - \sqrt{a}| < \epsilon. \tag{34}$$

Additionally, if $a > M > 0$, then $\sqrt{a} > \sqrt{M}$, so for every $\epsilon < \sqrt{M}$, since δ_ϵ is an increasing function, we conclude that

$$\operatorname*{ínf}_{a \in [M,\infty)} \delta_\epsilon(a) = 2\sqrt{M}\epsilon - \epsilon^2,$$

and from the Theorem 2.1, we deduce that $f(x) = \sqrt{x}$ is uniformly continuous on $[M, \infty)$.

5. *The function* $f(x) = \frac{x}{1+x}$. Since $f : (-1, \infty) \to (-\infty, 1)$, $f(x) = \frac{x}{1+x}$ is a continuous, bijective and increasing function, from Lemma 1.2 there exists

$$\delta_\epsilon(a) = \frac{\epsilon(a+1)^2}{1 + \epsilon(1 + a)} \tag{35}$$

so that, if

$$|x - a| < \delta, \quad \text{then} \quad \left| \frac{x}{1+x} - \frac{a}{1+a} \right| < \epsilon. \tag{36}$$

Similarly, from Theorem 2.1 it is not difficult to see that f is uniformly continuous on $[M, \infty)$, for any $M > -1$.

6. *The natural exponential function.* Since for every $\epsilon > 0$

$$\operatorname*{lím}_{a \to \infty} \ln(e^a + \epsilon) - a = 0,$$

then by Theorem 2.2 the function $f(x) = e^x$ is not uniformly continuous on $[M, \infty)$, where M denotes any real number.

7. *The natural logarithm function.* From Lemma 1.2, it is not difficult to check that the functions
$$\delta_\epsilon(a) = a(1 - e^{-\epsilon})$$
form a family of delta-epsilon functions for the natural logarithm function $\ln : (0, \infty) \to \mathbb{R}$. Furthermore, since for $M > 0$
$$\inf_{a \in [M,\infty)} \delta_\epsilon(a) = M(1 - e^{-\epsilon}) > 0,$$
then from Theorem 2.1 we conclude that the function $\ln : [M, \infty) \to \mathbb{R}$ is uniformly continuous.

4. **Conclusions**

In general, to obtain an epsilon-delta proof is hard work. Under certain assumptions, the methods we presented in Section 1 to deal with that issue give us a fast way to construct epsilon-delta proofs. In this way, the method in Lemma 1.2 is actually more useful than the method given in Lemma 1.1, where difficult estimates are needed.

The concept of delta-epsilon function introduced in Section 2 provides another way to characterize uniform continuity (see Theorem 2.1). In our opinion, this theorem gives us a good way to understand the concept of uniform continuity. In addition, Theorem 2.1 together with the study of epsilon-delta proofs provide a different approach to deal with uniform continuity. Although it might not be the best way to study uniform continuity of a function, our approach allows us to obtain new interesting results: see Theorem 2.2 and its corollaries.

On the other hand, Corollary 2.2 is not a new result, see [1]. Hence, it would be interesting to describe some other classic results about uniform continuity in terms of our new approach.

Theorem 2.1 can be rewritten in a more general setting, for instance, in terms of metric spaces. Thus, an interesting question would be how to obtain similar conditions as in Theorem 2.2 that would allow us to study uniform continuity of functions defined on metric spaces.

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