Ordered Semihypergroup Constructions

Construcciones de Semihipergrupos Ordenados

L. Kamali Ardekani[1,a], **B. Davvaz**[2,b]

Abstract. The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. In this paper, we study some aspects of hyperideals, bi-hyperideals and quasi-hyperideals of ordered semihypergroups. We investigate the notions of regular, intra-regular and completely regular ordered semihypergroups and give their characterizations in terms of hyperideals, bi-hyperideals and quasi-hyperideals. Also, the notion of duo ordered semihypergroups is introduced and some related results are discussed.

Keywords: Ordered semihypergroup, hyperideal, completely regular, duo ordered semihypergroup, bi-hyperideal, intra-regular, quasi-hyperideal.

Resumen. El concepto de semihipergrupos ordenados es una generalización del concepto de semigrupos ordenados. En este trabajo, estudiamos algunos aspectos de hiperideales, bi-hiperideales y cuasi hiperideales de semihipergrupos ordenados. Investigamos las nociones de semihipergrupos ordenados regulares ideales, intra-regulares y completamente regulares y damos sus caracterizaciones en términos de hiperideales, bi-hiperideales y cuasi-hiperideales. Además, se introduce la noción de semihipergrupos ordenados dúo y se discuten algunos resultados relacionados.

Palabras claves: Semihipergrupo ordenado, hiperideal, completamente regular, semihipergrupo ordenado dúo, bi-hiperideal, intra-regular, cuasi-hiperideal.

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1. Introduction

Ordered semigroups have been studied by many authors, for example [1, 4, 12, 13, 16, 17, 19, 18, 23, 24]. In [10], Kehayopulu defined the ideal and weakly prime ideal in po-semigroup (: ordered semigroups). Completely regular poe-semigroups (: ordered semigroups having a greatest element) have been considered by Kehayopulu in [11]. She showed the similarity between the theory of semigroups based on ideals and the theory of ordered semigroups based

[1]Faculty of Engineering, Ardakan University, Ardakan, Iran
[2]Department of Mathematics, Yazd University, Yazd, Iran
[a]l.kamali@ardakan.ac.ir
[b]davvaz@yazd.ac.ir

on ideals. Also, she considered the notion of duo ordered semigroups in [14]. Good and Hughes [6] introduced the notion of bi-ideals. In [25], prime and semiprime bi-ideals of ordered semigroups are defined and some related results are discussed. The notion of quasi-ideals was first introduced by Steinfeld [27] for rings and semigroups. Then, several authors studied these concepts, for example see [8, 9, 20, 26]. In 1992, Kehayopulu in [11] introduced quasi-ideals. Afterward, Kehayopulu and Tsingelis changed the definition of quasi-ideals in ordered semigroups [15, 28].

The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [22] and has been studied in the following decades and nowadays by many mathematicians. Algebraic hyperstructures are a generalization of classical algebraic structures. A mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on S, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of S. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b,\ \ x \circ A = \{x\} \circ A \text{ and } \ A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup\limits_{u\in x\circ y} u \circ z = \bigcup\limits_{v\in y\circ z} x \circ v$. A non-empty subset A of S is called a *subsemihypergroup* if $A \circ A \subseteq A$. In [7], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation " $\leq$ ", where " $\leq$ " is a partial order relation that satisfies the monotone condition. Indeed, an ordered semihypergroup $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order " $\leq$ " that is compatible with the hyperoperation, meaning that for all $x, y, z \in S$,

$$x \leq y \Longrightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for all $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Recently, Changphas and Davvaz introduced the notions of hyperideals, bi-hyperideals, quasi-hyperideals and pure hyperideals in ordered semihypergroups [3, 2], also see [5]. In this paper, by using the notion pseudoorder on a semihypergroup $(S, \circ)$, we obtain an ordered semihypergroup. Also, we extend the results given in [3, 2]. The paper is structured as follows. In Section 2 we remind some basic notions of ordered semihypergroups. In Section 3, we investigate the properties of hyperideals on ordered semihypergroups, especially regular and intra-regular ordered semihypergroups. In Section 4, we introduce the notion of duo ordered semihypergroups and some properties of them are discussed. In section 5, we present some results on quasi-hyperideals in ordered semihypergroups.

We tried to use elements instead of sets in the proofs of our results. Most of results of this paper are valid for semihypergroups without order (by setting

A instead of $(A]$), because if we define the relation " $\leq$ " on semihypergroup $(S, \circ)$ as $\leq := \{(x, y) \mid x = y\}$, then $(S, \circ, \leq)$ is an ordered semihypergroup.

2. Basic definitions and preliminary results

In this section, we remind some notions and definitions that are used in the following sections [3, 2, 5, 7].

An *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order " $\leq$ " that is compatible with the hyperoperation, meaning that for all $x, y, z \in S$,

$$x \leq y \Longrightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means that for all $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

An ordered semihypergroup $(S, \circ, \leq)$ is called *commutative* if $(S, \circ)$ is commutative.

Note that the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. Indeed, every ordered semigroup is an ordered semihypergroups.

For a non-empty subset A of an ordered semihypergroup $(S, \circ, \leq)$, we write

$$(A] = \{x \in S \mid x \leq a. \text{ for some } a \in A\}.$$

It is clear that $(S] = S$.

The following is easy to see for non-empty subsets A, B and C of an ordered semihypergroups $(S, \circ, \leq)$:

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) $A \subseteq B \Longrightarrow (A] \subseteq (B]$;

(4) $(A] \circ (B] \subseteq (A \circ B]$;

(5) $((A] \circ (B]] = (A \circ B]$;

(6) $(A] \cup (B] = (A \cup B]$;

(7) $((A] \circ (B] \circ (C]] = (A \circ B \circ C]$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. A non-empty subset A of S is called a *left* (respectively, *right*) *hyperideal* of S if it satisfies the following conditions:

(1) $S \circ A \subseteq A$ (respectively, $A \circ S \subseteq A$);

(2) For $x \in A$ and $y \in S$, $y \leq x$ implies that $y \in A$.

If A is a left or a right hyperideal of S, then it is called an *one-sided hyperideal* of S. If A is both a left and a right hyperideal of S, then it is called a *two-sided hyperideal* of S, or simply a *hyperideal* of S. Note that the condition (2) of above definition is equivalent to $A = (A]$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, we have

(1) If A and B are hyperideals of S, then $A \cap B$ and $A \cup B$ are hyperideals of S;

(2) If A is a left hyperideal and B is a right hyperideal of S, then $(A \circ B]$ is a hyperideal of S;

(3) For all $a \in S$, $(S \circ a \circ S]$ is a hyperideal of S.

Note that in case (1), if A and B are hyperideals of S, then there is $a \in A \neq \emptyset$ and $b \in B \neq \emptyset$. Therefore, $a \circ b \subseteq A \circ S \subseteq A$ and $a \circ b \subseteq S \circ B \subseteq B$. So, $a \circ b \subseteq A \cap B$ and this implies that $A \cap B \neq \emptyset$.

When we say that the hyperideals of S form a chain, we consider them endowed with the inclusion " $\subseteq$ ".

Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$. We have

$L(a) = (a \cup S \circ a]$: the left hyperideal of S generated by a;

$R(a) = (a \cup a \circ S]$: the right hyperideal of S generated by a;

$I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]$: the hyperideal of S generated by a.

Obviously, $L(a)$, $R(a)$ and $I(a)$ are a left hyperideal, a right hyperideal and a hyperideal of S, respectively. Obviously, $R(L(a)) = L(R(a))$, for all $a \in S$.

An ordered semihypergroup $(S, \circ, \leq)$ is called *left* (respectively, *right*) *simple* if it does not contain proper left (respectively, right) hyperideals. Note that an ordered semihypergroup $(S, \circ, \leq)$ is left (respectively, right) simple if and only if $(S \circ x] = S$ (respectively, $(x \circ S] = S$), for all $x \in S$.

An ordered semihypergroup $(S, \circ, \leq)$ is called *regular* (respectively, *intra-regular*) if for all $a \in S$, $a \in (a \circ S \circ a]$ (respectively, $a \in (S \circ a^2 \circ S]$, where we mean $a^2 = a \circ a$). Equivalently, if for all $A \subseteq S$, $A \in (A \circ S \circ A]$ (respectively, $A \in (S \circ A^2 \circ S]$). $(S, \circ, \leq)$ is called *left regular* (respectively, *right regular*) if for all $a \in S$, $a \in (S \circ a^2]$ (respectively, $a \in (a^2 \circ S]$). Equivalently, if for all $A \subseteq S$, $A \in (S \circ A^2]$ (respectively, $A \in (A^2 \circ S]$). $(S, \circ, \leq)$ is called *completely regular* if it is regular, left regular and right regular.

Note that if S is a left simple and right simple ordered semihypergroup, then S is regular. Because for all $x \in S$, $x \in (x \circ S] \subseteq (x \circ (S \circ x]] \subseteq (x \circ S \circ x]$.

A subset A of an ordered semihypergroup S is called *idempotent* if $a \in (a^2]$, for all $a \in A$. If S is idempotent, then S is regular, because $x \in (x^2] \subseteq ((x^2] \circ x] \subseteq (x^3] \subseteq (x \circ S \circ x]$.

Theorem 2.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, in the following cases S is intra-regular.*

(1) S is left regular;

(2) S is right regular and regular;

(3) S is completely regular.

Proof. (1) Suppose that S is left regular, then for all $a \in S$, we have

$$a \in (S \circ a^2] = (S \circ a \circ a] \subseteq (S \circ (S \circ a^2] \circ (S \circ a^2]] \subseteq (S \circ a^2 \circ S].$$

(2) Suppose that S is right regular and regular, then for all $a \in S$, we have

$$a \in (a^2 \circ S] = (a \circ a \circ S] \subseteq ((a \circ S \circ a] \circ (a \circ S]] \subseteq (S \circ a^2 \circ S].$$

(3) By (1), the proof is obvious. □

Theorem 2.2. *An ordered semihypergroup $(S, \circ, \leq)$ is completely regular if and only if $a \in (a^2 \circ S \circ a^2]$, for all $a \in S$.*

Proof. Suppose that S is completely regular. Then, for all $a \in S$, we have $a \in (a \circ S \circ a] \subseteq \big((a^2 \circ S] \circ S \circ (S \circ a^2]\big] \subseteq (a^2 \circ S \circ a^2]$.

Conversely, suppose that for all $a \in S$, we have $a \in (a^2 \circ S \circ a^2]$. Then, $a \in (a^2 \circ S \circ a^2] \subseteq (S \circ a^2]$ and $a \in (a^2 \circ S \circ a^2] \subseteq (a^2 \circ S]$. So, S is left regular and right regular. Also, we get $a \in (a^2 \circ S \circ a^2] \subseteq (a \circ S \circ S \circ S \circ a] \subseteq (a \circ S \circ a]$. This implies that S is regular. So, S is completely regular. □

A subset T of an ordered semihypergroup $(S, \circ, \leq)$ is called *prime* if for every $A, B \subseteq S$ such that $A \circ B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. Equivalently, if $a, b \in S$, $a \circ b \subseteq T$ implies $a \in T$ or $b \in T$. T is called *weakly prime* if for every hyperideals $A, B \subseteq S$ such that $A \circ B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. T is called *semiprime* if for every $A \subseteq S$ such that $A^2 \subseteq T$, we have $A \subseteq T$. Equivalently, if $a \in S$, $a^2 \subseteq T$ implies $a \in T$. T is called *weakly semiprime* if for every hyperideal $A \subseteq S$ such that $A^2 \subseteq T$, we have $A \subseteq T$.

Theorem 2.3. *A hyperideal T of an ordered semihypergroup $(S, \circ, \leq)$ is weakly prime if and only if for all hyperideals A and B of S such that $(A \circ B] \cap (B \circ A] \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.*

Proof. Suppose that T is a weakly prime hyperideal of S and A, B are hyperideals of S such that $(A \circ B] \cap (B \circ A] \subseteq T$. Since $(A \circ B]$ and $(B \circ A]$ are hyperideals of S, we get $(A \circ B] \circ (B \circ A] \subseteq (A \circ B] \circ S \subseteq (A \circ B]$ and $(A \circ B] \circ (B \circ A] \subseteq S \circ (B \circ A] \subseteq (B \circ A]$. So, $(A \circ B] \circ (B \circ A] \subseteq (A \circ B] \cap (B \circ A] \subseteq T$. Hence $(A \circ B] \subseteq T$ or $(B \circ A] \subseteq T$, since T is a weakly prime hyperideal. So, we have $A \circ B \subseteq T$ or $B \circ A \subseteq T$. This implies that $A \subseteq T$ or $B \subseteq T$, since T is a weakly prime hyperideal.

Conversely, suppose that A and B are hyperideals of S such that $A \circ B \subseteq T$. Then, $(A \circ B] \cap (B \circ A] \subseteq (A \circ B] \subseteq (T] = T$. Therefore $A \subseteq T$ or $B \subseteq T$, by hypothesis. This implies that T is a weakly prime hyperideal. □

Theorem 2.4. *A hyperideal of an ordered semihypergroup $(S, \circ, \leq)$ is prime if and only if it is both semiprime and weakly prime. In commutative ordered semihypergroup the prime and weakly prime hyperideals coincide.*

Proof. Suppose that S is an ordered semihypergroup and T is a hyperideal of S. If T is prime, then obviously T is semiprime and weakly prime.

Conversely, suppose that T is a semiprime and weakly prime hyperideal and let $A, B \subseteq S$ such that $A \circ B \subseteq T$. Then,

$$\begin{aligned}(B \circ S \circ A]^2 &\subseteq (B \circ S \circ (A \circ B) \circ S \circ A] \subseteq (S \circ (A \circ B) \circ S] \\ &\subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

So, $(B \circ S \circ A] \subseteq T$, since T is semiprime. Therefore,

$$(S \circ B \circ S] \circ (S \circ A \circ S] \subseteq (S \circ (B \circ S \circ A] \circ S] \subseteq (S \circ T \circ S] \subseteq T.$$

Hence, $(S \circ B \circ S] \subseteq T$ or $(S \circ A \circ S] \subseteq T$, since $(S \circ B \circ S], (S \circ A \circ S]$ are hyperideals of S and T is a weakly prime hyperideal. If $(S \circ A \circ S] \subseteq T$, then for all $a \in S$, we have

$$\begin{aligned}(I(a))^3 &= (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]^2 \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \circ S] \subseteq T.\end{aligned}$$

This implies that $(I(A))^3 \subseteq T$ and so $\left((I(A))^2\right] \circ I(A) \subseteq \left((I(A))^3\right] \subseteq T$. Hence, $\left((I(A))^2\right] \subseteq T$ or $I(A) \subseteq T$, since $\left((I(A))^2\right], I(A)$ are hyperideals of S and T is a weakly prime hyperideal. If $\left((I(A))^2\right] \subseteq T$, then $(I(A))^2 \subseteq T$. Therefore, $I(A) \subseteq T$, since T is a semiprime hyperideal. Then, in general we have $I(A) \subseteq T$ and so $A \subseteq T$. If $(S \circ B \circ S] \subseteq T$, then one can similarly prove that $B \subseteq T$. This shows that T is a prime hyperideal.

Suppose that S is commutative and T is a weakly prime hyperideal of S. Also, let $\emptyset \neq A, B \subseteq S$ such that $A \circ B \subseteq T$. Then,

$$\begin{aligned}I(A) \circ I(B) &= (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S] \circ (B \cup S \circ B \cup B \circ S \cup S \circ B \circ S] \\ &\subseteq (A \circ B \cup S \circ A \circ B] \subseteq I(A \circ B) \subseteq I(T) \\ &= (T \cup S \circ T \cup T \circ S \cup S \circ T \circ S] \\ &\subseteq (T] = T.\end{aligned}$$

So, $I(A) \subseteq T$ or $I(B) \subseteq T$, since $I(A), I(B)$ are hyperideals of S and T is a weakly prime hyperideal of S. This implies that $A \subseteq T$ or $B \subseteq T$. □

3. On hyperideals of regular and intra-regular ordered semihypergroups

In this section, we expand the results of [3] and we investigate some properties of hyperideals on regular and intra-regular ordered semihypergroups. Also, we give the relation between hyperideals and (semi)prime (weakly (semi)prime) ordered semihypergroups.

Theorem 3.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and L be a hyperideal of S. Then, the following statements are valid:*

(1) If S is left regular and L is a left hyperideal, then L is a left regular subsemihypergroup;

(2) If S is right regular and L is a right hyperideal, then L is a right regular subsemihypergroup;

(3) If S is regular, then L is a regular subsemihypergroup.

Proof. (1) Suppose that S is left regular and L is a left hyperideal. Then, for all $a \in L$, we have

$$a \in (S \circ a^2] \subseteq \big(S \circ (S \circ a^2] \circ a\big] \subseteq \big((S \circ a^2] \circ a\big] \subseteq (S \circ a^3] \subseteq (S \circ L \circ a^2] \subseteq (L \circ a^2].$$

So, L is left regular.

(2) The proof is similar to (1).

(3) Suppose that S is regular and L is a hyperideal of S. Then, for all $a \in L$, we have

$$a \in (a \circ S \circ a] \subseteq ((a \circ S \circ a] \circ S \circ a] \subseteq (a \circ S \circ a \circ S \circ a] \subseteq (a \circ S \circ L \circ S \circ a] \subseteq (a \circ L \circ a].$$

So, L is regular. □

Lemma 3.2. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the following are equivalent:*

(1) $(A^2] = A$, for every hyperideal A of S;

(2) $A \cap B = (A \circ B]$, for all hyperideals A and B of S;

(3) $I(a) \cap I(b) = (I(a) \circ I(b)]$, for all $a, b \in S$;

(4) $I(a) = ((I(a))^2]$, for all $a \in S$;

(5) $a \in (S \circ a \circ S \circ a \circ S]$, for all $a \in S$.

Proof. (1) $\longrightarrow$ (2) : Suppose that A and B are hyperideals of S. Then, $(A \circ B] \subseteq (A \circ S] \subseteq (A] = A$ and $(A \circ B] \subseteq (S \circ B] \subseteq (B] = B$. So, $(A \circ B] \subseteq A \cap B$. On the other hand, by hypothesis, we have

$$A \cap B = \big((A \cap B)^2\big] = (A \circ A \cap A \circ B \cap B \circ A \cap B \circ B] \subseteq (A \circ B].$$

So, $A \cap B = (A \circ B]$.

(2) $\longrightarrow$ (3) : By hypothesis, this statement is obvious since $I(a)$ and $I(b)$

are hyperideals of S.

(3) $\longrightarrow$ (4) : Put in (3), $a = b$.

(4) $\longrightarrow$ (5) : By hypothesis, we find that $(I(a))^2 = \left((I(a))^2\right] \circ I(a) \subseteq \left((I(a))^3\right]$. This implies that $(I(a))^3 = (I(a))^2 \circ I(a) \subseteq \left((I(a))^3\right] \circ (I(a)] \subseteq \left((I(a))^4\right]$. So,

$$(I(a))^4 = (I(a))^3 \circ I(a) \subseteq \left((I(a))^4\right] \circ (I(a)] \subseteq \left((I(a))^5\right].$$

Therefore, by hypothesis, we get

$$\begin{aligned} I(a) &= \left((I(a))^2\right] \subseteq \left(\left((I(a))^3\right]\right] = \left((I(a))^3\right] \subseteq \left(\left((I(a))^4\right]\right] = \left((I(a))^4\right] \\ &\subseteq \left(\left((I(a))^5\right]\right] = \left((I(a))^5\right] \subseteq (S \circ I(a)] = I(a). \end{aligned}$$

So,

$$I(a) = \left((I(a))^5\right]. \tag{1}$$

On the other hand, similar to the proof of Theorem 2.4, we can prove that $(I(a))^3 \subseteq (S \circ a \circ S]$. Then, $(I(a))^4 \subseteq (S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \subseteq (S \circ a \circ S \circ a \cup S \circ a \circ S \circ a \circ S]$. This implies that

$$(I(a))^5 = (S \circ a \circ S \circ a \cup S \circ a \circ S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \subseteq (S \circ a \circ S \circ a \circ S].$$

Then, by (1), $a \in I(a) = \left((I(a))^5\right] \subseteq (S \circ a \circ S \circ a \circ S]$.

(5) $\longrightarrow$ (1) : Suppose that $x \in (A^2]$, where A is an hyperideal of S. Then, there are $a, b \in A$ such that $x \leq a \circ b$. So, $x \in A$ since A is an hyperideal of S and $a \circ b \subseteq A$. This implies that $(A^2] \subseteq A$.

On the other hand, suppose that $a \in A$. Then, by hypothesis, $a \in (S \circ a \circ S \circ a \circ S]$. So, there are $t, h, k \in S$ such that $a \leq t \circ a \circ h \circ a \circ k$. We have $(t \circ a) \circ h \subseteq (S \circ A) \circ S \subseteq A \circ S \subseteq A$ and $a \circ k \subseteq A \circ S \subseteq A$. Therefore, $t \circ a \circ h \circ a \circ k \subseteq A^2$ and so $a \in (A^2]$. This showes that $A \subseteq (A^2]$. Hence, $(A^2] = A$. □

Note that if $(S, \circ, \leq)$ is a regular ordered semihypergroup, then for every hyperideal A of S, we have $(A^2] = A$, because $A \subseteq (A \circ S \circ A] \subseteq (A^2] \subseteq (A \circ S] \subseteq (A] = A$ and this implies that $(A^2] = A$, for every hyperideal A of S.

Also, in a right regular (left regular, intra regular) ordered semihypergroup $(S, \circ, \leq)$, we have $(A^2] = A$, for every hyperideal A of S.

Theorem 3.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the hyperideals of S are weakly prime if and only if they form a chain and the five equivalent conditions of Lemma 3.2 hold in S.*

Proof. Suppose that hyperideals of S are weakly prime. Also, let A, B be hyperideals of S. Then, $A \circ B \subseteq (A \circ B]$ and $(A \circ B]$ is an hyperideal of S. Hence, by hypothesis, we get $A \subseteq (A \circ B]$ or $B \subseteq (A \circ B]$. So, $A \subseteq (A \circ B] \subseteq (S \circ B] \subseteq B$ or $B \subseteq (A \circ B] \subseteq (A \circ S] \subseteq A$. This implies that the hyperideals of S form a chain.

If $x \in (A^2]$, then there are $a, b \in A$ such that $x \leq a \circ b \subseteq A \circ S \subseteq A$. So, $x \in A$, since A is an hyperideal of S. This implies that $(A^2] \subseteq A$. Also, by hypothesis, we have $A \subseteq (A^2]$. So, $(A^2] = A$, for every hyperideal A of S.

Conversely, suppose that the hyperideals of S form a chain and $(A \circ B] = A \cap B$, for all hyperideals A, B of S. Also, let A, B and T be hyperideals of S such that $A \circ B \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then $A = A \cap B = (A \circ B] \subseteq (T] = T$. Similarly, if $B \subseteq A$, then $B \subseteq T$. This completes the proof. $\square$

Theorem 3.4. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the hyperideals of S are prime if and only if they form a chain and S is intra-regular.*

Proof. Suppose that the hyperideals of S are prime. Then, the hyperideals of S are weakly prime and semiprime. Therefore, by Theorem 3.3, all the hyperideals of S form a chain. Also, by hypothesis, we have

$$a^4 \subseteq (S \circ a^2 \circ S] \Longrightarrow a^2 \subseteq (S \circ a^2 \circ S] \Longrightarrow a \in (S \circ a^2 \circ S],$$

since $(S \circ a^2 \circ S]$ is a hyperideal of S. Therefore, S is intra-regular.

Conversely, suppose that S is intra-regular and the hyperideals of S form a chain. We prove the statements in several steps.

Step (1): The hyperideals of S are semiprime.

Suppose that A is an hyperideal of S and $a \in S$ such that $a^2 \subseteq A$. Then, $a \in (S \circ a^2 \circ S] \subseteq (S \circ A \circ S] \subseteq A$, since S is intra-regular. So, A is semiprime.

Step (2): For all $x \in S$, $I(x) = (S \circ x \circ S]$.

Since $(S \circ x \circ S]$ is a hyperideal of S, then by Step (1) we have for all $x \in S$,

$$x^4 \subseteq (S \circ x \circ S] \Longrightarrow x^2 \subseteq (S \circ x \circ S] \Longrightarrow x \in (S \circ x \circ S].$$

Therefore, $I(x) = (x \cup S \circ x \circ S] \subseteq (S \circ x \circ S]$. Obviously, $(S \circ x \circ S] \subseteq I(x)$. This implies that $I(x) = (S \circ x \circ S]$.

Step (3): For all $x, y \in S$, $I(x \circ y) = I(x) \cap I(y)$.

We have $x \circ y \subseteq I(x) \circ S \subseteq I(x)$ and $x \circ y \subseteq S \circ I(y) \subseteq I(y)$. So, by Step (2), $I(x \circ y) = (S \circ x \circ y \circ S] \subseteq I(x), I(y)$. Therefore, $I(x \circ y) \subseteq I(x) \cap I(y)$.

Conversely, suppose that $t \in I(x) \cap I(y)$. Then, by Step (2) we get $t \in I(x) = (S \circ x \circ S]$ and $t \in I(y) = (S \circ y \circ S]$. Therefore, there are $a, b, c, d \in S$ such that

$$t^2 = t \circ t \leq c \circ y \circ d \circ a \circ x \circ b. \tag{2}$$

On the other hand, $(y \circ d \circ a \circ x)^2 \subseteq S \circ x \circ y \circ S \subseteq (S \circ x \circ y \circ S] = I(x \circ y)$. Hence, $y \circ d \circ a \circ x \subseteq I(x \circ y)$, since $I(x \circ y)$ is a semiprime hyperideal of S. So,

$$c \circ y \circ d \circ a \circ x \circ b \subseteq S \circ I(x \circ y) \circ S \subseteq I(x \circ y). \tag{3}$$

By (2) and (3), we find that $t^2 \leq I(x \circ y)$. Therefore, $t^2 \subseteq (I(x \circ y)] = I(x \circ y)$ and so $t \in I(x \circ y)$, since $I(x \circ y)$ is a semiprime hyperideal.

Now, we prove the statement of theorem. Suppose that T is a hyperideal of S and $a, b \in S$ are such that $a \circ b \subseteq T$. By hypothesis, we have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then by Step (3), we get $a \in I(a) = I(a) \cap I(b) = I(a \circ b) \subseteq T$. Similarly, one can prove that if $I(b) \subseteq I(a)$, then $b \in T$. So, T is prime. $\square$

Theorem 3.5. *A commutative ordered semihypergroup $(S, \circ, \leq)$ is regular if and only if every hyperideal of S is semiprime.*

Proof. Suppose that I is a hyperideal of a regular commutative ordered semihypergroup S and $a \in S$ is such that $a \circ a \subseteq I$. There is $x \in S$ such that $a \leq a \circ x \circ a$, since S is regular. We have $a \leq a \circ a \circ x \subseteq I \circ S \subseteq I$. So, $a \in I$ and this implies that I is semiprime.

Conversely, suppose that every hyperideal of S is semiprime and $a \in S$. Since S is commutative, then $(a^2 \circ S]$ is an hyperideal of S and so $(a^2 \circ S]$ is semiprime. Then,

$$a^4 \subseteq (a^2 \circ S] \Longrightarrow a^2 \subseteq (a^2 \circ S] \Longrightarrow a \in (a^2 \circ S] \Longrightarrow a \in (a \circ S \circ a].$$

This implies that S is regular. $\square$

Theorem 3.6. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and suppose T is a hyperideal of S. Then, the following are equivalent:*

(1) T is weakly prime;

(2) If $a, b \in S$ such that $(a \circ S \circ b] \subseteq T$, then $a \in T$ or $b \in T$;

(3) If $a, b \in S$ such that $I(a) \circ I(b) \subseteq T$, then $a \in T$ or $b \in T$;

(4) If A, B are right hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$;

(5) If A, B are left hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$;

(6) If A is a right hyperideal and B is a left hyperideal of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.

Proof. (1) $\longrightarrow$ (2) : Suppose that $a, b \in S$ such that $(a \circ S \circ b] \subseteq T$. Then,

$$(S \circ a \circ S] \circ (S \circ b \circ S] \subseteq (S \circ (a \circ S \circ b] \circ S] \subseteq (S \circ T \circ S] \subseteq T.$$

Since $(S \circ a \circ S]$ and $(S \circ b \circ S]$ are hyperideals of S, hence by hypothesis we get $(S \circ a \circ S] \subseteq T$ or $(S \circ b \circ S] \subseteq T$.

If $(S \circ a \circ S] \subseteq T$, then

$$\begin{aligned}
&I(a) \circ I(a) \circ I(a) \\
&\subseteq ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S) \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)] \\
&\qquad \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\
&\subseteq (S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\
&\subseteq (S \circ a \circ S] \subseteq T.
\end{aligned}$$

Therefore,

$$(I(a) \circ I(a)] \circ I(a) \subseteq (I(a) \circ I(a) \circ I(a)] \subseteq (T] = T.$$

Hence, by hypothesis, $(I(a) \circ I(a)] \subseteq T$ or $I(a) \subseteq T$, since $I(a)$ and $(I(a) \circ I(a)]$ are hyperideals of S. If $(I(a) \circ I(a)] \subseteq T$, then $I(a) \circ I(a) \subseteq (I(a) \circ I(a)] \subseteq T$ and so $I(a) \subseteq T$, by hypothesis. Therefore, in both case we have $I(a) \subseteq T$ and this implies that $a \in T$.

Similarly, $(S \circ b \circ S] \subseteq T$ implies that $b \in T$.

$(2) \longrightarrow (3)$: Suppose that $a, b \in T$ such that $I(a) \circ I(b) \subseteq T$. Then,

$$(a \circ S \circ b] \subseteq ((a] \circ (S \circ b]] \subseteq (I(a) \circ I(b)] \subseteq (T] = T.$$

So, by hypothesis we have $a \in T$ or $b \in T$.

$(3) \longrightarrow (4)$: Suppose that A and B are right hyperideals of S such that $A \circ B \subseteq T$ and $A \nsubseteq T$. Then, there is $a \in A$ such that $a \notin T$. We have

$$I(a) \subseteq (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S] \subseteq (A \cup S \circ A].$$

Also, similarly we obtain $I(b) \subseteq (B \cup S \circ B]$, for all $b \in B$. So,

$$\begin{aligned} I(a) \circ I(b) &\subseteq (A \cup S \circ A] \circ (B \cup S \circ B] \\ &\subseteq (A \circ B \cup A \circ S \circ B \cup S \circ A \circ B \cup S \circ A \circ S \circ B] \\ &\subseteq (A \circ B \cup S \circ A \circ B] \subseteq (T] = T. \end{aligned}$$

Therefore, by hypothesis, $b \in T$, for all $b \in B$, since $a \notin T$. This implies that $B \subseteq T$.

$(3) \longrightarrow (5)$: The proof is similar to the proof $(3) \longrightarrow (4)$.

$(3) \longrightarrow (6)$: Suppose that A is a right hyperideal and B is a left hyperideal of S such that $A \circ B \subseteq T$ and $A \nsubseteq T$. Then, there is $a \in A$ such that $a \notin T$. We have $I(a) \subseteq (A \cup S \circ A]$ and

$$I(b) \subseteq (B \cup S \circ B \cup B \circ S \cup S \circ B \circ S] \subseteq (B \cup B \circ S], \text{ for all } b \in B.$$

Therefore,

$$\begin{aligned} I(a) \circ I(b) &\subseteq (A \cup S \circ A] \circ (B \cup B \circ S] \\ &\subseteq (A \circ B \cup A \circ B \circ S \cup S \circ A \circ B \cup S \circ A \circ B \circ S] \subseteq T. \end{aligned}$$

Hence, by hypothesis, $b \in T$, for all $b \in B$, since $a \notin T$. This implies that $B \subseteq T$.

$(4), (5), (6) \longrightarrow (1)$: This is obvious. □

Theorem 3.7. *In Theorem 3.6, the conditions (4), (5) and (6) are equivalent respectively to the following conditions:*

(4′) If $a, b \in S$ such that $R(a) \circ R(b) \subseteq T$, then $a \in T$ or $b \in T$;

(5′) If $a, b \in S$ such that $L(a) \circ L(b) \subseteq T$, then $a \in T$ or $b \in T$;

(6′) If $a, b \in S$ such that $R(a) \circ L(b) \subseteq T$, then $a \in T$ or $b \in T$.

Proof. $(4) \longrightarrow (4')$: This is obvious.

$(4') \longrightarrow (4)$: Suppose that A and B are right hyperideals of S such that $A \circ B \subseteq T$ and $A \nsubseteq T$. Then, there is $a \in A$ such that $a \notin T$. For all $b \in B$, we have

$$R(a) \circ R(b) \subseteq (A \cup A \circ S] \circ (B \cup B \circ S] \subseteq (A \circ B] \subseteq T.$$

Therefore, by hypothesis, $b \in T$, for all $b \in B$, since $a \notin T$. This implies that $B \subseteq T$.

Similarly, one can prove that (5) and (6) are equivalent with (5′) and (6′), respectively. □

Corollary 3.8. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and suppose T is a hyperideal of S. T is weakly semiprime if and only if any of the following four equivalent conditions holds in S:*

(1) If $a \in S$ such that $(a \circ S \circ a] \subseteq T$, then $a \in T$;

(2) If $a \in S$ such that $I(a)^2 \subseteq T$, then $a \in T$;

(3) If A is a right hyperideal of S such that $A^2 \subseteq T$, then $A \subseteq T$;

(4) If A is a left hyperideal of S such that $A^2 \subseteq T$, then $A \subseteq T$.

Proof. The proof is obvious, by Theorem 3.6. □

4. On regular duo ordered semihypergroups

We begin this section with the following definition.

Definition 4.1. An ordered semihypergroup $(S, \circ, \leq)$ is called *right* (respectively, *left*) *duo* if the right (respectively, left) hyperideals of S are two-sided. S is called *duo* if it is right duo and left duo.

Example 4.2. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation " $\circ$ " and the order relation " $\leq$ " are defined by:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

Then, S is duo.

Theorem 4.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup.*

(1) If S is regular and right duo, then $(S \circ x] \subseteq (x \circ S]$, for all $x \in S$;

(2) If $(S \circ x] \subseteq (x \circ S]$, for all $x \in S$, then S is right duo.

Proof. (1) Suppose that $x \in S$, then

$$S \circ x \subseteq S \circ (x \circ S \circ x] \subseteq (S \circ x \circ S \circ x] \subseteq (S \circ (x \circ S] \circ x], \tag{4}$$

since S is regular. On the other hand, we have $(x \circ S]$ is a left hyperideal of S, because S is right duo. So, by (4), we get $S \circ x \subseteq ((x \circ S] \circ x] \subseteq (x \circ S \circ x] \subseteq (x \circ S]$. This implies that $(S \circ x] \subseteq ((x \circ S]] = (x \circ S]$.

(2) Suppose that A is a right hyperideal of S. Then, $S \circ A \subseteq (S \circ A] \subseteq (A \circ S] \subseteq (A] = A$. This implies that A is a left hyperideal of S. □

Theorem 4.4. *Let $(S, \circ, \leq)$ be an ordered semihypergroup.*

(1) If S is regular and left duo, then $(x \circ S] \subseteq (S \circ x]$, for all $x \in S$;

(2) If $(x \circ S] \subseteq (S \circ x]$, for all $x \in S$, then S is left duo.

Proof. The proof is similar to the proof of Theorem 4.3. □

Corollary 4.5. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If S is duo and regular, then $(S \circ x] = (x \circ S]$, for all $x \in S$. Conversely, if $(S \circ x] = (x \circ S]$, for all $x \in S$, then S is duo.*

Proof. By Theorems 4.3 and 4.4, the proof is obvious. □

Theorem 4.6. *Let $(S, \circ, \leq)$ be an ordered semihypergroup.*

(1) If S is regular and right duo, then S is right regular;

(2) If S is regular and left duo, then S is left regular;

(3) If S is regular and duo, then S is right regular, left regular and intra-regular.

Proof. (1) Suppose that $x \in S$. Since S is regular and right duo, it follows that

$$\begin{aligned} x \in (x \circ S \circ x] &\subseteq ((x \circ S \circ x] \circ S \circ x] \subseteq ((x \circ S \circ x] \circ (S \circ x]] \\ &\subseteq (x \circ S \circ x \circ S \circ x] \subseteq (x \circ S \circ (x \circ S] \circ x] \\ &\subseteq (x \circ (x \circ S] \circ x] \subseteq (x \circ x \circ S \circ x] \\ &\subseteq \left(x^2 \circ S\right]. \end{aligned}$$

So, S is right regular.

(2) The proof is similar to the proof (1).

(3) By parts (1) and (2), S is right regular and left regular. Also, for all $x \in S$, we have $x \in (S \circ x^2] \subseteq \left(S \circ (x^2 \circ S] \circ x\right] \subseteq (S \circ x^2 \circ S \circ x] \subseteq (S \circ x^2 \circ S]$. So, S is intra-regular. □

Theorem 4.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If S is right regular, left regular and left duo (right duo), then S is regular.*

Proof. Suppose that S is right regular, left regular and left duo. Then, for all $x \in S$,

$$\begin{aligned} x \in (x^2 \circ S] &\subseteq \big(x \circ (S \circ x^2] \circ S\big] \subseteq \big(x \circ (S \circ x^2]\big] \subseteq \big((x] \circ (S \circ x^2]\big] \\ &\subseteq \big((x \circ S \circ x^2]\big] = (x \circ (S \circ x) \circ x] \subseteq (x \circ S \circ x]. \end{aligned}$$

This implies that S is regular.

If S is right regular, left regular and right duo, then similarly one can show that S is regular. □

Corollary 4.8. *Let $(S, \circ, \leq)$ be a duo ordered semihypergroup. S is regular if and only if it is right regular and left regular.*

Proof. By Theorems 4.6 and 4.7, the proof is obvious. □

Corollary 4.9. *Let $(S, \circ, \leq)$ be a right duo (respectively, left duo) and regular ordered semihypergroup. Then, right hyperideals (respectively, left hyperideals) of S are semiprime.*

Proof. Suppose that S is right duo and regular. Also, let A be a right hyperideal and B be a subset of S such that $B \circ B \subseteq A$. By Theorem 4.6, S is right regular and so $B \subseteq (B^2 \circ S] \subseteq (A \circ S] \subseteq (A] = A$.

Now, suppose that S is left duo and regular and A is a left hyperideal of S such that $B \circ B \subseteq A$, where B is a subset of S. Then, by Theorem 4.6, S is left regular and so $B \subseteq (S \circ B^2] \subseteq (S \circ A] \subseteq (A] = A$. □

Theorem 4.10. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If right hyperideals (respectively, left hyperideals) of S are semiprime, then S is right regular (respectively, left regular).*

Proof. Suppose that right hyperideals of S are semiprime and let $x \in S$. Then, $x^2 \circ x^2 \subseteq x^2 \circ S \subseteq (x^2 \circ S]$. By hypothesis, the right hyperideal $(x^2 \circ S]$ is semiprime and so $x^2 \subseteq (x^2 \circ S]$. Hence, $x \in (x^2 \circ S]$.

The proof of the other case is similar. □

Corollary 4.11. *Let $(S, \circ, \leq)$ be a duo ordered semihypergroup. S is regular if and only if right hyperideals and left hyperideals of S are semiprime.*

Proof. Suppose that right hyperideals and left hyperideals of S are semiprime. By Theorem 4.10, S is right regular and left regular. On the other hand, by hypothesis S is duo. So, by Corollary 4.8, S is regular.

The converse is obvious by Corollary 4.9. □

Theorem 4.12. *Let $(S, \circ, \leq)$ be a duo and regular ordered semihypergroup. Then, for every right hyperideal A and for every left hyperideal B of S, we have $A \cap B = (A \circ B]$.*

Proof. Suppose that A is a right hyperideal and B is a left hyperideal of S. By hypothesis, we have $A \cap B \subseteq ((A \cap B) \circ S \circ (A \cap B)] \subseteq (A \circ S \circ B] \subseteq (A \circ B]$. On the other hand, by hypothesis we obtain $(A \circ B] \subseteq (A \circ S] \subseteq (A] = A$ and $(A \circ B] \subseteq (S \circ B] \subseteq (B] = B$. So, $(A \circ B] \subseteq A \cap B$ and this completes the proof. □

Theorem 4.13. *Let $(S, \circ, \leq)$ be an ordered semihypergroup such that $A \cap B = (B \circ A]$, for every right hyperideal A and every left hyperideal B of S. Then, S is duo and $x \in (S \circ x \circ S]$, for all $x \in S$.*

Proof. Suppose that A and B are a right hyperideal and a left hyperideal of S, respectively. Then, by hypothesis we have $S \circ A \subseteq (S \circ A] = S \cap A = A$ and $B \circ S \subseteq (B \circ S] = S \cap B = B$. This implies that A is a left hyperideal and B is a right hyperideal of S. So, S is duo.

Now, suppose that $x \in S$. Then, by hypothesis we have

$$\begin{aligned} x \in R(x) \cap L(x) &= (L(x) \circ R(x)] = ((x \cup S \circ x] \circ (x \cup x \circ S]] \\ &= (x \circ x \cup x^2 \circ S \cup S \circ x^2 \cup S \circ x^2 \circ S] \subseteq (x \circ x \cup S \circ x \circ S]. \end{aligned} \tag{5}$$

Therefore, $x^2 \subseteq (x \circ x \cup S \circ x \circ S] \circ x \subseteq (x^3 \cup S \circ x \circ S] \subseteq (S \circ x \circ S]$. So, by (5), $x \in (S \circ x \circ S]$. □

5. Bi-hyperideals and Quasi-hyperideals

In 1992, Kehayopulu introduced the notion of quasi-ideal in ordered semigroups as follows [11]: A non-empty subset Q of an ordered semigroup S is called a *quasi-ideal* if (1) $S \circ Q \cap Q \circ S \subseteq Q$, (2) for $x \in Q$ and $y \in S$, $y \leq x$ implies that $y \in Q$. Also, Sang Keun Lee and Young In Kwon in [21] based their results on quasi-ideals defined as above. Later Kehayopulu and Tsingelis changed the above definition as follows [15, 28]: A non-empty subset Q of an ordered semigroup S is called a *quasi-ideal* if (1) $(S \circ Q] \cap (Q \circ S] \subseteq Q$, (2) for $x \in Q$ and $y \in S$, $y \leq x$ implies that $y \in Q$, (for $H \subseteq S$, $(H] = \{t \in S \mid t \leq h, \text{ for some } h \in H\}$ [10]). The reason they changed the first definition was that the quasi-ideals should be intersections of right and left ideals, while the first definition failed to have this property [16].

So, by the definition of quasi-ideals in ordered semigroups, we consider the definition of quasi-hyperideals in ordered semihypergroups as follows:

Definition 5.1. A non-empty subset Q of an ordered semihypergroup $(S, \circ, \leq)$ is called a *quasi-hyperideal* of S if it satisfies the following conditions:

(1) $(S \circ Q] \cap (Q \circ S] \subseteq Q$;

(2) For $x \in Q$ and $y \in S$, $y \leq x$ implies that $y \in Q$.

It is clear that if Q is a quasi-hyperideal, then $(S \circ Q) \cap (Q \circ S) \subseteq Q$ and Q is a subsemihypergroup of S.

Suppose that a is an element of an ordered semihypergroup $(S, \circ, \leq)$. Then, $Q(a) = (a \cup ((a \circ S] \cap (S \circ a])]$ denotes the quasi-hyperideal of S generated by a. Obviously, $Q(a)$ is a quasi-hyperideal of S.

Example 5.2. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation " $\circ$ " and the order relation " $\leq$ " are defined by:

$\circ$	a	b	c	d	f
a	a	a	a	a	a
b	a	$\{a,b\}$	a	$\{a,d\}$	a
c	a	$\{a,f\}$	$\{a,c\}$	$\{a,c\}$	$\{a,f\}$
d	a	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
f	a	$\{a,f\}$	a	$\{a,c\}$	a

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(f,f),(a,b),(a,c),(a,d),(a,f)\}.$$

It is easy to see that the quasi-hyperideals of S are $\{a\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{a,f\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{a,b,f\}$, $\{a,c,f\}$.

Theorem 5.3. *An ordered semihypergroup $(S, \circ, \leq)$ is completely regular if and only if every quasi-hyperideal of S is a completely regular subsemihypergroup of S.*

Proof. Suppose that S is a completely regular ordered semihypergroup and Q is a quasi-hyperideal of S. Then, $\emptyset \neq Q \subseteq S$ and we have

$$Q \circ Q \subseteq (Q \circ S) \cap (S \circ Q) \subseteq (Q \circ S] \cap (S \circ Q] \subseteq Q.$$

So, Q is subsemihypergroup of S.

Also, for all $a \in Q$, we have $a \circ S \circ a \subseteq Q \circ S \circ Q \subseteq (Q \circ S) \cap (S \circ Q) \subseteq Q$. So, by Theorem 2.2,

$$a \in (a^2 \circ S \circ a^2] = (a \circ (a \circ S \circ a) \circ a] \subseteq (a \circ Q \circ a].$$

This implies that Q is regular. On the other hand,

$$\begin{aligned} a \in (a \circ S \circ a] &\subseteq \big(a \circ S \circ (S \circ a^2]\big] \subseteq (a \circ S \circ a^2] \subseteq \big(a \circ S \circ (S \circ a^2] \circ a\big] \\ &\subseteq \big((a \circ S \circ a) \circ a^2\big] \subseteq (Q \circ a^2]. \end{aligned}$$

This implies that Q is left regular. Also, we find that

$$\begin{aligned} a \in (a \circ S \circ a] &\subseteq \big((a^2 \circ S] \circ S \circ a\big] \subseteq (a^2 \circ S \circ a] = \big(a \circ (a^2 \circ S] \circ S \circ a\big] \\ &\subseteq \big(a \circ (a^2 \circ S) \circ S \circ a\big] \subseteq \big(a^2 \circ (a \circ S \circ a)\big] \subseteq (a^2 \circ Q]. \end{aligned}$$

So, Q is right regular and so Q is completely regular.

The converse is obvious. $\square$

Theorem 5.4. *An ordered semihypergroup $(S, \circ, \leq)$ is left regular and right regular if and only if every quasi-hyperideal of S is semiprime.*

Proof. Suppose that S is left and right regular and Q is a quasi-hyperideal of S. Also, suppose that $a \in S$ is such that $a^2 \subseteq Q$. So, we have $a \in (a^2 \circ S] \subseteq (Q \circ S]$ and $a \in (S \circ a^2] \subseteq (S \circ Q]$. So, $a \in (Q \circ S] \cap (S \circ Q] \subseteq Q$. Therefore, $a \in Q$ and this implies that Q is semiprime.

Conversely, suppose that every quasi-hyperideal of S is semiprime and let $a \in S$. By hypothesis, we have that $Q(a^2)$ is semiprime. So, $a \in Q(a^2)$, since $a^2 \subseteq Q(a^2)$. On the other hand, $Q(a^2) = L(a^2) \cap R(a^2) = \big(a^2 \cup (S \circ a^2)\big] \cap \big(a^2 \cup (a^2 \circ S)\big]$. Therefore, $Q(a^2) \subseteq \big(a^2 \cup (S \circ a^2)\big]$ and $Q(a^2) \subseteq \big(a^2 \cup (a^2 \circ S)\big]$. Then, $a^2 \subseteq a \circ \big(a^2 \cup (S \circ a^2)\big] \subseteq (a^3 \cup a \circ S \circ a^2] \subseteq (S \circ a^2]$. So,

$$a \in Q(a^2) \subseteq \big(a^2 \cup (S \circ a^2)\big] \subseteq \big((S \circ a^2] \cup (S \circ a^2)\big] = (S \circ a^2].$$

This implies that S is left regular. Also, we find that $a^2 \subseteq \big(a^2 \cup (a^2 \circ S)\big] \circ a \subseteq (a^3 \cup a^2 \circ S \circ a] \subseteq (a^2 \circ S]$. Hence,

$$a \in Q(a^2) \subseteq \big(a^2 \cup (a^2 \circ S)\big] \subseteq \big((a^2 \circ S] \cup (a^2 \circ S)\big] = (a^2 \circ S].$$

This implies that S is right regular. □

Definition 5.5. [2] A subsemihypergroup A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *bi-hyperideal* of S if it satisfies the following conditions:

(1) $A \circ S \circ A \subseteq A$;

(2) For $x \in A$ and $y \in S$, $y \leq x$ implies that $y \in A$.

Note that a left hyperideal (respectively, right hyperideal) is a bi-hyperideal. Also, every quasi-hyperideal B of an ordered semihypergroup $(S, \circ, \leq)$ is a bi-hyperideal, because $B \circ S \circ B \subseteq (B \circ S] \cap (S \circ B] \subseteq B$.

Suppose that a is an element of an ordered semihypergroup $(S, \circ, \leq)$. Then, $B(a) = (a \cup a^2 \cup a \circ S \circ a]$ denotes the bi-hyperideal of S generated by a. Obviously, $B(a)$ is a bi-hyperideal of S.

Theorem 5.6. *If the bi-hyperideals of an ordered semihypergroup* $(S, \circ, \leq)$ *are semiprime, then* S *is completely regular.*

Proof. Suppose that every bi-hyperideal of S is semiprime. At first, we show that $(a^2 \circ S \circ a^2]$ is a bi-hyperideal. It is clear that $\emptyset \subseteq (a^2 \circ S \circ a^2] \subseteq S$. Also, we have $(a^2 \circ S \circ a^2] \circ S \circ (a^2 \circ S \circ a^2] \subseteq (a^2 \circ S \circ a^2]$. If $x \in (a^2 \circ S \circ a^2]$ and $S \ni y \leq x$, then, $y \in (a^2 \circ S \circ a^2]$. Therefore, $(a^2 \circ S \circ a^2]$ is a bi-hyperideal of S and so $(a^2 \circ S \circ a^2]$ is semiprime, by hypothesis. On the other hand, we have $a^8 \subseteq (a^2 \circ S \circ a^2]$. Since $(a^2 \circ S \circ a^2]$ is semiprime, we find that

$$a^4 \subseteq (a^2 \circ S \circ a^2] \Longrightarrow a^2 \subseteq (a^2 \circ S \circ a^2] \Longrightarrow a \in (a^2 \circ S \circ a^2].$$

This implies that S is completely regular, by Theorem 2.2. □

Theorem 5.7. *Let* $(S, \circ, \leq)$ *be a regular ordered semihypergroup and suppose* B *is a bi-hyperideal of* S*. Then,* $B^n \subseteq B$*, for all* $n \in \mathbb{N}$*.*

Proof. We prove the statement by induction on n. Suppose that B is a bi-hyperideal of S. Then, $B \subseteq (B \circ S \circ B] \subseteq B$. So, $(B \circ S \circ B] = B$ and this implies that $B^2 = (B \circ S \circ B] \circ (B \circ S \circ B] \subseteq (B \circ S \circ B] = B$. Suppose that the statement is valid for $n = k$, that is $B^k \subseteq B$. For $n = k + 1$, we have $B^{k+1} = B^k \circ B \subseteq B \circ B \subseteq B$ and this completes the proof. □

Theorem 5.8. *A semihypergroup $(S, \circ)$ is completely regular if and only if every quasi-ideal of S is a completely regular subsemihypergroup of S.*

Proof. Suppose that $(S, \circ)$ is a completely regular semihypergroup and Q is a quasi-ideal of S. We define the relation " $\leq$ " on S as $\leq := \{(x, y) \mid x = y\}$. Then, $(S, \circ, \leq)$ is a completely regular ordered semihypergroup. We prove that Q is a quasi-hyperideal of $(S, \circ, \leq)$. Suppose that $t \in (Q \circ S] \cap (S \circ Q]$. So, there are $q_1, q_2 \in Q$ and $s_1, s_2 \in S$ such that $t \leq q_1 \circ s_1$ and $t \leq s_2 \circ q_2$. So, $t \in Q \circ S,\ S \circ Q$. Then, $t \in (Q \circ S) \cap (S \circ Q) \subseteq Q$. So, by Theorem 5.3, Q is a completely regular subsemihypergroup of S.

The converse is obvious. □

Theorem 5.9. *Let $(S, \circ)$ be a completely regular semihypergroup. Then, every quasi-ideal of S is semiprime.*

Proof. Suppose that Q is a quasi-ideal of $(S, \circ)$ and " $\leq$ " is defined as the proof of Theorem 5.8. Then, $(S, \circ, \leq)$ is a completely regular ordered semihypergroup. Also, by the proof of Theorem 5.8, Q is a quasi-hyperideal of $(S, \circ, \leq)$. Hence, by Theorem 5.4, Q is semiprime. □

Definition 5.10. Let $(S, \circ, \leq)$ be an ordered semihypergroup. A bi-hyperideal B of S is called *prime* if $x \circ S \circ y \subseteq B$ implies that $x \in B$ or $y \in B$. Equivalently, for subsets $C, D \subseteq S$, $C \circ S \circ D \subseteq B$ implies that $C \subseteq B$ or $D \subseteq B$.

Definition 5.11. Let $(S, \circ, \leq)$ be an ordered semihypergroup. A bi-hyperideal B of S is called *semiprime* if $x \circ S \circ x \subseteq B$ implies that $x \in B$. Equivalently, for a subset $C \subseteq S$, $C \circ S \circ C \subseteq B$ implies $C \subseteq B$.

Theorem 5.12. *A bi-hyperideal B of an ordered semihypergroup $(S, \circ, \leq)$ is prime if and only if for all right hyperideals R and left hyperideals L of S such that $R \circ L \subseteq B$, we have $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that B is a prime bi-hyperideal of S and $R \circ L \subseteq B$ and $R \nsubseteq B$. Then, there is $r \in R$ such that $r \notin B$. For all $x \in L$, we have $r \circ S \circ x \subseteq R \circ L \subseteq B$. So, $x \in B$, since B is prime and $r \notin B$. This implies that $L \subseteq B$.

Conversely, suppose that for all right hyperideals R and left hyperideals L of S such that $R \circ L \subseteq B$, we have $R \subseteq B$ or $L \subseteq B$. Also, suppose $x \circ S \circ y \subseteq B$, for all $x, y \in S$. Then, $(x \circ S] \circ (S \circ y] \subseteq (x \circ S \circ y] \subseteq B$. Therefore, by hypothesis $(x \circ S] \subseteq B$ or $(S \circ y] \subseteq B$. If $(x \circ S] \subseteq B$, then $R(x) \circ L(x) = (x \cup x \circ S] \circ (x \cup S \circ x] \subseteq (x \circ x \cup x \circ S \circ x]$. So, for all $z \in R(x) \circ L(x)$, we have $z \in (x \circ x \cup x \circ S \circ x]$. Therefore, there exists

$t \in x \circ x \subseteq B$ or $t \in x \circ S \circ x \subseteq x \circ S \subseteq (x \circ S] \subseteq B$ such that $z \leq t$. This implies that $z \in B$ and so $R(x) \circ L(x) \subseteq B$. By hypothesis, we get $R(x) \subseteq B$ or $L(x) \subseteq B$ and this implies that $x \in B$. If $(S \circ y] \subseteq B$, then, one can similarly prove that $y \in B$. So, B is a prime hyperideal of S. □

Theorem 5.13. *A prime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$ is a prime one-sided hyperideal of S.*

Proof. Suppose that B is a prime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $(B \circ S] \circ (S \circ B] \subseteq (B \circ S \circ B] \subseteq B$. So, by Theorem 5.12, $B \circ S \subseteq (B \circ S] \subseteq B$ or $S \circ B \subseteq (S \circ B] \subseteq B$. On the other hand, if $x \in B$, $S \ni y \leq x$, then $y \in B$, since B is a bi-hyperideal. Therefore, B is a right hyperideal or left hyperideal of S and this completes the proof. □

Definition 5.14. Let B be a bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. We define

$$L(B) = \{x \in B \mid S \circ x \subseteq B\} \quad \text{and} \quad H(B) = \{x \in L(B) \mid x \circ S \subseteq L(B)\}.$$

Lemma 5.15. *Let B be a bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $L(B)$ is a left hyperideal of S.*

Proof. Suppose that $x \in L(B)$. Then, for all $z \in S$, $z \circ x \subseteq S \circ x \subseteq B$. On the other hand, $S \circ (z \circ x) = (S \circ z) \circ x \subseteq S \circ x \subseteq B$. So, $z \circ x \subseteq L(B)$ and this implies that $S \circ L(B) \subseteq L(B)$. Suppose that $S \ni y \leq x$ and $x \in L(B) \subseteq B$. Then, $y \in B$, since B is a bi-hyperideal. On the other hand, $z \circ y \leq z \circ x$, for all $z \in S$. Hence, $z \circ y \subseteq B$, since $z \circ x \subseteq B$ and B is a bi-hyperideal of S. This implies that $y \in L(B)$. □

Theorem 5.16. *Let B be a bi-hyperideal of ordered semihypergroup $(S, \circ, \leq)$. Then, $H(B)$ is the unique largest hyperideal of S contained in B.*

Proof. We have $H(B) \subseteq L(B) \subseteq B$. Suppose that $x \in H(B)$. Then, $x \circ S \subseteq L(B)$ and $S \circ x \subseteq B$. It is clear that $x \circ y \subseteq L(B)$ and $x \circ y \circ S \subseteq x \circ S \subseteq L(B)$, for all $y \in S$. So, $x \circ y \subseteq H(B)$. This implies that $H(B) \circ S \subseteq H(B)$. Also, we have $y \circ x \subseteq B$ and $S \circ y \circ x \subseteq S \circ x \subseteq B$. So, $y \circ x \subseteq L(B)$, for all $y \in S$. On the other hand, by Lemma 5.15, $y \circ x \circ S \subseteq S \circ x \circ S \subseteq S \circ L(B) \subseteq L(B)$. This implies that $y \circ x \subseteq H(B)$, for all $y \in S$. So, $S \circ H(B) \subseteq H(B)$.

Suppose that $x \in H(B)$, $S \ni y \leq x$. Then, $x \in L(B)$ and by Lemma 5.15, we get $y \in L(B)$. Also, we have $y \circ z \leq x \circ z \subseteq x \circ S \subseteq L(B)$, for all $z \in S$. So, $y \circ z \subseteq L(B)$, by Lemma 5.15. This implies that $y \circ S \subseteq L(B)$ and so $y \in H(B)$. Then, $H(B)$ is a hyperideal of S.

Now, suppose that I is a hyperideal of S such that $I \subseteq B$. We have $S \circ u \subseteq I \subseteq B$, for all $u \in I$. So, $u \in L(B)$ and this implies that $I \subseteq L(B)$. On the other hand, $u \circ S \subseteq I \subseteq L(B)$. Therefore, $u \in H(B)$ and so $I \subseteq H(B)$. This completes the proof. □

Theorem 5.17. *Let B be a prime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $H(B)$ is a weakly prime hyperideal of S.*

Proof. By Theorem 5.16, we have that $H(B)$ is a hyperideal of S. Suppose that $a, b \in S$ are such that $I(a) \circ I(b) \subseteq H(B) \subseteq B$. Then, by Theorem 5.12, we get $I(a) \subseteq B$ or $I(b) \subseteq B$. By Theorem 5.16, $H(B)$ is the largest hyperideal of S contained in B. Therefore, $I(a) \subseteq H(B)$ or $I(b) \subseteq H(B)$. This implies that $a \in H(B)$ or $b \in H(B)$. So, by Theorem 3.6, $H(B)$ is weakly prime. □

Lemma 5.18. *Let B be a semiprime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, for every left hyperideal L (respectively, right hyperideal R) of S, $L^2 \subseteq B$ (respectively, $R^2 \subseteq B$) implies $L \subseteq B$ (respectively, $R \subseteq B$).*

Proof. The proof is similar to Proposition 10 of [6]. □

Theorem 5.19. *Let B be a semiprime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $H(B)$ is a weakly semiprime hyperideal of S.*

Proof. By Theorem 5.16, $H(B)$ is an hyperideal of S. We prove that $H(B)$ is weakly semiprime. Suppose that $a \in S$ is such that $(I(a))^2 \subseteq H(B) \subseteq B$. By Lemma 5.18, we obtain $I(a) \subseteq B$. So, by Theorem 5.16, $I(a) \subseteq H(B)$. This implies that $a \in H(B)$. Now, Corollary 3.8 implies that $H(B)$ is weakly semiprime. □

Theorem 5.20. *Let B be a semiprime bi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, B is a quasi-hyperideal of S.*

Proof. Suppose that $y \in (B \circ S] \cap (S \circ B]$. Then, there are $t \in B \circ S$ and $r \in S \circ B$ such that $y \leq t$ and $y \leq r$. We have

$$y \circ S \circ y \leq t \circ S \circ r \subseteq (B \circ S) \circ S \circ (S \circ B) \subseteq (B \circ S \circ B) \subseteq B.$$

So, $y \circ S \circ y \subseteq B$. Hence, $y \in B$, since B is semiprime. This implies that $(B \circ S] \cap (S \circ B] \subseteq B$. Suppose that $x \in B$ and $S \ni y \leq x$. Then, $y \in B$, since B is a bi-hyperideal. Therefore, B is a quasi-hyperideal of S. □

Theorem 5.21. *An ordered semihypergroup $(S, \circ, \leq)$ is regular if and only if every bi-hyperideal of S is semiprime.*

Proof. Suppose that B is a bi-hyperideal of a regular ordered semihypergroup S and $a \in S$ such that $a \circ S \circ a \subseteq B$. Since B is regular, there is $x \in S$ such that $a \leq a \circ x \circ a \subseteq B$. This implies that $a \in B$, since B is a bi-hyperideal.

Conversely, suppose that every bi-hyperideal of S is semiprime and let $a \in S$. Since $(a \circ S \circ a] \circ S \circ (a \circ S \circ a] \subseteq (a \circ S \circ a]$, hence $(a \circ S \circ a]$ is a bi-hyperideal of S. Therefore, by hypothesis, $(a \circ S \circ a]$ is semiprime. Then, $a \circ S \circ a \subseteq (a \circ S \circ a]$ implies that $a \in (a \circ S \circ a]$. Therefore, S is regular. □

Theorem 5.22. *Let $(S, \circ, \leq)$ is an ordered semihypergroup. If for all $a \in S$ we have $R(a) \cap Q(a) \cap L(a) \subseteq (L(a) \circ Q(a) \circ R(a)]$, then S is intra-regular.*

Proof. By hypothesis we get

$$\begin{aligned}
a \in & R(a) \cap Q(a) \cap L(a) \subseteq (L(a) \circ Q(a) \circ R(a)] \\
&= ((a \cup S \circ a) \circ (a \cup ((a \circ S] \cap (S \circ a])) \circ (a \cup a \circ S)] \\
&\subseteq ((a \cup S \circ a) \circ (a \cup (a \circ S]) \circ (a \cup a \circ S)] \\
&= \big(a^3 \cup (a^2 \circ S] \circ a \cup S \circ a^3 \cup (S \circ a^2 \circ S] \circ a \cup a^3 \circ S \cup (a^2 \circ S] \circ a \circ S \\
&\qquad \cup S \circ a^3 \circ S \cup (S \circ a^2 \circ S] \circ (a \circ S)\big] .
\end{aligned} \tag{6}$$

On the other hand, $(a^2 \circ S] \circ a \subseteq (a^2 \circ S]$, $(S \circ a^2 \circ S] \circ a \subseteq (S \circ a^2 \circ S]$, $(a^2 \circ S] \circ a \circ S \subseteq (a^2 \circ S]$ and $(S \circ a^2 \circ S] \circ (a \circ S) \subseteq (S \circ a^2 \circ S]$. So, by (6), we have

$$a \in \big(a^3 \cup (a^2 \circ S] \cup (S \circ a^2 \circ S]\big] . \tag{7}$$

Then,

$$\begin{aligned}
a^3 \quad & \subseteq \big(a^3 \cup (a^2 \circ S] \cup (S \circ a^2 \circ S]\big] \circ a^2 \\
& \subseteq \big(a^5 \cup (a^2 \circ S] \circ a^2 \cup (S \circ a^2 \circ S] \circ a^2\big] .
\end{aligned} \tag{8}$$

On the other hand, $a^5 \subseteq S \circ a^2 \circ S$, $(a^2 \circ S] \circ a^2 \subseteq (a^2 \circ S]$ and $(S \circ a^2 \circ S] \circ a^2 \subseteq (S \circ a^2 \circ S]$. Therefore, by (8),

$$a^3 \subseteq \big(S \circ a^2 \circ S \cup (a^2 \circ S] \cup (S \circ a^2 \circ S]\big] \subseteq \big((a^2 \circ S] \cup (S \circ a^2 \circ S]\big] .$$

Hence, by (7),

$$\begin{aligned}
a \quad & \in \big(\big((a^2 \circ S] \cup (S \circ a^2 \circ S]\big] \cup (a^2 \circ S] \cup (S \circ a^2 \circ S]\big] \\
& \subseteq ((a^2 \circ S] \cup (S \circ a^2 \circ S]].
\end{aligned} \tag{9}$$

Then,

$$a^2 \subseteq a \circ \big((a^2 \circ S] \cup (S \circ a^2 \circ S]\big] \subseteq \big((a^3 \circ S] \cup (a \circ S \circ a^2 \circ S]\big] \subseteq (S \circ a^2 \circ S].$$

Therefore $(a^2 \circ S] \subseteq (S \circ a^2 \circ S]$. So, by (9), $a \in (S \circ a^2 \circ S]$ and this implies that S is intra-regular. □

Theorem 5.23. *Let $(S, \circ, \leq)$ be an intra-regular ordered semihypergroup. Then, $R(a) \cap B(a) \cap L(a) \subseteq (L(a) \circ B(a) \circ R(a)]$, for all $a \in S$.*

Proof. Suppose that S is intra-regular. Then, for all $a \in S$, we have

$$a \in (S \circ a^2 \circ S] \subseteq (S \circ (S \circ a^2 \circ S] \circ (S \circ a^2 \circ S] \circ S] \subseteq (S \circ a^2 \circ S \circ a^2 \circ S].$$

Therefore,

$$\begin{aligned}
R(a) \cap B(a) \cap L(a) \quad & \subseteq (S \circ (R(a) \cap B(a) \cap L(a)] \\
& \qquad \circ ((R(a) \cap B(a) \cap L(a)] \circ S \circ (R(a) \cap B(a) \cap L(a)]) \\
& \qquad \circ (R(a) \cap B(a) \cap L(a)] \circ S] \\
& \subseteq (L(a) \circ B(a) \circ R(a)] .
\end{aligned}$$

This completes the proof. □

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