On spectral properties and extensions of bounded linear operators

Sobre propiedades espectrales y extensiones de operadores lineales acotados

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Abstract. If T is a bounded linear operator on some Banach space and T has a bounded extension $\overline{T}$ on another space. In general, almost nothing can be said concerning the relationship between the spectral properties of T and $\overline{T}$. In this paper, we give several sufficient conditions for which a large number of spectral properties introduced recently are transmitted from an operator T to $\overline{T}$ and vice-versa.

Keywords: Spectral properties, single-valued extension property, poles of the resolvent, semi Fredholm operator.

Resumen. Si T es un operador lineal acotado en algún espacio de Banach y T tiene una extensión limitada $\overline{T}$ en otro espacio. En general, casi nada se puede decir sobre la relación entre las propiedades espectrales de T y $\overline{T}$. En este trabajo, damos varias condiciones suficientes para que un gran número de propiedades espectrales introducidas recientemente se transmiten de un operador T a $\overline{T}$ y viceversa.

Palabras claves: Propiedades espectrales, propiedad de la extensión univaluada, polos del resolvente, operador semi Fredholm.

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1. Introduction and preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T and by $R(T) = T(X)$ the range of T. We denote by $\alpha(T) = \dim N(T)$ the nullity of T and by $\beta(T) = \operatorname{codim} R(T) = \dim X/R(T)$ the defect of T. Other two classical quantities in operator theory are the *ascent* $p = p(T)$ of an operator T, defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the *descent* $q = q(T)$, defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [15, Prop. 50.2]. An operator $T \in L(X)$ is said to be *Fredholm* (resp. *upper semi -Fredholm*, *lower semi-Fredholm*), if $\alpha(T)$, $\beta(T)$ are both finite (resp. $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. If T is semi-Fredholm then the *index* of T defined by $\operatorname{ind} T = \alpha(T) - \beta(T)$. Other two important classes of operators in Fredholm theory are the classes of upper/lower semi-Browder operators. These classes are defined as follows: $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is a Fredholm (resp. *upper semi-Fredholm, lower semi-Fredholm*) operator and both $p(T)$ and $q(T)$ are finite (resp. $p(T) < \infty$, $q(T) < \infty$). A operator $T \in L(X)$ is said to be *upper semi-Weyl* (resp. *lower semi-Weyl*) if T is upper Fredholm (resp. lower semi-Fredholm) operator and index $\operatorname{ind} T \leq 0$ (resp. $\operatorname{ind} T \geq 0$). $T \in L(X)$ is said to be *Weyl* if T is both upper and lower semi-Weyl, i.e. T is a Fredholm operator having index 0.

The classes of operators defined above generate the following spectra. The *Fredholm spectrum* is defined by

$$\sigma_{\mathrm{f}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$

and the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{uf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Fredholm}\}.$$

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{b}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, the *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{ub}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

and

$$\sigma_{\rm uw}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. According [5] and [6], $T \in L(X)$ is said to be semi *B-Fredholm* (resp. *B-Fredholm*, *upper semi B-Fredholm*, *lower semi B-Fredholm*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as a operator from the space $R(T^n)$ into itself, is a semi-Fredholm (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm) operator. Analogously, $T \in L(X)$ is said to be *B-Browder* (resp. *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder (resp. upper semi-Browder, lower semi -Browder) operator. If T_n is a semi-Fredholm operator, it follows from [6, Proposition 2.1] that also T_m is semi-Fredholm for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of a semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n. Thus, $T \in L(X)$ is said to be a *B-Weyl operator* if T is a B-Fredholm operator having index 0. $T \in L(X)$ is said to be *upper semi B-Weyl* (resp. *lower semi B-Weyl*) if T is upper semi B-Fredholm (resp. lower semi B-Fredholm) with index $\operatorname{ind} T \leq 0$ (resp. $\operatorname{ind} T \geq 0$). Note that if T is B-Fredholm then also T^* is B-Fredholm with $\operatorname{ind} T^* = -\operatorname{ind} T$. An operator $T \in L(X)$ is said to be *left Drazin invertible* (resp. *right Drazin invertible*) if $p(T) < \infty$ (resp. $q(T) < \infty$) and $R(T^{p(T)+1})$ (resp. $R(T^{q(T)})$) is closed. $T \in L(X)$ is called *Drazin invertible* if the ascent and the descent of T are both finite. It is proved in [5, Theorem 3.6] that T is a B-Browder operator (resp. upper semi B-Browder, lower semi B-Browder) if and only if T is a Drazin invertible (resp. left Drazin invertible, right Drazin invertible) operator.

Another spectra related with semi B-Fredholm operators are defined as follows. The *Drazin invertible spectrum* is defined by

$$\sigma_{\rm d}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

The *B-Weyl spectrum* is defined by

$$\sigma_{\rm bw}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

while the *B-Browder spectrum* is defined by

$$\sigma_{\rm bb}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}.$$

Clearly, by [5, Theorem 3.6], $\sigma_{\rm d}(T) = \sigma_{\rm bb}(T)$.

Now, we introduce an important property in local spectral theory. The localized version of this property has been introduced by Finch [13], and in the framework of Fredholm theory this property has been characterized in several ways, see Chapter 3 of [3]. $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T) f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [3, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{2}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \ \text{ is not bounded below}\}.$$

Note that if $\sigma_{\mathrm{s}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{s}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \ \text{ is not onto}\},$$

then $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$, $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ap}}(T^*)$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{s}}(T)$.

It is easily seen from definition of localized SVEP that

$$\lambda \notin \operatorname{acc}\sigma_{\mathrm{ap}}(T) \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

where $\operatorname{acc}\sigma_{\mathrm{ap}}(T)$ means the set of all accumulation points of $\sigma_{\mathrm{ap}}(T)$, and if T^* denotes the dual of T then

$$\lambda \notin \operatorname{acc}\sigma_{\mathrm{s}}(T) \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{4}$$

Remark 1.1. The implications (1), (2), (3) and (4) are actually equivalences whenever $T \in L(X)$ is semi-Fredholm. Moreover, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc}\sigma(T)$, $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{acc}\sigma_{ap}(T)$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \Xi(T)$, where $\Xi(T)$ denote the set $\{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}$ (see [3, Chapter 3]).

Denote by $\operatorname{iso} K$ the set of all isolated points of $K \subseteq \mathbb{C}$. Let $T \in L(X)$, define

$$\begin{aligned}
p_{00}(T) &= \sigma(T) \setminus \sigma_{\mathrm{b}}(T),\\
p_{00}^a(T) &= \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ub}}(T),\\
\pi_{00}(T) &= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},\\
\pi_{00}^a(T) &= \{\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T) : 0 < \alpha(\lambda I - T) < \infty\},
\end{aligned}$$

Observe that, for every $T \in L(X)$, we have $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$.

Now, we describe several spectral properties introduced recently in [8], [9], [7], [12], [14], [18], [17], [19], [20] and [21].

Definition 1.2. An operator $T \in L(X)$ is said to satisfy property:

(i) (w), if $\sigma_{ap}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ ([18]);

(ii) (aw), if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi^a_{00}(T)$ ([8]);

(iii) (b), if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T)$ ([7],[9]);

(iv) (ab), if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p^a_{00}(T)$ ([8]);

(v) (z) if $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$ ([20]);

(vi) (az), if $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = p^a_{00}(T)$ ([20]);

(vii) (h), if $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ ([19],[21]);

Also, T is said to satisfy:

(viii) Browder's theorem, if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ ([14]);

(ix) a-Browder's theorem, if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$ ([17]);

(x) generalized Browder's theorem, if $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T)$ ([14]);

(xi) Weyl's theorem, if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$ ([12]);

(xii) a-Weyl's theorem, if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$ ([18]).

In the sequel of this paper, according B. Barnes [4], we always assume that Y is a Banach space and X is a subspace of Y which is also a Banach space. Also, we assume $X \neq Y$ and X continuously embedded in Y. Suppose that $T \in L(X)$ admits an extension $\overline{T} \in L(Y)$ such that $\overline{T}(Y) \subseteq X$. We denote

$$\mathcal{M}(X,Y) = \{T \in L(X) : T \text{ has an extension } \overline{T} \in L(Y) \text{ and } \overline{T}(Y) \subseteq X\}.$$

It is easily seen that $0 \in \sigma(\overline{T})$ for all extension $\overline{T}$ of $T \in \mathcal{M}(X,Y)$, since $R(\overline{T}) = \overline{T}(Y) \subseteq X \neq Y$. However, $\sigma(T)$ and $\sigma(\overline{T})$ may differ only in 0. Also, X is a $\overline{T}$-invariant subspace, because $\overline{T}(X) \subseteq \overline{T}(Y) \subseteq X$.

Specific spectral properties have been studied by several authors, through restrictions ([10], [11]) and extensions ([1], [4]). In this paper, under some conditions, we show that all spectral properties given in Definition 1.2 are transmitted from an operator $T \in \mathcal{M}(X,Y)$ to an extension $\overline{T}$ and vice versa.

2. Relations between the spectra of T and $\overline{T}$

In this section, we establish several lemmas that will be used throughout the paper. We begin examining some algebraic relations between an operator $T \in \mathcal{M}(X,Y)$ and its extensions $\overline{T} \in L(Y)$.

Lemma 2.1. *Let $\overline{T} \in L(Y)$ be an extension of $T \in \mathcal{M}(X,Y)$. Then, for all $\lambda \neq 0$:*

(i) $N((\lambda I-\overline{T})^m)=N((\lambda I-T)^m)$*, for any* m*;*

(ii) $R((\lambda I-T)^m)=R((\lambda I-\overline{T})^m)\cap X$*, for any* m*;*

(iii) $\alpha(\lambda I-\overline{T})=\alpha(\lambda I-T)$*;*

(iv) $p(\lambda I-\overline{T})=p(\lambda I-T)$*;*

(v) $\beta(\lambda I-\overline{T})=\beta(\lambda I-T)$*.*

Proof. (i) For $m=0$, the equality $N((\lambda I-\overline{T})^m)=N((\lambda I-T)^m)$ holds trivially. Let $y\in N((\lambda I-\overline{T})^m)$, $m\geq 1$, then

$$\begin{aligned} 0 &= (\lambda I-\overline{T})^m y \\ &= \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}\overline{T}^k y \\ &= \lambda^m y+\sum_{k=1}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}\overline{T}^k y. \\ &= \lambda^m y+\overline{T}\left(\sum_{k=1}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}\overline{T}^{k-1} y\right) \end{aligned}$$

Therefore $y=-\lambda^{-m}\overline{T}\left(\sum_{k=1}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}\overline{T}^{k-1}y\right)\in\overline{T}(Y)\subseteq X$, and since X is a $\overline{T}$-invariant subspace, we conclude that

$$\begin{aligned} 0 &= \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}\overline{T}^k y \\ &= \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}(-1)^k\lambda^{m-k}T^k y \\ &= (\lambda I-T)^m y \end{aligned}$$

So $y\in N((\lambda I-T)^m)$, from which we obtain

$$N((\lambda I-\overline{T})^m)\subseteq N((\lambda I-T)^m).$$

On the other hand, since T is the restriction of $\overline{T}$ on X, we have

$$N((\lambda I-T)^m)\subseteq N((\lambda I-\overline{T})^m).$$

From which, we obtain that $N((\lambda I-\overline{T})^m)=N((\lambda I-T)^m)$.

(ii) Since T is the restriction of $\overline{T}$ on X, then

$$R((\lambda I-T)^m)\subseteq R((\lambda I-\overline{T})^m)\cap X.$$

Now, we show the inclusion $R((\lambda I-\overline{T})^m)\cap X\subseteq R((\lambda I-T)^m)$. For this it will suffice to show that for every $m\in\mathbb{N}$ the implication

$$(\lambda I-\overline{T})^m y\in X\Rightarrow y\in X,$$

holds. For $m = 1$. Let $(\lambda I - \overline{T})y \in X$. Then there exists $x \in X$ such that $(\lambda I - \overline{T})y = x$, so $y = \lambda^{-1}(x + \overline{T}y)$. But since $\overline{T}y \in X$, because $\overline{T}(Y) \subseteq X$, we have that $y = \lambda^{-1}(x + \overline{T}y) \in X$ and $y \in X$.

By the above reasoning, we conclude that, for $m = 1$, the implication

$$(\lambda I - \overline{T})y \in X \Rightarrow y \in X$$

holds. Now, suppose that for $m \geq 1$,

$$(\lambda I - \overline{T})^m y \in X \Rightarrow y \in X.$$

If $(\lambda I - \overline{T})^{m+1} y \in X$, then $(\lambda I - \overline{T})((\lambda I - \overline{T})^m y) \in X$. From the proof of case $m = 1$, we conclude that $(\lambda I - \overline{T})^m y \in X$. Therefore by inductive hypothesis, $y \in X$. Then, by mathematical induction, we conclude that for all $m \in \mathbb{N}$

$$(\lambda I - \overline{T})^m y \in X \Rightarrow y \in X,$$

holds. Finally, if $x \in R((\lambda I - \overline{T})^m) \cap X$ there exists $y \in Y$ such that $(\lambda I - \overline{T})^m y = x \in X$. Then $(\lambda I - \overline{T})^m y \in X$. As above, we conclude that $y \in X$. Thus

$$\begin{aligned} x &= (\lambda I - \overline{T})^m y \\ &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} \overline{T}^k y \\ &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} T^k y \\ &= (\lambda I - T)^m y, \end{aligned}$$

and hence $x \in R((\lambda I - T)^m)$. This shows that,

$$R((\lambda I - \overline{T})^m) \cap X \subseteq R((\lambda I - T)^m).$$

Consequently, $R((\lambda I - T)^m) = R((\lambda I - \overline{T})^m) \cap X$.

(iii) and (iv), it follows immediately from the equality,

$$N((\lambda I - \overline{T})^m) = N((\lambda I - T)^m), \quad \forall m \in \mathbb{N}.$$

(v) Observe that $R(\lambda I - T)$ is a subspace of X, then there exists at least one algebraic complement for $R(\lambda I - T)$ (see [15, Prop. 4.1]). Let M be a complement for $R(\lambda I - T)$, then M is a subspace of X such that $X = R(\lambda I - T) \oplus M$. Since $R(\lambda I - T) = R(\lambda I - \overline{T}) \cap X$, we have

$$\{0\} = R(\lambda I - T) \cap M = R(\lambda I - \overline{T}) \cap X \cap M = R(\lambda I - \overline{T}) \cap M.$$

Thus $R(\lambda I - \overline{T}) \cap M = \{0\}$. Now, we show that $Y = R(\lambda I - \overline{T}) + M$.

Since the spectrum $\sigma(\overline{T})$ is a compact and nonempty subset of the complex plane, the resolvent set $\mathbb{C} \setminus \sigma(\overline{T}) \neq \emptyset$. Let $\mu \in \mathbb{C}$ such that $\mu I - \overline{T}$ is invertible in $L(Y)$. Then, if $z \in Y$ there exists $y \in Y$ such that $z = (\mu I - \overline{T})y$. Thus $z = \mu y - \overline{T}y$, then we can write $z = u + v$, where $u = \mu y \in Y$ and $v = -\overline{T}y \in X$. From this decomposition, for any $\lambda \neq 0$, since $R(\lambda I - T) \subseteq R(\lambda I - \overline{T})$, we obtain that

$$\begin{aligned} z &= u + v \\ &= \lambda^{-1}(\lambda I - \overline{T})u + \lambda^{-1}\overline{T}u + v \\ &= \lambda^{-1}(\lambda I - \overline{T})u + (\lambda^{-1}\overline{T}u + v) \in R(\lambda I - \overline{T}) + X \\ &= \lambda^{-1}(\lambda I - \overline{T})u + (\lambda^{-1}\overline{T}u + v) \in R(\lambda I - \overline{T}) + R(\lambda I - T) + M \\ &= \lambda^{-1}(\lambda I - \overline{T})u + (\lambda^{-1}\overline{T}u + v) \in R(\lambda I - \overline{T}) + M \end{aligned}$$

Therefore, we have that $Y \subseteq R(\lambda I - \overline{T}) + M$, consequently $Y = R(\lambda I - \overline{T}) + M$. But since $R(\lambda I - \overline{T}) \cap M = \{0\}$, then we have $Y = R(\lambda I - \overline{T}) \oplus M$, which implies that $\beta(\lambda I - \overline{T}) = \dim M = \beta(\lambda I - T)$. This shows that $\beta(\lambda I - \overline{T}) = \beta(\lambda I - T)$. □

The following result provides an important relationship between $R(\lambda I - \overline{T})$ and $R(\lambda I - T)$. In the proof of this lemma we use the notion of paraclosed (or paracomplete) subspace and the Neubauer Lemma (see [16]). Also, we used the notion of *reduced minimum modulus* of a non-zero operator $T \in L(X)$,

$$\gamma(T) = \inf_{x \notin N(T)} \frac{\| Tx \|}{\text{dist } (x, N(T))},$$

and the equivalence, $T(X)$ is closed if and only if $\gamma(T) > 0$ ([15, Prop 36.1]).

Lemma 2.2. *Let $\overline{T} \in L(Y)$ be an extension of $T \in \mathcal{M}(X, Y)$. Suppose that X is a proper dense subspace of Y or X is closed in Y. If $\lambda \neq 0$, then $R(\lambda I - \overline{T})$ is closed in Y if and only if $R(\lambda I - T)$ is closed in X.*

Proof. (Sufficiency) By Lemma 2.1(i), for all $x \in X$

$$\begin{aligned} \| (\lambda I - T)x \| &= \| (\lambda I - \overline{T})x \| \\ &\geq \gamma(\lambda I - \overline{T}) \text{ dist } (x, N(\lambda I - \overline{T})) \\ &= \gamma(\lambda I - \overline{T}) \text{ dist } (x, N(\lambda I - T)) \end{aligned}$$

Thus,

$$\gamma(\lambda I - T) = \inf_{x \notin N(\lambda I - T)} \frac{\| (\lambda I - T)x \|}{\text{dist } (x, N(\lambda I - T))} \geq \gamma(\lambda I - \overline{T}).$$

Therefore, we obtain that $\gamma(\lambda I - T) \geq \gamma(\lambda I - \overline{T})$. Then $\gamma(\lambda I - \overline{T}) > 0$ implies $\gamma(\lambda I - T) > 0$. This shows, since both X and Y are Banach spaces, that

$$R(\lambda I - \overline{T}) \text{ closed in } Y \Rightarrow R(\lambda I - T) \text{ closed in } X.$$

(Necessity) Suppose that $R(\lambda I - T)$ is closed in X. Consider two different cases.

Case I: X is a proper dense subspace of Y.
In this case,

$$\| (\lambda I - T)x \| \geq \gamma(\lambda I - T) \,\mathrm{dist}\, (x, N(\lambda I - T)), \quad \forall x \in X.$$

Suppose now that $y \in Y$. Since X is assumed to be dense in Y, there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ such that $y = \lim_{n\to\infty} x_n$. From this, and by Lemma 2.1(i), we obtain

$$\begin{aligned}
\| (\lambda I - \overline{T})y \| &= \| (\lambda I - \overline{T})(\lim_{n\to\infty} x_n) \| \\
&= \lim_{n\to\infty} \| (\lambda I - T)(x_n) \| \\
&\geq \lim_{n\to\infty} \gamma(\lambda I - T) \,\mathrm{dist}\, (x_n, N(\lambda I - T)) \\
&= \gamma(\lambda I - T) \,\mathrm{dist}\, (\lim_{n\to\infty} x_n, N(\lambda I - T)) \\
&= \gamma(\lambda I - T) \,\mathrm{dist}\, (y, N(\lambda I - \overline{T}))
\end{aligned}$$

From which,

$$\gamma(\lambda I - \overline{T}) = \inf_{y \notin N(\lambda I - \overline{T})} \frac{\| (\lambda I - \overline{T})y \|}{\mathrm{dist}\, (y, N(\lambda I - \overline{T}))} \geq \gamma(\lambda I - T).$$

Therefore $\gamma(\lambda I - \overline{T}) \geq \gamma(\lambda I - T)$. Since $R(\lambda I - T)$ is closed, then $\gamma(\lambda I - T) > 0$. Thus, $\gamma(\lambda I - \overline{T}) > 0$. This shows that $R(\lambda I - \overline{T})$ is closed in Y.

Case II: X is closed in Y.
By Lemma 2.1(ii), $R(\lambda I - \overline{T}) \cap X = R(\lambda I - T)$ is closed in X. But since, X is closed in Y, we have that $R(\lambda I - \overline{T}) \cap X$ is closed in Y. Also, if $\lambda \neq 0$ the polynomials $\lambda - z$ and z have not common divisors, so there exist two polynomials u and v such that $1 = (\lambda - z)u(z) + zv(z)$, for all $z \in \mathbb{C}$. Hence $\overline{T}^0 = (\lambda I - \overline{T})u(\overline{T}) + \overline{T}v(\overline{T})$ and so $Y \subseteq R(\lambda I - \overline{T}) + R(\overline{T}) \subseteq R(\lambda I - \overline{T}) + X \subseteq Y$. Thus $R(\lambda I - \overline{T}) + X = Y$ is closed in Y. Since both $R(\lambda I - \overline{T})$ and X are paraclosed subspaces and both $R(\lambda I - \overline{T}) \cap X$ and $R(\lambda I - \overline{T}) + X$ are closed in Y, using Neubauer Lemma [16, Prop. 2.1.2], we have that $R(\lambda I - \overline{T})$ is closed in Y. □

B. Barnes [4] studied some relationships between an operator $T \in \mathcal{M}(X, Y)$ and its extensions $\overline{T} \in L(Y)$ and proved the following result.

Theorem 2.3. *([4, Theorem 4]) Let $\overline{T} \in L(Y)$ be an extension of $T \in \mathcal{M}(X, Y)$. Then:*

(i) $\sigma(T) \setminus \{0\} = \sigma(\overline{T}) \setminus \{0\}$.

(ii) $\sigma_f(T) \setminus \{0\} = \sigma_f(\overline{T}) \setminus \{0\}$.

(iii) $\sigma_w(T) \setminus \{0\} = \sigma_w(\overline{T}) \setminus \{0\}$.

(iv) If $\lambda \notin \sigma_f(T)$, then ind $(\lambda I - T) =$ ind $(\lambda I - \overline{T})$ for all $\lambda \neq 0$.

(v) *If X is a proper dense subspace of Y, then $\sigma(T)=\sigma(\overline{T})$, $\sigma_f(T)=\sigma_f(\overline{T})$, and $\sigma_w(T)=\sigma_w(\overline{T})$.*

(vi) *If X is closed in Y, but X has no closed complement in Y, then $\sigma(T)=\sigma(\overline{T})$, $\sigma_f(T)=\sigma_f(\overline{T})$, and $\sigma_w(T)=\sigma_w(\overline{T})$.*

Proof. For the proof see [4, Theorem 4] □

Atkinson's well known theorem says that $T\in L(X)$ is a Fredholm operator if and only if its projection in the algebra $L(X)/\mathcal{F}(X)$ is invertible, where $\mathcal{F}(X)$ is the ideal of finite rank operators in the algebra $L(X)$. More generally, T is upper (resp., lower) semi-Fredholm operator if and only if its projection in the algebra $L(X)/\mathcal{F}(X)$ is left (resp., right) invertible. From this, and by Lemmas 2.1 and 2.2, Theorem 2.3 may be extended to other spectra as follows.

Theorem 2.4. *Let $\overline{T}\in L(Y)$ be an extension of $T\in\mathcal{M}(X,Y)$. Then:*

(i) $\sigma_{\mathrm{ap}}(T)\setminus\{0\}=\sigma_{\mathrm{ap}}(\overline{T})\setminus\{0\}$.

(ii) $\sigma_{\mathrm{uf}}(T)\setminus\{0\}=\sigma_{\mathrm{uf}}(\overline{T})\setminus\{0\}$.

(iii) $\sigma_{\mathrm{uw}}(T)\setminus\{0\}=\sigma_{\mathrm{uw}}(\overline{T})\setminus\{0\}$.

(iv) $\sigma_{\mathrm{ub}}(T)\setminus\{0\}=\sigma_{\mathrm{ub}}(\overline{T})\setminus\{0\}$.

(v) $\sigma_{\mathrm{b}}(T)\setminus\{0\}=\sigma_{\mathrm{b}}(\overline{T})\setminus\{0\}$.

(vi) *If X is a proper dense subspace of Y, then $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{ap}}(\overline{T})$ and $\sigma_{\mathrm{b}}(T)=\sigma_{\mathrm{b}}(\overline{T})$.*

(vii) *If X is closed, but X has no closed complement in Y, then $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{ap}}(\overline{T})$ and $\sigma_{\mathrm{b}}(T)=\sigma_{\mathrm{b}}(\overline{T})$.*

(viii) *If X is a proper dense subspace of Y and $T(X)$ is closed in X, then $\sigma_{\mathrm{uf}}(T)=\sigma_{\mathrm{uf}}(\overline{T})$, $\sigma_{\mathrm{uw}}(T)=\sigma_{\mathrm{uw}}(\overline{T})$ and $\sigma_{\mathrm{ub}}(T)=\sigma_{\mathrm{ub}}(\overline{T})$.*

(ix) *If X is closed, but X has no closed complement in Y and $T(X)$ is closed in X, then $\sigma_{\mathrm{uf}}(T)=\sigma_{\mathrm{uf}}(\overline{T})$, $\sigma_{\mathrm{uw}}(T)=\sigma_{\mathrm{uw}}(\overline{T})$ and $\sigma_{\mathrm{ub}}(T)=\sigma_{\mathrm{ub}}(\overline{T})$.*

Proof. (i) Suppose that $\lambda\in\sigma_{\mathrm{ap}}(T)\setminus\{0\}$, then $\lambda I-T$ is not bounded below and $\lambda\neq 0$. Thus $p(\lambda I-T)>0$ or $R(\lambda I-T)$ is not closed in X. If $p(\lambda I-T)>0$, by Lemma 2.1, $p(\lambda I-\overline{T})=p(\lambda I-T)>0$. Now, since X is a Banach subspace of Y and Y is a Banach space, X is closed in Y. Thus, by Lemma 2.2, if $R(\lambda I-T)$ is not closed in X then $R(\lambda I-\overline{T})$ is not closed in Y. Therefore, $p(\lambda I-\overline{T})>0$ or $R(\lambda I-\overline{T})$ is not closed in Y. From which $\lambda I-\overline{T}$ is not bounded below and $\lambda\neq 0$, then $\lambda\in\sigma_{\mathrm{ap}}(\overline{T})\setminus\{0\}$. Similarly, we can prove the inclusion $\sigma_{\mathrm{ap}}(\overline{T})\setminus\{0\}\subseteq\sigma_{\mathrm{ap}}(T)\setminus\{0\}$.

The proofs of (ii), (iii), (iv) and (v) are analogous to (i).

(vi) By part (i), $\sigma_{\mathrm{ap}}(T)\setminus\{0\}=\sigma_{\mathrm{ap}}(\overline{T})\setminus\{0\}$. Now, if $0\notin\sigma_{\mathrm{ap}}(T)$ then T is injective. Consequently T has SVEP at 0, and $0\notin\Xi(T)$. But since

$\sigma_{\rm ap}(T) \cup \Xi(T) = \sigma(T) = \sigma(\overline{T})$, we have $0 \notin \sigma(\overline{T})$, a contradiction. Similarly, $0 \notin \sigma_{\rm ap}(\overline{T})$ implies $\overline{T}$ injective. Thus $\overline{T}$ has SVEP at 0, and $0 \notin \Xi(\overline{T})$. Again, since $\sigma_{\rm ap}(\overline{T}) \cup \Xi(\overline{T}) = \sigma(\overline{T})$, we have $0 \notin \sigma(\overline{T})$, a contradiction. Thus $0 \in \sigma_{\rm ap}(T)$ and $0 \in \sigma_{\rm ap}(\overline{T})$, so the equality $\sigma_{\rm ap}(T) = \sigma_{\rm ap}(\overline{T})$ holds.

For the equality $\sigma_{\rm b}(T) = \sigma_{\rm b}(\overline{T})$, observe that

$$\sigma_{\rm b}(T) = \sigma_{\rm w}(T) \cup \operatorname{acc} \sigma(T) = \sigma_{\rm w}(\overline{T}) \cup \operatorname{acc} \sigma(\overline{T}) = \sigma_{\rm b}(\overline{T}).$$

(vii) The proof is analogous to that of part (vi).

(viii) To show the equality $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(\overline{T})$, note first that $\sigma_{\rm uf}(T) \setminus \{0\} = \sigma_{\rm uf}(\overline{T}) \setminus \{0\}$. Suppose that $0 \notin \sigma_{\rm uf}(T)$. Then T is upper semi-Fredholm, so its projection in the algebra $L(X)/\mathcal{F}(X)$ is left invertible. Therefore there exist $S \in L(X)$ and $F \in \mathcal{F}(X)$ such that $ST - F$ is the identity on X. Then, $\overline{P} = \overline{S}\overline{T} - \overline{F}$ is a bounded projection of Y on X. Thus, $Y = X \oplus N(\overline{P})$ and X is closed in Y. Hence $X = Y$, a contradiction. Consequently, $0 \in \sigma_{\rm uf}(T)$. We now show that also $0 \in \sigma_{\rm uf}(\overline{T})$. Suppose that $0 \notin \sigma_{\rm uf}(\overline{T})$. Then $\overline{T}$ is upper semi-Fredholm, this implies that $\alpha(\overline{T}) < \infty$. But, since $N(T) = N(\overline{T}) \cap X \subseteq N(\overline{T})$, it follows that $\alpha(T) < \infty$. Now, by hypothesis, $T(X)$ is closed in X. Therefore, T is upper semi-Fredholm and as it has been proved before this is impossible. Thus $0 \in \sigma_{\rm uf}(\overline{T})$ and $0 \in \sigma_{\rm uf}(T)$, so the equality $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(\overline{T})$ holds.

To show the equality $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(\overline{T})$, by part (iii), we know that $\sigma_{\rm uw}(T) \setminus \{0\} = \sigma_{\rm uw}(\overline{T}) \setminus \{0\}$. Proceeding as in the first part, we see that $0 \in \sigma_{\rm uw}(T) \cap \sigma_{\rm uw}(\overline{T})$. Suppose that $0 \notin \sigma_{\rm uw}(T)$. Then T is upper semi-Fredholm, since it is upper Weyl. As above, it then follows that X has a closed complement in Y, contradicting our assumption. Similarly, if $0 \notin \sigma_{\rm uw}(\overline{T})$ then $\overline{T}$ is upper semi-Fredholm. As above, it then follows that T is upper semi-Fredholm, again contradicting the assumption that X has no closed complement in Y. Thus $0 \in \sigma_{\rm uw}(T) \cap \sigma_{\rm uw}(\overline{T})$ and $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(\overline{T})$.

Finally, by part (vi), we concluded that

$$\sigma_{\rm ub}(T) = \sigma_{\rm uw}(T) \cup \operatorname{acc} \sigma_{\rm ap}(T) = \sigma_{\rm uw}(\overline{T}) \cup \operatorname{acc} \sigma_{\rm ap}(\overline{T}) = \sigma_{\rm ub}(\overline{T}).$$

Then $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(\overline{T})$.

(ix) Similarly as in the part (viii), for the equality $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(\overline{T})$, it suffices to examine the case $\lambda = 0$. Suppose that $0 \notin \sigma_{\rm uf}(T)$. Proceeding as in the part (viii), there exist a bounded projection $\overline{P}$ of Y on X. Thus, $Y = X \oplus N(\overline{P})$ and hence X has a closed complement in Y, a contradiction. Consequently, $0 \in \sigma_{\rm uf}(T)$. The proof of $0 \in \sigma_{\rm uf}(\overline{T})$, is analogous to that of part (viii). Then $0 \in \sigma_{\rm uf}(\overline{T})$ and $0 \in \sigma_{\rm uf}(T)$, so the equality $\sigma_{\rm uf}(T) = \sigma_{\rm uf}(\overline{T})$ holds.

Again proceeding as in the part (viii), we obtain $\sigma_{\rm uw}(T) = \sigma_{\rm uw}(\overline{T})$ and $\sigma_{\rm ub}(T) = \sigma_{\rm ub}(\overline{T})$. □

Remark 2.5. By the parts (viii) and (ix) of the proof of Theorem 2.4, we concluded that $0 \notin \sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T)$, $0 \notin \sigma(T) \setminus \sigma_{\rm uw}(T)$ and $0 \notin \sigma(T) \setminus \sigma_{\rm w}(T)$. Similarly, $0 \notin \sigma_{\rm ap}(\overline{T}) \setminus \sigma_{\rm uw}(\overline{T})$, $0 \notin \sigma(\overline{T}) \setminus \sigma_{\rm uw}(\overline{T})$ and $0 \notin \sigma(\overline{T}) \setminus \sigma_{\rm w}(\overline{T})$.

As immediate consequence of Lemma 2.1, and Theorems 2.3 and 2.4.

Lemma 2.6. *Let $\overline{T} \in L(Y)$ be an extension of $T \in \mathcal{M}(X, Y)$. Suppose that X is a proper dense subspace of Y or X is closed, but X has no closed complement in Y. Then, if $T(X)$ is closed in X or 0 is not an isolated point of $\sigma(T)$, the following statements are true:*

(i) $p_{00}(\overline{T}) = p_{00}(T)$.

(ii) $p^a_{00}(\overline{T}) = p^a_{00}(T)$.

(iii) $\pi_{00}(\overline{T}) = \pi_{00}(T)$.

(iv) $\pi^a_{00}(\overline{T}) = \pi^a_{00}(T)$.

Proof. (i) and (ii) follows from Theorem 2.4.

(iii) To show the equality $\pi_{00}(\overline{T}) = \pi_{00}(T)$, by Lemma 2.1(iii), it suffices to examine the case $\lambda = 0$. We claim that $0 \notin \pi_{00}(T)$. To see this, suppose that $0 \in \pi_{00}(T)$. Then $\alpha(T) < \infty$, assuming that $T(X)$ is closed in X, T is upper semi-Fredholm. As in the proof of Theorem 2.4, we have a contradiction. Therefore, $0 \notin \pi_{00}(T)$. Similarly, $0 \in \pi_{00}(\overline{T})$ implies that $\alpha(T) < \infty$, since $N(T) = N(\overline{T}) \cap X \subseteq N(\overline{T})$. Again T is upper semi-Fredholm, so by the same argument above we have a contradiction. Thus $0 \notin \pi_{00}(\overline{T})$. Now, by Theorem 2.3, $\sigma(T) = \sigma(\overline{T})$. Then $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(\overline{T})$. Also, by Lemma 2.1(iii), $\alpha(\lambda I - T) = \alpha(\lambda I - \overline{T})$ for all $\lambda \neq 0$. Consequently, we have the equality $\pi_{00}(\overline{T}) = \pi_{00}(T)$. On the other hand, assuming that 0 is not an isolated point of $\sigma(T)$, then $0 \notin \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(\overline{T})$. Again, by Lemma 2.1(iii), we get $\pi_{00}(\overline{T}) = \pi_{00}(T)$.

(iv) The proof is analogous to that of part (iii). □

3. Spectral properties and extensions

In this section we state the main result of the paper.

Theorem 3.1. *Let $\overline{T} \in L(Y)$ be an extension of $T \in \mathcal{M}(X, Y)$. Suppose that X is a proper dense subspace of Y or X is closed, but X has no closed complement in Y. If $T(X)$ is closed in X or 0 is not an isolated point of $\sigma(T)$, then property (i) (resp.,(ii)-(xii)) in Definition 1.2 holds for T if and only if property (i)(resp.,(ii)-(xii)) in Definition 1.2 holds for $\overline{T}$.*

Proof. By Theorems 2.3, 2.4 and Lemma 2.6 (and by using Remark 2.5), we obtain readily the result. For the property (x), observe the equivalence between Browder's theorem and generalized Browder's theorem proved in [2]. □

We give one illustrative example for the behavior of an operator T and its extensions $\overline{T}$, when the subspace X does not satisfy the hypothesis of Theorem 3.1.

Example 3.2. Let Y be a Banach space, and assume that X and Z are proper closed subspaces of Y with $Y = X \oplus Z$. Let $\overline{P}$ be the projection of Y on X which is zero on Z. Since $\overline{P}$ is a projection operator, i.e $\overline{P}^2 = \overline{P}$, then $\sigma(\overline{P}) = \{0, 1\}$. Also, the operator $P = \overline{P}|_X$ is the identity operator on X, so $\sigma(P) = \{1\}$. Now, if X is infinite dimensional and Z is finite dimensional, then P satisfy the properties (i)-(vii) and (xi)-(xii) in Definition 1.2. But, $\overline{P}$ does not satisfy the properties (i)-(vii) and (xi)-(xii) in Definition 1.2.

Remark 3.3. Theorem 2.3 has been proved in [4] by using different arguments. This result is also an easy consequence of Lemmas 2.1 and 2.2. Aiena et al., showed in [1] a particular case of Theorem 3.1. They showed the equivalence $\overline{T}$ satisfies Weyl's theorem if and only if T satisfies Weyl's theorem, when X is a proper dense subspace of a Hilbert space H and $T(X)$ is closed in X. In Theorem 3.1, we extend this equivalence to other spectral properties. On the other hand, every isolated point of $\sigma(T)$ is a boundary point of $\sigma(T)$. Thus, in Theorem 3.1, there are more weak ways to express the hypothesis. We may replace the assumption 0 is not an isolated point of $\sigma(T)$ by $0 \notin \partial\sigma(T)$, $p(T) = \infty$ or $q(T) = \infty$. Also, since in a Banach space we have that every subspace of finite codimension is closed, we may replace the assumption $T(X)$ closed in X by $\beta(T) < \infty$.

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