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Existence and regularity of 1D-solitons for a hyperelastic dispersive model

Existencia y regularidad de 1D-solitones para un modelo hiperelástico dispersivo

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Abstract. We show the existence, regularity and analyticity of one-dimensional solitons for a dispersive type equation that models the deformations of a hyperelastic compressible plate. We follow a variational approach by characterizing solitons as critical points of a suitable functional. Our method involves the Mountain Pass Theorem.

Keywords: Hypereslactic Equation, Solitons, Variational Approach, Mountain Pass Theorem.

Resumen. En este trabajo mostramos la existencia, regularidad y analiticidad de solitones uno-dimensionales para una ecuación de tipo dispersivo que modela las deformaciones de una placa hiperelástica. Seguimos una aproximación variacional, en la cual caracterizamos solitones como puntos críticos de una funcional adecuada. Nuestro método involucra el Teorema de Paso de Montaña.

Palabras claves: Ecuación Hiperelástica, Solitones, Aproximación Variacional, Teorema de Paso de Montaña.

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1. Introduction

In this work we consider the following generalized two-dimensional nonlinear dispersive elastic equation

$$\partial_x \left(I - \partial_x^2 + \mu\partial_x^4\right) \partial_t u + \partial_x^2 \left(\frac{p+2}{p+1}u^{p+1} - \gamma\Big[u\partial_x(\partial_x u)^p + \frac{p}{p+1}(\partial_x u)^{p+1}\Big]\right) - \alpha\partial_y^2 u + \beta\partial_x^2\partial_y^2 u = 0. \quad (1)$$

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Equation (1) was derived by R. M. Chen in [6] as a model for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state. In this model u represents vertical displacement of the plate relative to a uniformly pre-stressed state, while x and y are rescaled longitudinal and lateral coordinates in the horizontal plane. To reduce the full three-dimensional field equation to an approximate two-dimensional plate equation, an assumption has been made that the thickness of the plate is small in comparison to the other dimensions. It is also assumed that the small perturbations superimposed on the pre-stressed state only appear in the vertical direction (the z-direction) and in one horizontal direction (the x-direction). Hence the variation of waves in the transverse direction (the y-direction) is small. Equation (1) is obtained under the additional assumption that the wavelength in the x-direction is short. On the other hand, if the wavelength is large, we obtain the Kadomtsev-Petviashvili (KP) equation.

The parameters in equation (1) are all material constants. The scalar μ describes the stiffness of the plate which is nonnegative. The coefficients α and β are material constants that measure weak transverse effects. The material constant γ occurs as a consequence of the balance between the nonlinear and dispersive effects. Note that there is no dissipation in this model.

Equation (1) generalizes several well-known equations including the BBM equation [2] when $\mu=\alpha=\beta=\gamma=0$, the regularized long-wave Kadomtsev-Petviashvili (KP) equation [3] (also referred as KP-BBM equation, see [8]) when $\mu=\beta=\gamma=0$, and the Camassa-Holm (CH) equation [5] when $\delta=\alpha=\beta=0, \gamma=1$. In contrast to the derivation in [6] of nonlinear dispersive waves in a hyperelastic plate, these particular equations are usually derived as models of water waves. In equation (1), the two spatial dimensions make the analysis very different from the CH equation. The $\mu-$terms include a nonlinear term of fourth order, which makes equation (1) very different from the KP-BBM equation.

For equations that model the evolution of nonlinear waves, it is very important to determine the existence and uniqueness of solution for the associated initial value problem, and the existence of special solutions as the travelling waves. For instance, travelling wave solutions are important in the study of dynamics of wave propagation in many applied models such as fluid dynamics, acoustic, oceanography, and weather forecasting. An important application is the use of solitons (travelling wave of finite energy) as an efficient means of long-distance communication.

For $\gamma \in \mathbb{R}, \mu, \alpha, \beta > 0$ and $p=1$, R. M. Chen (see [7]) showed, in the "nonstandard" space Sobolev type $W(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_W^2 = \int_{\mathbb{R}} \big[u^2 + u_x^2 + u_{xx}^2 + (\partial_x^{-1} u_y)^2 + u_y^2\big] dxdy,$$

the existence (in the weak sense) and stability of two-dimensional travelling wave solutions (2D-solitons) which propagate with speed wave $c > 0$, i.d. solutions of the form $u(x,y,t) = v(x-ct, y)$. For this, R. M. Chen used the Concentration-Compactness Theorem. Formally $\partial_x^{-1}u_y$ is defined via the Fourier transform as

$$\widehat{\partial_x^{-1}u_y} = \frac{\eta}{\xi}\widehat{u}(\xi,\eta).$$

In this paper, using the Mountain Pass Theorem, for $p = p_1/p_2$ with $(p_1, p_2) = 1$ (p_2 odd), $\gamma \in \mathbb{R}$ and $\mu, \alpha, \beta > 0$ we establish the existence, regularity and analyticity of one-dimensional travelling wave solutions (1D-solitons) for the equation (1). This is, solutions of the form

$$u(x, y, t) = v(x + y - ct), \tag{2}$$

which propagate with wavefront normal to $z = (1, 1) \in \mathbb{R}^2$, velocity $\frac{c}{|z|}$, and profile v. First we show the existence of 1D-solitons in the "standard" Sobolev space $H^2(\mathbb{R})$ equipped with the norm

$$\|v\|_{H^2}^2 = \int_{\mathbb{R}} \big[v^2 + (v')^2 + (v'')^2\big]\,dx.$$

We will use a variational approach for which travelling waves corresponding to critical points of a suitable energy functional. Next, we prove that this solutions are regular functions. Moreover, we prove that this solutions admit a Taylor expansion (analytic solutions). These characteristics make the 1D-solitons very interesting from the physical and numerical view points.

1.1. Variational approach

By a 1D-soliton for the equation (1) we shall mean a solution of the form (2), where c denote the speed of the wave. Then one see that the travelling wave profile v should satisfy the ordinary differential equation

$$\Big[(c+\alpha)v - (c+\beta)v' + c\mu v'''' - \frac{p+2}{p+1}v^{p+1} + \gamma\Big(v\big[(v')^p\big]' + \frac{p}{p+1}(v')^{p+1}\Big)\Big]'' = 0. \tag{3}$$

Among all the travelling wave solutions of (1) we shall focus on solutions which have the additional property that the waves are localized and that v and its derivatives decay at infinity, that is,

$$v^{(k)}(y) \to 0 \quad \text{as} \quad |y| \to \infty, \quad 0 \le k \le 6.$$

We denote by $C_0^k(\mathbb{R})$ the space of real functions with k continuous derivatives vanishing at infinity, with the obvious sense when $k = \infty$. Under this decay assumption the travelling wave equation takes the form

$$(c+\alpha)v - (c+\beta)v'' + c\mu v'''' - \frac{p+2}{p+1}v^{p+1} + \gamma\Big(v\,[(v')^p]' + \frac{p}{p+1}(v')^{p+1}\Big) = 0. \tag{4}$$

A classical solution of (4) is a $C_0^4(\mathbb{R})$ function satisfying (4) in the usual sense. Multiply the travelling wave equation (4) with a test function $w \in C_0^\infty(\mathbb{R})$, after integration by parts, we obtain

$$\int_{\mathbb{R}} \Big[(c+\alpha)vw + (c+\beta)v'w' + c\mu v''w''\Big]dx \\ - \int_{\mathbb{R}} \Big[\frac{p+2}{p+1}v^{p+1}w + \gamma\Big(v(v')^p w' + \frac{1}{p+1}(v')^{p+1}w\Big)\Big]dx = 0. \quad (5)$$

Note that (5) makes sense as soon as $v \in C_0^2(\mathbb{R})$, whereas (4) requires four derivatives on v. Let us say (provisionally) that a $C_0^2(\mathbb{R})$ function v that satisfies (5) is a weak solution of (4).

The following program outlines the main steps of the *variational approach* in the theory of partial differential equations (see Section 8.1 in [4]):

Step A. The notion of weak solution is made precise. This involves Sobolev spaces.

Step B. Existence of a weak solution is established by a variational method, via the Mountain Pass Theorem in our case.

Step C. Weak solution is proved to be of class $C_0^4(\mathbb{R})$ (for example): this is a regularity result.

Step D. A classical solution is recovered by showing that any weak solution that is $C_0^4(\mathbb{R})$ is a classical solution.

To carry out Step D is very simple. In fact, suppose that $v \in C_0^4(\mathbb{R})$ satisfies (5). Then integrating by parts (5) we obtain for all $w \in C_0^\infty(\mathbb{R})$ that

$$\int_{\mathbb{R}} \Big[(c+\alpha)v - (c+\beta)v'' + c\mu v'''' - \frac{p+2}{p+1}v^{p+1}\Big]w\,dx \\ + \gamma\int_{\mathbb{R}} \Big[v\big((v')^p\big)' + \frac{p}{p+1}(v')^{p+1}\Big]w\,dx = 0.$$

Hence, using that the space $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the equation (4) holds a.e. on $\mathbb{R}$ and thus everywhere on $\mathbb{R}$, since $v \in C_0^4(\mathbb{R})$. Finally, we notice that if $v \in C_0^6(\mathbb{R})$ is a solution for the equation (4) then v is a classical solution for the equation (3).

Throughout this work $\|\cdot\|_X$ denotes norm in the Hilbert space X, $<,>_X$ is its inner product and X' represents the dual space. C denotes a generic constant whose value may change from instance to instance.

2. Existence of weak solutions

In this section we will establish the existence of a solution of (4) in the weak sense by using a variational approach in which weak solutions correspond to critical points of a suitable functional. We begin by defining the appropriate functional spaces. The usual Sobolev space $H^k(\mathbb{R}), k \in \mathbb{Z}^+ \cup \{0\}$, is the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R})$ with respect to the inner product

$$\langle v, w\rangle_{H^k} = \sum_{n=0}^{k} \int_{\mathbb{R}} v^{(n)} \cdot w^{(n)} dx,$$

and the norm

$$\|v\|_{H^k}^2 = \sum_{n=0}^{k} \int_{\mathbb{R}} \left[v^{(n)}\right]^2 dx.$$

Now, if we assume that $v, w \in H^2(\mathbb{R})$, from the Young inequality, we see that

$$\int_{\mathbb{R}} \Big[(c+\alpha)vw + (c+\beta)v'w' + c\mu v''w''\Big] dx \le C(c,\alpha,\beta,\mu) \left(\|v\|_{H^2}^2 + \|w\|_{H^2}^2\right).$$

So that, the first integral in (5) is well defined. In a similar way, using the Hölder inequality and the fact that for $q \ge 2$ the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous, we see that

$$\begin{aligned}
\int_{\mathbb{R}} v^{p+1} w\, dx, \int_{\mathbb{R}} (v')^{p+1} w\, dx &\le \left(\|v\|_{L^{2(p+1)}}^{p+1} + \|v'\|_{L^{2(p+1)}}^{p+1}\right) \|w\|_{L^2} \\
&\le C \left(\|v\|_{H^1}^{p+1} + \|v'\|_{H^1}^{p+1}\right) \|w\|_{L^2} \\
&\le 2C \|v\|_{H^2}^{p+1} \|w\|_{H^2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\int_{\mathbb{R}} v(v')^p w\, dx &\le \|v\|_{L^4} \|v'\|_{L^{4p}}^p \|w\|_{L^2} \\
&\le C \|v\|_{L^4} \|v'\|_{H^1}^p \|w\|_{L^2} \\
&\le C \|v\|_{H^2}^{p+1} \|w\|_{H^2}.
\end{aligned}$$

Therefore, the second integral in (5) is well defined. Then we have the following definition.

Definition 2.1. We say that $v \in H^2(\mathbb{R})$ is a weak solution of (4) if for all $w \in H^2(\mathbb{R})$ the integral equation (5) holds.

Next, we will see that weak solutions of the equation (4) corresponds to critical points of the functional J_c defined as

$$J_c(v) = I_c(v) + G(v),$$

where

$$\begin{aligned}
I_c(v) &= \frac{1}{2} \int_{\mathbb{R}} \left[(c+\alpha)v^2 + (c+\beta)(v')^2 + c\mu(v'')^2\right] dx, \\
G(v) &= \frac{1}{p+1} \int_{\mathbb{R}} \left[v^{p+2} + \gamma v (v')^{p+1}\right] dx.
\end{aligned}$$

First we will show in the following lemma some properties for I_c and G, assuming that $p = p_1/p_2$ with $(p_1, p_2) = 1$ (p_2 odd), $\gamma \in \mathbb{R}$ and $\mu, \alpha, \beta > 0$.

Lemma 2.2. *Let $c > 0$. Then*

1. *The functionals I_c and G are well defined in $H^2(\mathbb{R})$.*
2. *The functional I_c is nonnegative. Moreover, there are $C_1(\alpha, \beta, \mu, c) < C_2(\alpha, \beta, \mu, c)$ such that*

$$C_1\|v\|_{H^2}^2 \le I_c(v) \le C_2\|v\|_{H^2}^2. \tag{6}$$

Proof. 1. I_c is clearly well defined for $v \in H^2(\mathbb{R})$. In addition, note that if $v \in H^2(\mathbb{R})$ then, using the fact that the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous for $q \ge 2$, we see that there is a constant $C = C(p, \gamma) > 0$ such that

$$|G(v)| \le C\left(\|v\|_{L^{p+2}}^{p+2} + \|v\|_{L^2}\|v'\|_{L^{2(p+1)}}^{p+1}\right) \le C\|v\|_{H^2}^{p+2}. \tag{7}$$

So, G is well defined.
2. This property is straightforward. We define C_1, C_2 by

$$C_1 = \min\{c+\alpha, c+\beta, c\mu\}, \quad C_2 = \max\{c+\alpha, c+\beta, c\mu\}.$$

□

Proposition 2.3. *If v is a nontrivial critical point for the functional J_c in the space $H^2(\mathbb{R})$, then v is a nontrivial weak solution for the equation (4).*

Proof. If $v, w \in H^2(\mathbb{R})$, a direct calculation shows that

$$\langle I_c'(v), w\rangle = \int_{\mathbb{R}}\Big[(c+\alpha)vw + (c+\beta)v'w' + c\mu v''w''\Big]dx,$$
$$\langle G'(v), w\rangle = \int_{\mathbb{R}}\Big[\frac{p+2}{p+1}v^{p+1}w + \gamma\Big(v(v')^p w' + \frac{1}{p+1}(v')^{p+1}w\Big)\Big]dx.$$

In particular, if $v \in H^2(\mathbb{R})$ is a critical point for the functional J_c we see that for all $w \in H^2(\mathbb{R})$,

$$\langle I_c'(v), w\rangle - \langle G'(v), w\rangle = \langle J_c'(v), w\rangle = 0. \tag{8}$$

Therefore, v is a solution of the integral equation (5). □

Our approach to show the existence of a nontrivial critical point for the functional J_c is to use the Mountain Pass Theorem without the Palais-Smale condition (see M. Willem [10], A. Ambrosetti *et al.* [1]) to build a Palais-Smale sequence for J_c for a minimax value and use a local embedding result to obtain a critical point for J_c as a weak limit of such Palais-Smale sequence.

Theorem 2.4. *(Mountain Pass Theorem) Let X be a Hilbert space, $\varphi \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\|_X > r$ and*

$$\vartheta = \inf_{\|v\|_X = r} \varphi(v) > \varphi(0) \ge \varphi(e).$$

Then, given $n \in \mathbb{N}$*, there is* $v_n \in X$ *such that*

$$\varphi(v_n) \to \delta, \quad \varphi'(v_n) \to 0 \quad \text{in } X' \quad \text{and} \quad \delta \geq \vartheta, \tag{PS}$$

where

$$\delta = \inf_{\pi \in \Pi} \max_{t \in [0,1]} \varphi(\pi(t)), \quad \text{and} \quad \Pi = \{\pi \in C([0,1], X) : \pi(0) = 0, \quad \pi(1) = e\}.$$

Before we go further, we establish an important result for our analysis. $B_r(\zeta)$ denotes the ball in $\mathbb{R}$ of center ζ and radius $r > 0$.

Lemma 2.5. *If* $(v_n)_n$ *is a bounded sequence in* $H^2(\mathbb{R})$ *and there is a positive constant* $r > 0$ *such that*

$$\lim_{n \to \infty} \sup_{\zeta \in \mathbb{R}} \int_{B_r(\zeta)} (v_n)^2 \, dx = 0.$$

Then we have that

$$\lim_{n \to \infty} G(v_n) = 0. \tag{9}$$

***Proof*.** Let $\zeta \in \mathbb{R}$ and $r > 0$. Using the Hölder inequality and the fact that the embedding $H^1(B_r(\zeta)) \hookrightarrow L^q(B_r(\zeta))$ is continuous for $q \geq 2$, we see that

$$\begin{aligned}
\int_{B_r(\zeta)} |v_n||v_n'|^{p+1} dx &\leq \|v_n\|_{L^2(B_r(\zeta))} \|v_n'\|^{p+1}_{L^{2(p+1)}(B_r(\zeta))} \\
&\leq C \|v_n\|_{L^2(B_r(\zeta))} \|v_n'\|^{p+1}_{H^1(B_r(\zeta))} \\
&\leq C \|v_n\|^{p+1}_{H^2(B_r(\zeta))} \left(\sup_{\zeta \in \mathbb{R}} \int_{B_r(\zeta)} |v_n|^2 dx \right)^{1/2}.
\end{aligned}$$

Now, covering $\mathbb{R}$ by balls of radius r in such a way that each point of $\mathbb{R}$ is contained in at most two balls, we find

$$\int_{\mathbb{R}} |v_n||v_n'|^{p+1} dx \leq 2C \|v_n\|^{p+1}_{H^2(\mathbb{R})} \left(\sup_{\zeta \in \mathbb{R}} \int_{B_r(\zeta)} |v_n|^2 dx \right)^{1/2}.$$

Thus, under the assumptions of the lemma,

$$\lim_{n \to \infty} \int_{\mathbb{R}} |v_n||v_n'|^{p+1} dx = 0.$$

In a similar fashion we obtain that $\lim_{n \to \infty} \int_{\mathbb{R}} v_n^{p+2} dx = 0$. So that

$$\lim_{n \to \infty} G(v_n) = \frac{1}{p+1} \lim_{n \to \infty} \int_{\mathbb{R}} \left[v_n^{p+2} + \gamma v_n (v_n')^{p+1} \right] dx = 0.$$

□

Now, we want to verify the Mountain Pass Theorem hypotheses given in Theorem 2.4 and to build a Palais-Smale sequence for J_c.

Lemma 2.6. *Let $c > 0$. Then*

1. *There exists $\rho > 0$ small enough such that*
$$\vartheta(c) := \inf_{\|z\|_{H^2(\mathbb{R})}=\rho} J_c(z) > 0.$$
2. *There is $e \in H^2(\mathbb{R})$ such that $\|e\|_{H^2} \geq \rho$ and $J_c(e) \leq 0$.*
3. *If $\delta(c)$ is defined as*
$$\delta(c) = \inf_{\pi\in\Pi} \max_{t\in[0,1]} J_c(\pi(t)),\ \Pi = \{\pi \in C\big([0,1], H^2\big) : \pi(0) = 0,\ \pi(1) = e\},$$
then $\delta(c) \geq \vartheta(c)$ and there is a sequence $(v_n)_n$ in $H^2(\mathbb{R})$ such that
$$J_c(v_n) \to \delta, \quad J_c'(v_n) \to\ 0 \ \ in \ \ \big(H^2(\mathbb{R})\big)'. \tag{10}$$

Proof. From inequalities (6)-(7), we have for any $v \in H^2(\mathbb{R})$ that

$$\begin{aligned} J_c(v) &\geq C_1\|v\|_{H^2}^2 - C(\lambda, p)\|v\|_{H^2}^{p+2} \\ &\geq \left(C_1 - C\|v\|_{H^2}^p\right)\|v\|_{H^2}^2. \end{aligned}$$

Then for $\rho > 0$ small enough such that

$$C_1 - \rho^p C > 0, \tag{11}$$

we conclude for $\|v\|_{H^2(\mathbb{R})} = \rho$ that

$$J_c(v) \geq (C_1 - \rho^p C)\,\rho^2 := \epsilon > 0.$$

In particular, we also have that

$$\vartheta(c) = \inf_{\|z\|_{H^2}=\rho} J_c(z) \geq \epsilon > 0.$$

For any $t \in \mathbb{R}$ we see that

$$J_c(tv_0) = t^2\left[I_c(v_0) - \frac{t^p}{p+1}\int_{\mathbb{R}}\left(v_0^{p+2} + \gamma v_0(v_0')^{p+1}\right)dx\right].$$

Using the hypothesis, it is not hard to prove that there exist $v_0 \in C_0^\infty(\mathbb{R}) \subset H^2(\mathbb{R})$ such that $G(v_0) > 0$. So that,

$$\lim_{t\to\infty} J_c(tv_0) = -\infty,$$

because $0 \leq I_c(v_0) \leq C_2(c)\|v_0\|_{H^2}^2$. Then, there is $t_0 > 0$ such that $e = t_0 v_0 \in H^2(\mathbb{R})$ satisfies that

$$t_0\|v_0\|_{H^2} = \|e\|_{H^2} > \rho$$

and that $J_c(e) \leq J_c(0) = 0$. The third part follows by direct applying Theorem 2.4. □

The following theorem is our main result in this section.

Theorem 2.7. *Let $c > 0$. The equation (4) admits nontrivial weak solutions in the space $H^2(\mathbb{R})$.*

Proof. We will see that $\delta(c)$ is in fact a critical value of J_c. Let $(v_n)_n$ be the sequence in $H^2(\mathbb{R})$ given by previous lemma. First note from (12) that

$$\delta(c) \geq \vartheta(c) \geq \epsilon > 0. \tag{12}$$

Using the definition of J_c we have that

$$\begin{aligned} \langle J_c'(v), v\rangle &= 2I_c(v) - (p+2)G(v) \\ &= 2J_c(v) - pG(v). \end{aligned} \tag{13}$$

Then we obtain that

$$I_c(v_n) = \frac{p+2}{p} J_c(v_n) - \frac{1}{p}\left\langle J_c'(v_n), v_n\right\rangle.$$

But from (6) we conclude for n large enough that

$$C_1\|v_n\|_{H^2}^2 \leq I_c(v_n) \leq \frac{p+2}{p}(\delta(c)+1) + \|v_n\|_{H^2}.$$

Then we have shown that $(v_n)_n$ is a bounded sequence in $H^2(\mathbb{R})$. We claim that

$$\epsilon^* = \overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \int_{B_1(\zeta)} (v_n)^2 dx > 0.$$

Suppose that

$$\overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \int_{B_1(\zeta)} (v_n)^2 dx = 0.$$

Hence, from Lemma 2.5 we conclude that

$$\lim_{n\to\infty} G(v_n) = 0.$$

Then, we have from (12)-(13) that

$$\begin{aligned} 0 < \epsilon \leq \delta(c) &= J_c(v_n) - \frac{1}{2}\left\langle J_c'(v_n), v_n\right\rangle + o(1) \\ &\leq \frac{p}{2}G(v_n) + o(1) \\ &\leq o(1), \end{aligned}$$

but this is a contradiction. Thus, there is a subsequence of $(v_n)_n$, denoted the same, and a sequence $\zeta_n \in \mathbb{R}$ such that

$$\int_{B_1(\zeta_n)} (v_n)^2 dx \geq \frac{\epsilon^*}{2}. \tag{14}$$

Now, we define the sequence $\tilde{v}_n(x) = v_n(x+\zeta_n)$. For this sequence we also have that

$$\|\tilde{v}_n\|_{H^2} = \|v_n\|_{H^2}, \quad J_c(\tilde{v}_n) \to \delta \quad \text{and} \quad J_c'(\tilde{v}_n) \to 0 \ \text{ in } \ \left(H^2(\mathbb{R})\right)'.$$

Then $(\tilde{v}_n)_n$ is a bounded sequence in $H^2(\mathbb{R})$. Thus, for some subsequence of $(\tilde{v}_n)_n$, denoted the same, and for some $v \in H^2(\mathbb{R})$ we have that

$$\tilde{v}_n \rightharpoonup v, \quad \text{as } n \to \infty \quad (\text{weakly in } H^2(\mathbb{R})).$$

Since the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact for all bounded open set Ω, we see that

$$\tilde{v}_n \to v \ \text{ in } L^2_{loc}(\mathbb{R}).$$

Note that $v \neq 0$ because using (14) we have that

$$\int_{B_1(0)} v^2 dx = \lim_{n\to\infty} \int_{B_1(0)} (\tilde{v}_n)^2 dx = \lim_{n\to\infty} \int_{B_1(\zeta_n)} (v_n)^2 dx \geq \frac{\epsilon^*}{2}.$$

Moreover, if $w \in C_0^\infty(\mathbb{R})$, then for $K = supp\, w$ we have that

$$\begin{aligned}
\langle I_c'(v), w\rangle &= \int_K \left[(c+\alpha)vw + (c+\beta)v'w' + c\mu v''w''\right] dx \\
&= \lim_{n\to\infty} \int_K \left[(c+\alpha)\tilde{v}_n w + (c+\beta)\tilde{v}_n' w' + c\mu \tilde{v}_n'' w''\right] dx \\
&= \lim_{n\to\infty} \langle I_c'(\tilde{v}_n), w\rangle .
\end{aligned}$$

Now (taking a subsequence, if necessary) noting that

$$(\tilde{v}_n)^{p+1} \rightharpoonup v^{p+1}, \quad (\tilde{v}_n')^{p+1} \rightharpoonup (v')^{p+1}, \quad \tilde{v}_n(\tilde{v}_n')^p \rightharpoonup v(v')^p \quad \text{in} \quad L^2_{loc}(\mathbb{R}),$$

we have that

$$\int_K (\tilde{v}_n)^{p+1} w\, dx \to \int_K v^{p+1} w\, dx, \quad \int_K (\tilde{v}_n')^{p+1} w\, dx \to \int_K (v')^{p+1} w dx$$

and

$$\int_K \tilde{v}_n(\tilde{v}_n')^p w' dx \to \int_K v(v')^p w' dx.$$

Then we conclude that

$$\langle G'(v), w\rangle = \lim_{n\to\infty} \langle G'(\tilde{v}_n), w\rangle ,$$

and also that

$$\langle J_c'(v), w\rangle = \lim_{n\to\infty} \langle J_c'(\tilde{v}_n), w\rangle = 0.$$

If $w \in H^2(\mathbb{R})$, by using density, there is $w_k \in C_0^\infty(\mathbb{R})$ such that $w_k \to w$ in $H^2(\mathbb{R})$. Hence,

$$\begin{aligned}
|\langle J_c'(v), w\rangle| &\leq |\langle J_c'(v), w - w_k\rangle| + |\langle J_c'(v), w_k\rangle| \\
&\leq \|J_c'(v)\|_{(H^2)'} \|w - w_k\|_{H^2} + |\langle J_c'(v), w_k\rangle| \to 0.
\end{aligned}$$

Thus, we have already established that

$$J_c'(v) = 0.$$

In other words, $v \in H^2(\mathbb{R})$ is a nontrivial weak solution for the equation (4). $\square$

3. Regularity and analyticity of solutions

In this section, we will establish that any weak solution of the equation (4) is a regular and analytic function. For this, we will use that for $s \geq 0$ there are $K_1, K_2 > 0$ such that for all $v \in H^s(\mathbb{R})$,

$$K_1 \|v\|_{H^s}^2 \leq \int_{\mathbb{R}} \left(1+\xi^2\right)^s |\widehat{v}|^2 d\xi \leq K_2 \|v\|_{H^s}^2,$$

where the Fourier transform of a function v defined on $\mathbb{R}$ is given by

$$\widehat{v}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix\xi} v(x) dx.$$

In addition, we will use the following result (see Proposition 16 (c) in F. H. Soriano [9]).

Proposition 3.1. *For $s > 1$ there exists $C_3 > 0$ such that for all p and $k \in \mathbb{Z}^+$,*

$$\sum_{k_1+\cdots+k_p=k} \frac{1}{(k_1+1)^s \cdots (k_p+1)^s} \leq \frac{C_3^{p-1}}{(k+1)^s}.$$

Theorem 3.2. *Let $c > 0$. If $v \in H^2(\mathbb{R})$ is a weak solution of the equation (4) then v is a classic solution. Moreover, v is a analytic function, this is, for each $\zeta_0 \in \mathbb{R}$ there is $R > 0$ such that*

$$\sum_k \frac{D^k v(\zeta_0)}{k!} (x-\zeta_0)^k$$

converges absolutely in $\mathbb{R}$ to $v(x)$ for all $x \in B_R(\zeta_0)$.

Proof. First we will establish that $v \in H^l(\mathbb{R})$ for any $l \geq 0$, if $v \in H^2(\mathbb{R})$ is a weak solution of (4). Thus, using that if $f \in H^l(\mathbb{R}), l \geq 1+k$, then $f \in C_0^k(\mathbb{R})$, we conclude that v is a classical solution.

From the facts $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $H^1(\mathbb{R}) \hookrightarrow L^{2(p+1)}(\mathbb{R})$ we have that the functions

$$f = \frac{p+2}{p+1} v^{p+1}, \quad g = -\gamma v \left[(v')^p\right]', \quad h = -\frac{\gamma p}{p+1} (v')^{p+1} \tag{15}$$

belong to $L^2(\mathbb{R})$. Since

$$\int_{\mathbb{R}} \left[v(v')^{p-1} v''\right]^2 dx \leq \|v\|_{L^\infty}^2 \|v'\|_{L^\infty}^{2(p-1)} \|v''\|_{L^2}^2 \leq C \|v\|_{H^2}^{2(p+1)}$$

and

$$\int_{\mathbb{R}} v^{2(p+1)}dx, \int_{\mathbb{R}} (v')^{2(p+1)}dx \leq C\|v\|_{H^2}^{2(p+1)}.$$

Taking Fourier transform on the equation (4) we obtain that $\widehat{v}$ satisfies

$$\widehat{v}(\xi) = \frac{\widehat{f}(\xi) + \widehat{g}(\xi) + \widehat{h}(\xi)}{(c+\alpha) + (c+\beta)\xi^2 + c\mu\xi^4}.$$

Hence, there exists $M = M(\alpha, \beta, \mu, c)$ such that

$$\int_{\mathbb{R}} \left(1+\xi^2\right)^4 |\widehat{v}|^2 d\xi \leq M \int_{\mathbb{R}} \left(|\widehat{f}|^2 + |\widehat{g}|^2 + |\widehat{h}|^2\right) d\xi = M \left(\|f\|_{L^2}^2 + \|g\|_{L^2}^2 + \|h\|_{L^2}^2\right).$$

This implies that $v \in H^4(\mathbb{R})$. Now, since $H^l(\mathbb{R})$ is a algebra for $l \geq 1$, by using (15) we see that $f, g, h \in H^2(\mathbb{R})$. Then, repeating the previous argument we have that $v \in H^8(\mathbb{R})$. A simple bootstrapping argument then yields that $v \in H^l(\mathbb{R})$ for all $l \geq 0$.

Next, we will prove the analyticity of v. First we establish the result under the assumption of the existence of $R > 0$ such that for all $k \in \mathbb{N}_0$,

$$\left\|D^k v\right\|_{H^2} \leq C\frac{k!}{(k+1)^2}R^k, \tag{16}$$

where $D^k v = v^{(k)}$. If $\zeta_0 \in \mathbb{R}$, we will show that there exists $r > 0$ such that we have the following Taylor expansion for v in $B_r(\zeta_0)$,

$$v(x) = \sum_k \frac{D^k v(\zeta_0)}{k!}(x-\zeta_0)^k.$$

If we set $\zeta = x - \zeta_0$, then by the Taylor Theorem (with remainder) we have that

$$v(x) = \sum_{k=0}^{N-1} \frac{D^k v(\zeta_0)}{k!}\zeta^k + \mathcal{E}_N(x), \quad \mathcal{E}_N(x) = \frac{D^N v(\zeta_0 + t\zeta)}{N!}\zeta^N.$$

On the other hand, using (16) and the embedding $H^l(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $l \geq 1$, if $k \in \mathbb{N}_0$ we have that

$$|D^k v(x)| \leq ||D^k v||_{L^\infty} \leq ||D^k v||_{H^2} \leq CR^k \frac{k!}{(k+1)^2}.$$

If we take $r > 0$ in such a way that $2rR < 1$, we conclude for $|\zeta| < r$ that

$$|\mathcal{E}_N(x)| \leq C\frac{(rR)^N}{(N+1)^2} \leq C(rR)^N \leq C2^{-N}.$$

In other words, the Taylor series for v converges in $B_R(\zeta_0)$.

To complete the proof, we only need to prove that there exists $R > 0$ such that (16) holds for all $k \geq 0$. We will argue by induction on k. Since $v \in H^l(\mathbb{R}), l \geq 0$, we have the result for $k = 0, 1$. Now, suppose that (16) holds for fixed $k \in \mathbb{Z}^+$ and R (which will be chosen later). If we apply the operator D^k to equation (4), then multiply with $D^k v$, after integration by parts, we obtain that

$$2I_c(D^k v) = \frac{p+2}{p+1}\left\langle D^k(v^{p+1}), D^k v\right\rangle_{L^2} + \gamma\Big(\left\langle D^k\left[v(v')^p\right], D^k v'\right\rangle_{L^2} + \frac{1}{p+1}\left\langle D^k\left[(v')^{p+1}\right], D^k v\right\rangle_{L^2}\Big). \tag{17}$$

But by using Hölder inequality,

$$\begin{aligned}\left|\left\langle D^k\left[v(v')^p\right], D^k v'\right\rangle_{L^2}\right| &\leq \|D^k\left[v(v')^p\right]\|_{L^2}\|D^k v'\|_{L^2}\\ &\leq C\|D^k\left[v(v')^p\right]\|_{L^2}\|D^k v\|_{H^2}.\end{aligned}$$

In addition, we see that

$$\left|\left\langle D^k(v^{p+1}), D^k v\right\rangle_{L^2}\right| \leq \|D^k(v^{p+1})\|_{L^2}\|D^k v\|_{L^2}.$$

Also we have that

$$\left|\left\langle D^k\left[(v')^{p+1}\right], D^k v\right\rangle_{L^2}\right| \leq \left\|D^k\left[(v')^{p+1}\right]\right\|_{L^2}\|D^k v\|_{L^2}.$$

Then applying in (17) the previous estimates and inequality (6) we obtain that

$$\|D^k v\|_{H^2(\mathbb{R})} \leq C_1\Big(\|D^k(v^{p+1})\|_{L^2} + \|D^k\left[v(v')^p\right]\|_{L^2} + \|D^k\left[(v')^{p+1}\right]\|_{L^2}\Big).$$

We want to estimate the terms of right hand side. For simplicity first we consider $p = 1$. Thus, note that if $u, w \in H^l$ for any $l \geq 1$, we have for $k \in \mathbb{Z}^+$ that

$$D^k(uw) = \left(D^k u\right)w + \sum_{m=1}^{k-1}\binom{k}{m}(D^{k-m}u)D^m(w) + uD^k w.$$

Then we see, for example, that

$$D^k\left[(v')^2\right] = 2D^k(v')v' + \sum_{m=1}^{k-1}\binom{k}{m}D^{k-m}(v')D^m(v'). \tag{18}$$

Using the induction hypothesis we have that

$$\begin{aligned}\|D^k(v')v'\|_{L^2} &\leq \|D^k(v')\|_{L^2}\|v'\|_{L^\infty}\\ &\leq C_2\|D^{k+1}v\|_{L^2}\|v'\|_{H^1}\\ &\leq C_2\|D^{k-1}v\|_{H^2}\|v\|_{H^2}\\ &\leq C_2C^2R^{k-1}\frac{(k-1)!}{k^2}\\ &\leq \left(C\frac{(k+1)!}{(k+2)^2}R^{k+1}\right)\left(C_2CR^{-2}\frac{(k+2)^2}{k^2(k+1)}\right).\end{aligned}$$

Also we obtain that

$$\begin{aligned}\|D^{k-m}(v')D^m(v')\|_{L^2} &\leq \|D^{k-m+1}v\|_{L^2}\|D^m(v')\|_{L^\infty}\\ &\leq C_2\|D^{k-m-1}v\|_{H^2}\|D^m(v')\|_{H^1}\\ &\leq C_2\|D^{k-m-1}v\|_{H^2}\|D^m v\|_{H^2}.\end{aligned}$$

Hence, using induction hypothesis on the right hand side and Proposition 3.1, we obtain that

$$\begin{aligned}\sum_{m=1}^{k-1}\binom{k}{m}\|D^{k-m}(v')D^m(v')\|_{L^2} &\leq C_2\sum_{m=1}^{k-1}\frac{k!}{(k-m)!m!}\|D^{k-m-1}v\|_{H^2}\|D^m v\|_{H^2}\\ &\leq C_2C^2k!R^{k-1}\sum_{m=1}^{k-1}\frac{1}{(k-m)^3(m+1)^2}\\ &\leq C_2C^2k!R^{k-1}\sum_{k_1+k_2=k-1}\frac{1}{(k_1+1)^2(k_2+1)^2}\\ &\leq \left(C\frac{(k+1)!}{(k+2)^2}R^{k+1}\right)\left(C_2C_3CR^{-2}\frac{(k+2)^2}{k^2(k+1)}\right).\end{aligned}$$

Note that there exists $M>0$ such that $\frac{(k+2)^2}{k^2(k+1)}<M$. So, taking $R>0$ large enough such that

$$C_2C_3CMR^{-2}<1,$$

we conclude that

$$\|D^k[(v')^2]\|_{L^2}\leq C\frac{(k+1)!}{(k+2)^2}R^{k+1}.$$

Next, we see that

$$\begin{aligned}\|\left(D^k v\right)v'\|_{L^2} &\leq \|D^k v\|_{L^2}\|v'\|_{L^\infty}\\ &\leq C_2\|D^{k-1}v\|_{H^1}\|v'\|_{H^1}\\ &\leq C_2\|D^{k-1}v\|_{H^2}\|v\|_{H^2},\end{aligned}$$

and also that

$$\|D^k(v')v\|_{L^2}\leq \|D^{k+1}v\|_{L^2}\|v\|_{L^\infty}\leq C_2\|D^{k-1}v\|_{H^2}\|v\|_{H^2}.$$

Moreover,

$$\|D^{k-m}(v)D^m(v')\|_{L^2}\leq \|D^{k-m-1}v\|_{H^1}\|D^m(v')\|_{H^1}\leq C_2\|D^{k-m-1}v\|_{H^2}\|D^m v\|_{H^2}.$$

Then we obtain that

$$\|D^k(vv')\|_{L^2}\leq C\frac{(k+1)!}{(k+2)^2}R^{k+1}.$$

In a similar fashion we obtain that

$$\|D^k(v^2)\|_{L^2} \leq C\frac{(k+1)!}{(k+2)^2}R^{k+1}.$$

In other words, we have shown for R large enough that

$$\|D^k(v^2)\|_{L^2} + \|D^k\,(vv')\,\|_{L^2} + \|D^k\left[(v')^2\right]\|_{L^2} \leq C\frac{(k+1)!}{(k+2)^2}R^{k+1}.$$

Now we consider the case $p \geq 1$. To illustrate the type of computation we only consider the typical term $D^k[(v')^{p+1}]$, for this we will use the general rule

$$D^k[(v')^{p+1}] = \sum_{k_0+k_1+\cdots+k_p=k} \binom{k}{k_0, k_1 \ldots, k_p} (D^{k_0}v')(D^{k_1}v')\cdots(D^{k_p}v'),$$

where the sum extends over all $(p+1)$-tuples $(k_0, k_1, \ldots, k_p)$ of non-negative integers with

$$\sum_{i=0}^{p} k_i = k \quad \text{and} \quad \binom{k}{k_0, k_1, \ldots, k_p} = \frac{k!}{k_0!k_1!\cdots k_p!}.$$

Define the set $A(k,p)$ as

$$A(k,p) = \Big\{ (k_0, k_1, \ldots, k_p) \ : \ 0 \leq k_i \leq k-1 \text{ and} \sum_{i=0}^{p} k_i = k \Big\}.$$

Then we see that

$$D^k[(v')^{p+1}] = (p+1)(D^kv')(v')^p + \sum_{A(k,p)} \binom{k}{k_0, k_1 \ldots, k_p} (D^{k_0}v')(D^{k_1}v')\cdots(D^{k_p}v').$$

But we have that

$$\begin{aligned}
\|(D^kv')(v')^p\|_{L^2} &\leq \|D^k(v')\|_{L^2}\|v'\|_{L^\infty}^p \\
&\leq (C_2)^p\|D^{k-1}v\|_{H^2}\|v\|_{H^2}^p \\
&\leq \left(C\frac{(k+1)!}{(k+2)^2}R^{k+1}\right)\left((C_2C)^pR^{-2}\frac{(k+2)^2}{k^2(k+1)}\right),
\end{aligned}$$

and also that

$$\begin{aligned}
&\sum_{A(k,p)} \binom{k}{k_0,\dots,k_p} \left\|D^{k_0}(v')\cdots D^{k_p}(v')\right\|_{L^2} \\
&\quad\le (C_2)^p \sum_{A(k,p)} \frac{k!}{k_0!\cdots k_p!} \|D^{k_0-1}v\|_{H^2}\|D^{k_1}v\|_{H^2}\cdots\|D^{k_p}v\|_{H^2} \\
&\quad\le (C_2)^p C^{(p+1)} k! R^{k-1} \sum_{A(k,p)} \frac{1}{k_0^3(k_1+1)^2\cdots(k_p+1)^2} \\
&\quad\le (C_2)^p C^{(p+1)} k! R^{k-1} \sum_{k_0+k_1+\cdots+k_p=k-1} \frac{1}{(k_0+1)^2(k_1+1)^2\cdots(k_p+1)^2} \\
&\quad\le \left(C\frac{(k+1)!}{(k+2)^2}R^{k+1}\right)\left(C_3(C_2C)^pR^{-2}\frac{(k+2)^2}{k^2(k+1)}\right).
\end{aligned}$$

Then we can see for R large enough that

$$\|D^k(v^{p+1})\|_{L^2} + \|D^k\left[v\,(v')^p\right]\|_{L^2} + \|D^k\left[(v')^{p+1}\right]\|_{L^2} \le C\frac{(k+1)!}{(k+2)^2}R^{k+1}. \quad (19)$$

By using (19) we will establish that

$$\left\|D^{k+1}v\right\|_{H^2} \le C\frac{(k+1)!}{k+2}R^{k+1}.$$

To do this, we apply operator D^{k+1} to equation (4) and compute the L^2- inner product with $D^{k+1}v$. Thus, we have that

$$\begin{aligned}
2I_c(D^{k+1}v) &= \frac{p+2}{p+1}\left\langle D^{k+1}(v^{p+1}), D^{k+1}v\right\rangle_{L^2} \qquad (20)\\
&\quad + \gamma\left(\left\langle D^{k+1}\left[v(v')^p\right], D^{k+1}v'\right\rangle_{L^2} + \frac{1}{p+1}\left\langle D^{k+1}\left[(v')^{p+1}\right], D^{k+1}v\right\rangle_{L^2}\right).
\end{aligned}$$

Next, we see that

$$\begin{aligned}
\left|\left\langle D^{k+1}\left[v\,(v')^p\right], D^{k+1}v'\right\rangle_{L^2}\right| &= \left|\left\langle D^{k}\left[v\,(v')^p\right], D^{k+3}v\right\rangle_{L^2}\right| \\
&\le \|D^k\left[v\,(v')^p\right]\|_{L^2}\|D^{k+3}v\|_{L^2} \\
&\le \|D^k\left[v\,(v')^p\right]\|_{L^2}\|D^{k+1}v\|_{H^2}.
\end{aligned}$$

In a similar way we have that

$$\left|\left\langle D^{k+1}(v^{p+1}), D^{k+1}v\right\rangle_{L^2}\right| \le \|D^k(v^{p+1})\|_{L^2}\|D^{k+1}v\|_{H^2},$$

and

$$\left|\left\langle D^{k+1}\left[(v')^{p+1}\right], D^{k+1}v\right\rangle_{L^2}\right| \le \|D^k\left[(v')^{p+1}\right]\|_{L^2}||D^{k+1}v||_{H^2}.$$

Therefore,

$$
\begin{aligned}
&I_c(D^{k+1}v) \\
&\leq C_1 \left(\|D^k(v^{p+1})\|_{L^2} + \|D^k\left[v\,(v')^p\right]\|_{L^2} + \|D^k\left[(v')^{p+1}\right]\|_{L^2} \right) \|D^{k+1}v\|_{H^2}.
\end{aligned}
$$

Then, from (6) and (19) we conclude that

$$
\begin{aligned}
\|D^{k+1}v\|_{H^2} &\leq C_1 \left(\|D^k(v^{p+1})\|_{L^2} + \|D^k\left[v\,(v')^p\right]\|_{L^2} + \|D^k\left[(v')^{p+1}\right]\|_{L^2} \right) \\
&\leq C \frac{(k+1)!}{k+2} R^{k+1},
\end{aligned}
$$

as desired. □

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