ERROR ESTIMATION OF BILINEAR GALERKIN FINITE ELEMENT METHOD FOR 2D THERMAL PROBLEMS

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ABSTRACT

This study demonstrates a two-dimensional steady state heat conduction Laplace partial differential equation solution using the bilinear Galerkin finite element method. Heat transfer analysis is of vital importance in many engineering applications and devising computationally inexpensive numerical methods while maintaining accuracy is one of the primary concerns. The method uses structured mesh grid over a two-dimensional rectangular domain and solved using a stiffness matrix for the bilinear elements, calculated using the proposed modified numerical scheme. Several numerical experiments are conducted by controlling the number of nodes and changing element sizes of the presented scheme, and comparison made between analytical solution and software generated solution.

KEYWORDS

Galerkin method, Bilinear element, Heat conduction, Error analysis, Partial differential equation.

1. INTRODUCTION

Finite element Analysis is a robust numerical approach for solving partial differential equations (PDEs) used in various branches of science such as solid mechanics, fluid mechanics, electromagnetics, thermodynamics etc. (Kreyszig, 2011; Deb, Babuška, & Oden, 2001). One of the most powerful techniques for solving PDEs with weak formulation is using a weighed residual method called the Galerkin finite element method (GFEM). The formulation requires generating a basis function (Ainsworth & Oden, 1997) based on the elemental boundary conditions. This trial function is substituted in the partial differential equation and the first derivative of the trial function is taken for each nodal variable (Burden & Faires, 2001) to construct the residual function. The weighed form of the residual function for the whole domain is integrated by setting it equal to zero. Green's theorem can be applied over the boundary if necessary (Afzal, Sulaeman, & Okhunov, 2016). The corresponding numerical model is set up by discretizing the rectangular domain into smaller elements where each element consists of nodal coordinates and nodal variables, which are used to perform the Galerkin approximation of the PDEs. This is followed by the generation of an element matrix and vector matrix of the boundary by integrating the total number of elements. The set of linear equations represented by the matrices are consecutively solved using the Galerkin approach. For comparison purposes, an exact solution already available for the non-linear PDE (time independent and no heat source) used in 2D heat conduction rectangular domain with both Dirichlet and Neumann boundary condition, is presented. Finally, a stiffness matrix applicable for homogenous rectangular domain consisting of structured mesh grid elements is presented, the solution scheme of which significantly reduces the CPU performance cost.

2. MATHEMATICAL CONSTRUCTION

A homogeneous domain represented by a time independent heat conduction problem with zero heat extraction is considered and can be mathematically formulated by Laplace equation with Neumann

or Dirichlet boundary conditions as shown in Figure 1 (Gerald & Wheatly, 2003; Gockenbach, 2002; Burden & Faires, 2001; Kreyszig, 2011).
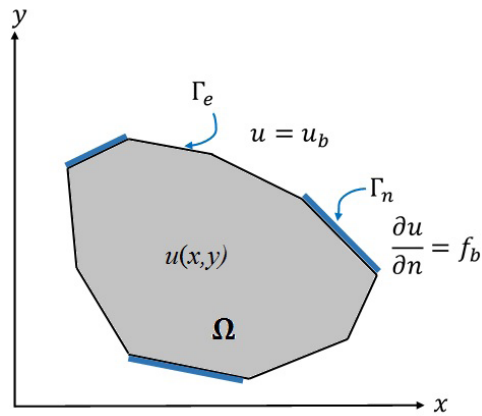

Figure 1. Physical domain of Ω bounded by Γ.

Two dimensional time independent heat-conduction problem can be represented by the following partial differential equation.

$$\nabla^2 u \ = \frac{\partial^2 u}{\partial x^2}(x,\,y) \ + \frac{\partial^2 u}{\partial y^2}(x,\,y) = 0, in \ \Omega \tag{1}$$

where, $0<x<a$ and $0<y<b$

The boundary conditions are given by:

$$u = u_b \ \text{on} \ \Gamma_e$$
$$\frac{\partial u}{\partial n} = f_b \tag{2}$$

where u_b and f_b are Dirichlet and Neumann boundary conditions respectively.

The weighed residual w is applied on Eqs (1) and (2) to generate its strong formulation as given:

$$I = \int_{\Omega} w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega + \oint_{\Gamma} w \frac{\partial u}{\partial n} d\Gamma \tag{3}$$

Consequently, the weak formulation is generated using integration by part over eq. (3) as given below:

$$I = -\int_{\Omega} w \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \right) d\Omega + \oint_{\Omega} w \frac{\partial u}{\partial n} d\Gamma \tag{4}$$

The stiffness matrix is generated by discretising the solution domain into smaller elements and performing integration for each element.

3. BASIS FUNCTION AND STIFFNESS MATRIX

The 2D computational domain is represented by bilinear rectangular elements where each element is constructed with a node at each of its four edges as shown in Figure 2. The nodal points are represented by (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), while the nodal variables are represented by u_1, u_2, u_3 and u_4. The nodal variable u at any given location (x,y) for a bilinear element is approximated by the basis function, which is written as:

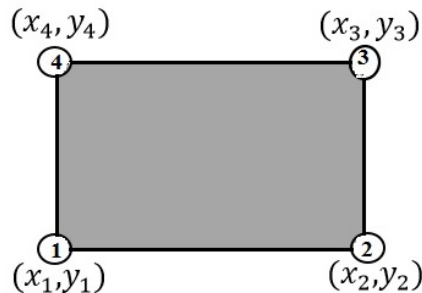$$u = a_i + a_i x + a_i y + a_i xy \tag{5}$$

Figure 2. Bilinear element.

The above-mentioned basis function can be represented in matrix form:

$$u = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{6}$$

Considering each bilinear element composed of four nodal points and nodal variables, the corresponding basis function in matrix form is represented as:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{7}$$

The corresponding shape function is formulated as:

$$u = H_1(x,y)u_1 + H_2(x,y)u_2 + H_3(x,y)u_3 + H_4(x,y)u_4 \tag{8}$$

The individual shape function is based on the four geometric bilinear coordinates for each element located in a structured mesh grid:

$$H_1 = \frac{1}{4A}[(x_3 y_3 - x_2 y_2 + x_4 y_4 + 4x^2) + 2x_1 - 2x_4 + \ldots + 2y_1 - 2y_2)x] \tag{9}$$

$$H_2 = \frac{1}{4A}[(x_3 y_3 - x_2 y_2 - x_4 y_4 + x_1 y_1) + 2(y_2 - y_1)x \ldots + 2(x_1 - x_4)y - 4xy] \tag{10}$$

$$H_3 = \frac{1}{4A}[(x_3 y_3 - x_2 y_2 - x_4 y_4 + x_1 y_1) + 2(y_2 - y_1)x \ldots + 2(x_4 - x_1)y + 4xy] \tag{11}$$

$$H_4 = \frac{1}{4A}[(x_3 y_3 - x_2 y_2 - x_4 y_4 + x_1 y_1) + 2(y_1 - y_2)x \ldots + 2(x_4 - x_1)y - 4xy] \tag{12}$$

Where the rectangle Area, A is given by:

$$A = x_3 y_3 - x_4 y_4 - x_2 y_2 + x_1 y_1 \qquad (13)$$

As it can be observed from Eq. (5) to (11), the trial function u is dependent on the nodal variables at the corners of the bilinear finite elements and the shape functions. The first derivative of this trial function gives the weighed function w in Eq. (4)

$$w = \frac{\partial u}{\partial u_i} \qquad (14)$$

In order to determine the elemental stiffness matrix, the first integral of the weak formulation in Eq. (4) is carried out sequentially given below:

$$k^e = \int_{\Omega^e} \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega \qquad (15)$$

$$k^e = \int_{\Omega^e} \left(\begin{bmatrix} \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial x} \\ \frac{\partial H_3}{\partial x} \\ \frac{\partial H_4}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial x} \end{bmatrix} + \begin{bmatrix} \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial y} \\ \frac{\partial H_3}{\partial y} \\ \frac{\partial H_4}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial y} & \frac{\partial H_2}{\partial y} & \frac{\partial H_3}{\partial y} & \frac{\partial H_4}{\partial y} \end{bmatrix} \right) d\Omega \qquad (16)$$

The matrix generated is a *4x4* matrix as given by:

$$\left[K^e \right] = \frac{1}{4A} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \tag{17}$$

The diagonal values as follows:

$$K_{11} = \frac{1}{3} \frac{x_2^2 + y_2^2 - 2x_2 x_4 - 2y_2 y_4 + x_4^2 + y_4^2}{x_2 y_4 - x_2 y_2 + x_4 y_2 - x_4 y_4} \tag{18}$$

$$k_{12} = \frac{1}{3} \frac{\frac{1}{2}x_2^2 - y_2^2 - x_2 x_4 + 2y_2 y_4 + \frac{1}{2}x_4^2 - y_4^2}{x_2 y_4 - x_2 y_2 + x_4 y_2 - x_4 y_4} \tag{19}$$

$$k_{13} = \frac{1}{6} \frac{-x_2^2 + y_2^2 + 2x_2 x_4 - 2y_2 y_4 - x_4^2 + y_4^2}{x_2 y_4 - x_2 y_2 + x_4 y_2 - x_4 y_4} \tag{20}$$

$$k_{14} = \frac{1}{3} \frac{-x_2^2 + \frac{1}{2}y_2^2 + 2x_2 x_4 - y_2 y_4 - x_4^2 + \frac{1}{2}y_4^2}{x_2 y_4 - x_2 y_2 + x_4 y_2 - x_4 y_4} \tag{21}$$

Since:

$k_{22}=k_{11}$, $k_{23}=k_{14}$, $k_{24}=k_{13}$, $k_{33}=k_{11}$, $k_{34}=k_{12}$ and $k_{44}=k_{11}$ the matrix is symmetric about its diagonal.

$$\left[K^e \right] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} & k_{23} & k_{24} \\ & & k_{33} & k_{34} \\ & symmetry & & k_{44} \end{bmatrix} \tag{22}$$

4. RESULTS AND DISCUSSION

In this section a set of numerical experiments are carried out for the evaluation and analysis of the results. A rectangular 2D domain is developed and steady state heat conduction with no heat source is applied. The domain measures 5 units and 10 units in the x and y direction respectively. GFEM is used by implementing the weighed residual approximation approach on each bilinear element for solving the Laplace PDE governing the 2D heat conduction problem. Both Dirichlet and Neumann boundary conditions are specified for the solution domain as given in Figure 3. The boundary conditions are below:

$$u(0, y) = 0 \quad \& \quad u(x,0) = 0$$

$$u(x,10) = 100\mathrm{Sin}(0.1\pi x) \quad \text{and} \quad \frac{\partial u}{\partial x}(5, y) = 0$$
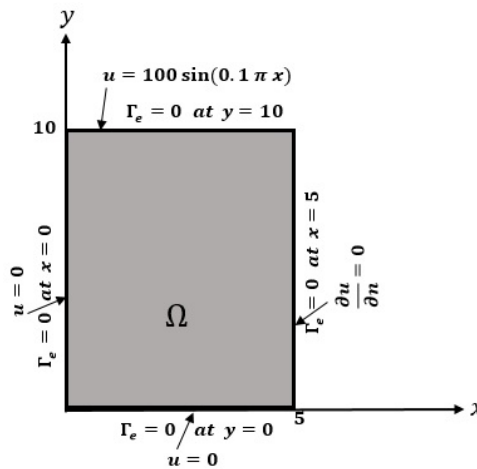

Figure 3. Steady state heat conduction problem of rectangular domain.

Solving the above mentioned problem required calculating the stiffness matrix of the bilinear elements using the Galerkin finite element approach. The accuracy of the approach largely depends on the element size and number of elements considered. It has been observed increasing the number of elements and decreasing the element size increases the accuracy of the solution by achieving higher convergence rates.

Comparison is made with the exact solution of the Eq. (23) below and Figure 4 shows the graphical analysis.

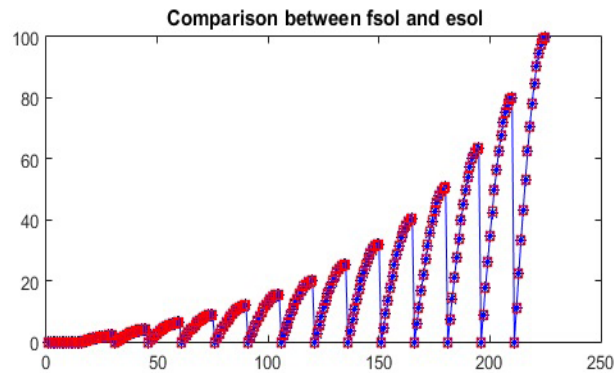$$u = 100 \sin(0.1\pi x) \sinh(0.1\pi y) sinh\pi \qquad (23)$$

Figure 4. Comparison between Galerkin FE solution and exact solution.

The Figure 5 represents the solution domain discretized into a numbers of smaller bilinear elements each consisting of nodes at the corners.
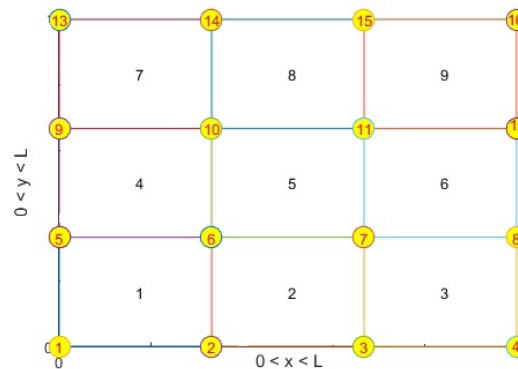

Figure 5. Physical domain with bilinear mesh.

The errors analysis of the Galerkin approach solution and analytical solution are given by the following equations:

$$H^1 = \frac{\sum_{i=1}^{N}\left|\widetilde{u_i} - \overline{u_i}\right|}{N} \tag{24}$$

$$L^2 = \sqrt{\frac{\sum_{i=1}^{N}\left|\widetilde{u_i} - \overline{u_i}\right|^2}{N}} \tag{25}$$

$$L^\infty = max\sum_{i=1}^{N}\left|\widetilde{u_i} - \overline{u_i}\right| \tag{26}$$

Table 1 below lists the set of errors analysis for different numerical experiments using various number of elements and element sizes. The equation form $\left\|\tilde{u}_i - \overline{u}_i\right\|_0$ represents the L^2 norm, $\left\|\tilde{u}_i - \overline{u}_i\right\|_1$ represents H^1 norm and $\left\|\tilde{u}_i - \overline{u}_i\right\|_\infty$ represents L^∞ norm (He, Lin, & Lin, 2010).

Table 1. L^∞, L^2 and H^1 errors of the Galerkin method solution.

Number of elements	$\|\tilde{u}_i - \overline{u}_i\|_0$	$\|\tilde{u}_i - \overline{u}_i\|_1$	$\|\tilde{u}_i - \overline{u}_i\|_\infty$
4×4	5.06E-01	8.90E-01	2.20E+00
8×8	1.19E-01	1.68E-01	3.71E-01
16×16	2.85E-02	3.74E-02	2.85E-02
32×32	6.96E-03	8.88E-03	1.86E-02
64×64	1.72E-03	2.17E-03	4.49E-03
128×128	4.27E-04	5.35E-04	1.11E-03
256×256	1.06E-04	1.33E-04	2.74E-04
512×512	2.66E-05	3.31E-05	6.83E-05

It can be observed from Table 1 that increasing the number of elements while decreasing the element size has a converging on both the exact and GFEM solution on all three of the calculated norms, H^1, L^2 and L^∞ with convergence rates of $O(h^2)$, $O(h)$ and $O(h^2)$ respectively. This validates convergence towards the expected exact solution and accuracy of the presented Galerkin finite element scheme. Figure 6 below show the graphical illustration of the convergence of decreasing the size and increasing the number of elements on the rate of errors respectively for the generated scheme.
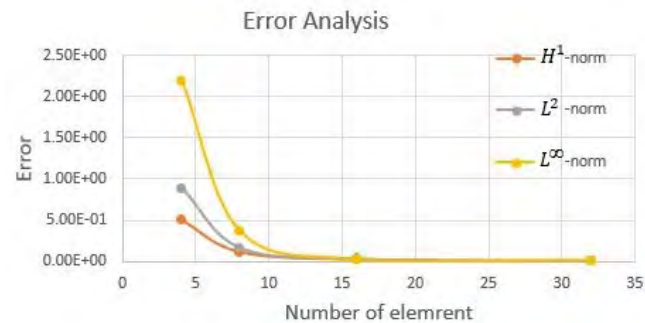

Figure 7. Error analysis vs number of element.

Figures 8 and 9 shows the 2D rectangular solution domain where, the surface plot and contour plot are taken from the results. The temperature distribution is governed by time independent heat conduction with no heat source.
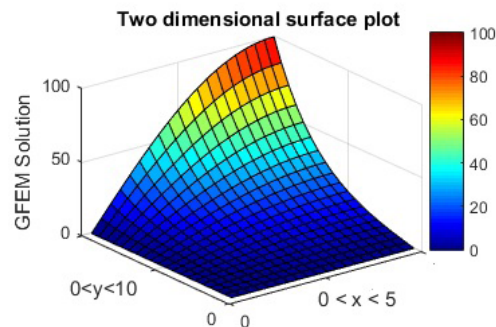

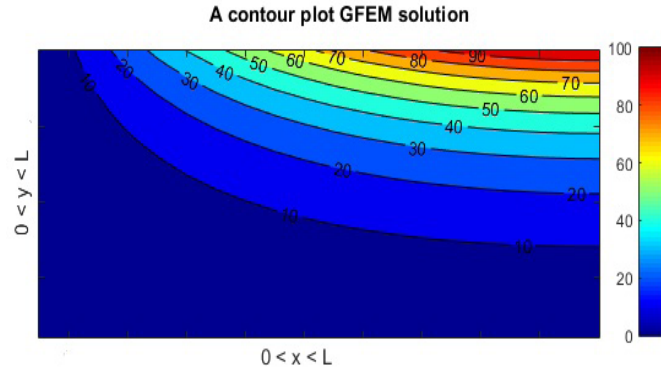**Figure 8.** Surface plot of temperature variation in the GFEM solution domain using Matlab.

Figure 9. Contour plot using Matlab.

5. CONCLUSION

It is shown that GFEM is an inexpensive and powerful technique to accurately model and has demonstrated solving a time dependent and external heat source independent 2D heat conduction Laplace partial differential equation, governing a rectangular planar domain, using bilinear elements. Increasing the number of elements while decreasing the element size representing the solution domain shows optimal convergence towards the exact solution, thereby validating the accuracy of the scheme.

ACKNOWLEDGEMENT

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