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Pureza de un qubit en términos de la transformada discreta de Wigner

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Abstract

An explanation and an illustration of the meaning of a discrete phase-space is given. The class of a discrete Wigner transform (DWT) for the specific case of a one-qubit state is introduced. We derive the one-qubit state formalism around its formulation in terms of the DWT in detail. A novel structure of a one-qubit purity in terms of the DWT is introduced. We find a criterion for stating when a one-qubit state is either *mixed* or *pure*.

Keywords: Discrete phase space, Hilbert space, unbiased bases, discrete Wigner transform, purity.

Resumen

Se proporciona e ilustra una explicación del significado de espacio fase discreto dirigida a un lector no especialista. Se presenta también la clase de la transformada discreta de Wigner (TDW) para el caso específico de un estado de un *qubit*. Asimismo, derivamos detalladamente el formalismo involucrado en la formulación del estado de un *qubit* en términos de la TDW. En este contexto, se introduce una estructura novedosa de la pureza de un *qubit* en términos de la TDW y se halla un criterio para decidir cuando el estado de un *qubit* es puro o mixto.

Palabras clave: espacio fase discreto, espacio de Hilbert, bases imparciales, transformada discreta de Wigner, pureza.

Introduction

The real-valued Wigner function $W(q,p)$ play the role of a quasi-probability distribution for continuous-variable quantum systems in continuous coordinates (q) versus momentum (p) (phase) space (Wigner, 1932; Hillary *et al*., 1984). In spite of the fact that $W(q,p)$ allows us to calculate properties of a system through phase-space integrals weighted by it, however this cannot be interpreted as the positive-valued probability of simultaneously measuring observables $\hat{p}$ and $\hat{q}$ with eigenvalues p_0 and q_0. In fact, $W(q,p)$ could be negative in some phase-space regions (from there the term quasi-probability).

Buot (1974), Hannay and Berry (1980) were the first to propose the novel idea of the analogous Wigner function for a discrete (finite-dimensional) Hilbert space. Later on, such findings were rediscovered by Cohen and Scully (1986) and Feynman (1987) who defined a discrete Wigner function W for a single qubit. The above works were extended by Wootters (1987) and Galetti and De Toledo Piza (1988) by introducing a Wigner function for prime-dimensional Hilbert spaces.

These extensions have been employed for teleportation protocols (Koniorczyk *et al*., 2001; Paz, 2002), quantum algorithms (Bianucci *et al*., 2002; Miquel *et al*., 2002), and decoherence (Lopez, 2003).

There is scarce (almost null) information in the literature about the concept of purity of a single qubit from the point of view of the discrete Wigner transform. Within such a formalism it is difficult to find a criteria for stating whether a qubit state is pure. In the present paper we discuss both the discrete Wigner function for a single qubit and introduce the concept of purity for it.

1. One-qubit in terms of a discrete Wigner function

The conventional definition of continuous phase space is a plane where the horizontal axis denotes the continuous position coordinate q while the vertical axis represents the continuous linear momenta variable p. At this stage it arises the following question: what does a discrete phase space mean? By discrete phase space (DPS) we mean that the coordinate variable q takes a finite set of real values $Q = \{q_1, q_2,....,q_n\}$ while the momenta variable takes also a finite set of real values $P = \{p_1, p_2,....,p_n\}$ in such a way that the DPS is the set $\{(q_i, p_j)|q_i \in Q, p_j \in P\}$. We can state that the discrete analogue of phase space in a d–dimensional Hilbert space is a $d \times d$ real array. Thus, for the case of one qubit one has $d = 2$. Consequently a 2–dimensional Hilbert discrete phase space can be represented by the following four real numbers denoted by:

$$\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{matrix} \tag{1}$$

In the continuous phase space (continuous XY –plane) we can draw continuous lines while in the discrete case this is not the case. We define a "line" in the d–dimensional discrete phase space as a set of d points in discrete phase space. Thus, a "line" in the case of a 2–dimensional discrete phase space is a set of 2 points in the discrete phase space of Eq. (1). The discrete phase space of Eq. (1) will contain the following six "lines"

$$\begin{aligned} \alpha_{1,1} &= \{x_{11}, x_{21}\}, \\ \alpha_{1,2} &= \{x_{11}, x_{22}\}, \\ \alpha_{1,3} &= \{x_{11}, x_{12}\}, \\ \\ \alpha_{2,1} &= \{x_{12}, x_{21}\}, \\ \alpha_{2,2} &= \{x_{12}, x_{22}\}, \\ \alpha_{2,3} &= \{x_{21}, x_{22}\}. \end{aligned} \tag{2}$$

The discrete phase space can then be partitioned into a collection of parallel lines. By a parallel line it is understood disjoint sets of $d = 2$ phase space points. Such partitions are called striations (Wootters, 1987). According to Eq. (2), in the present case there will be the following $d + 1 = 2 + 1 = 3$ striations

$$\begin{aligned} S_1 &= \{\alpha_{1,1}, \alpha_{2,2}\}, \\ S_2 &= \{\alpha_{1,2}, \alpha_{2,1}\}, \\ S_3 &= \{\alpha_{1,3}, \alpha_{2,3}\}. \end{aligned} \tag{3}$$

Wootters' definition of discrete Wigner functions employs a special set of $d + 1 = 2 + 1 = 3$ bases for a d–dimensional Hilbert space. Such bases can be defined in terms of eigen-states of generalized Pauli operators. In this way, for the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the respective eigen-states should be:

$$\begin{aligned} |e_{x1}\rangle &= \frac{1}{\sqrt{2}} \ (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ |e_{x2}\rangle &= \frac{1}{\sqrt{2}} \ (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned} \tag{4}$$

where we have employed the matrix notation $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. In the above equation by eigenstates of the Pauli operator σ_x we mean the following $\sigma_x|e_{x1}\rangle = |e_{x1}\rangle$ and $\sigma_x|e_{x2}\rangle = -|e_{x2}\rangle$ where we have used that $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$.

For the Pauli matrix $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ the eigen-states are:

$$|e_{y1}\rangle = \frac{1}{\sqrt{2}}\,(i|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix},$$

$$|e_{y2}\rangle = \frac{1}{\sqrt{2}}\,(-i\,|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix}, \tag{5}$$

Clearly, $\sigma_y|e_{y1}\rangle = -|e_{y1}\rangle$ and $\sigma_y|e_{y2}\rangle = |e_{y2}\rangle$.

While for $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ the respective eigen-states will be:

$$\begin{aligned}|e_{z1}\rangle &= |0\rangle,\\ |e_{z2}\rangle &= |1\rangle.\end{aligned} \tag{6}$$

Clearly $\sigma_z|e_{z1}\rangle = |e_{z1}\rangle$ and $\sigma_z|e_{z2}\rangle = -|e_{z2}\rangle$.

A set $B = \{|e_1\rangle, |e_2\rangle\}$ is a basis for a Hilbert space if any state $|\varphi\rangle$ of the space can be written as $|\varphi\rangle = a|e_1\rangle + b|e_2\rangle$ where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$. The basis B is orthonormal if $\langle e_i|e_j\rangle = \delta_{ij}$ where $\delta_{ij} = \begin{cases}0 \text{ } if\ i \neq j\\ 1 \text{ } if\ i = j\end{cases}$. Let us observe that $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

The states given by (5), (6), (7) define the following $d + 1 = 2 + 1 = 3$ different bases

$$\begin{aligned}B_x &= \{|e_{x1}\rangle, |e_{x2}\rangle\},\\ B_y &= \{|e_{y1}\rangle, |e_{y2}\rangle\},\\ B_z &= \{|e_{z1}\rangle, |e_{z2}\rangle\}.\end{aligned} \tag{7}$$

We must observe that there is a one-to-one correspondence between the striations of Eq. (3) and the bases of Eq. (7), that is:

$$\begin{aligned}S_1 &\to B_x,\\ S_2 &\to B_y,\\ S_3 &\to B_z.\end{aligned} \tag{8}$$

On the other hand, the bases of Eq. (7) are *unbiased*, that is

$$|\langle e_{i,\,j}\mid e_{k,\,l}\rangle|^2 = \frac{1}{d} = \frac{1}{2} \qquad i \neq k \tag{9}$$

2. Discrete Wigner functions

We have the necessary ingredients to define a class of discrete Wigner functions. On one hand, there is a $d + 1 = 2 + 1 = 3$ mutually unbiased bases $\{B_x, B_y, B_z\}$ and on the other a set of $d + 1 = 2 + 1 = 3$ striations $\{S_1, S_2, S_3\}$ of the $d \times d = 2 \times 2 = 4$ phase space into $d = 2$ parallel lines. We need to choose a one-to-one maps as follows:

(I) Each basis set B_i is associated with one striation S_i.
(II) Each basis vector $|e_{i,\, j}\rangle$ is associated with a line $\alpha_{i,\, j}$ (the jth line of the ith striation).

With the above association, the Wigner function W is uniquely defined if we demand that

$$Tr(|e_{i,\, j}\rangle\langle e_{i,\, j}|\rho) = \sum_{e\in\propto_{i,j}} W_e, \tag{10}$$

where $|e_{i,\, j}\rangle$ are the basis states of Eqs. (5)-(7) and ρ is the one-qubit density state (Nielsen, & Chuang, 2000). Let us observe that Eq. (10) gives rise to the following Wigner functions associated to the four real numbers of Eq. (1)

$$Tr(|e_{x,1}\rangle\langle e_{x,1}|\rho) = \sum_{e\in\propto_{1,1}} W_e = W(x_{11}) + W(x_{21}) = p_{1,1},$$

$$Tr(|e_{x,2}\rangle\langle e_{x,2}|\rho) = \sum_{e\in\propto_{1,2}} W_e = W(x_{11}) + W(x_{22}) = p_{1,2},$$

$$Tr(|e_{y,1}\rangle\langle e_{y,1}|\rho) = \sum_{e\in\propto_{1,3}} W_e = W(x_{11}) + W(x_{12}) = p_{2,1},$$

$$Tr(|e_{y,2}\rangle\langle e_{y,2}|\rho) = \sum_{e\in\propto_{2,1}} W_e = W(x_{12}) + W(x_{21}) = p_{2,2},$$

$$Tr(|e_{z,1}\rangle\langle e_{z,1}|\rho) = \sum_{e\in\propto_{2,2}} W_e = W(x_{12}) + W(x_{22}) = p_{3,1},$$

$$Tr(|e_{z,2}\rangle\langle e_{z,2}|\rho) = \sum_{e\in\propto_{2,3}} W_e = W(x_{21}) + W(x_{22}) = p_{3,2}. \tag{11}$$

where the probabilities satisfy:

$$\sum_{i,j} p_{i,j} = 1 \tag{12}$$

Eq. (11) define uniquely the Wigner function W in terms of the probabilities:

$$\begin{aligned}
W(x_{11}) &= \frac{1}{2}\,(p_{1,1} + p_{2,1} + p_{3,1} - 1),\\
W(x_{12}) &= \frac{1}{2}\,(p_{1,1} + p_{2,2} + p_{3,2} - 1),\\
W(x_{21}) &= \frac{1}{2}\,(p_{1,2} + p_{2,1} + p_{3,2} - 1),\\
W(x_{22}) &= \frac{1}{2}\,(p_{1,2} + p_{2,2} + p_{3,1} - 1).
\end{aligned} \tag{13}$$

3. Wigner formulation of one-qubit purity

In the literature is scarce the information on the one-qubit purity. We then propose the following formulation of the purity in terms of the Wigner function:

$$\pi_{i,j} = Tr(|e_{i,j}\rangle\langle e_{i,j}|\rho^2) = \sum_{e\in\propto_{i,j}} c_e\, W_e, \tag{14}$$

where $0 \leq c_e \leq 1$ are real numbers. Eq. (14) can be expanded in terms of the following six equations as follows

$$Tr(|e_{x,1}\rangle\langle e_{x,1}|\rho^2|) = \sum_{e\in\propto_{1,1}} c_e W_e = c_{11}W(x_{11}) + c_{21}W(x_{21}) = \pi_{1,1},$$

$$Tr(|e_{x,2}\rangle\langle e_{x,2}|\rho^2) = \sum_{e\in\propto_{1,2}} c_e W_e = c_{11}W(x_{11}) + c_{22}W(x_{22}) = \pi_{1,2},$$

$$Tr(|e_{y,1}\rangle\langle e_{y,1}|\rho^2) = \sum_{e\in\propto_{1,3}} c_e W_e = c_{11}W(x_{11}) + c_{12}W(x_{12}) = \pi_{2,1},$$

$$Tr(|e_{y,2}\rangle\langle e_{y,2}|\rho^2) = \sum_{e\in\propto_{2,1}} c_e W_e = c_{12}W(x_{12}) + c_{21}W(x_{21}) = \pi_{2,2},$$

$$Tr(|e_{z,1}\rangle\langle e_{z,1}|\rho^2) = \sum_{e\in\propto_{2,2}} c_e W_e = c_{12}W(x_{12}) + c_{22}W(x_{22}) = \pi_{3,1},$$

$$Tr(|e_{z,2}\rangle\langle e_{z,2}|\rho^2) = \sum_{e\in\propto_{2,3}} c_e W_e = c_{21}W(x_{21}) + c_{22}W(x_{22}) = \pi_{3,2}. \quad (15)$$

We say that one qubit is in a *mixed state* if

$$\sum_{i,j} \pi_{i,j} < 1. \quad (16)$$

On the other hand, one qubit is in a *pure state* if

$$\sum_{i,j} \pi_{i,j} = 1. \quad (17)$$

Conclusions

We have formulated the one-qubit state in terms of the Wigner transform operating on a discrete phase-space. In classical approaches the phase-space is usually understood as the composition of both the continuous spatial coordinates $\{X\}$ and the continuous momentum space $\{P\}$, that is, a $\{X, P\}$ continuous generalized coordinates space. In the present approach we focus on two spatial coordinates $\{q_1, q_2\}$ versus two momenta coordinates $\{p_1, p_2\}$ is such a way that the two above sets generates the following four elements set $\{q_1 p_1, q_1 p_2, q_2 p_1, q_2 p_2\} \equiv \{x_{11}, x_{12}, x_{21}, x_{22}\}$ defining the discrete phase-space of Eq. (1). An example of a one possible phase-space is the following $\{-17.21, 8.3, -2.1, -0.17\}$.

The striations of Eq. (3) can be generalized for a prime d–dimensional phase-space. In particular, we have considered a one-qubit state where $d = 2$. Let us note that for the one-qutrit state one must have $d = 3$ Eq. (10) defines uniquely a one-qubit state in terms of the Wigner discrete transform W. Such a definition is equivalent to the conventional definition of a one-qubit state if one observe from Eq. (10) that $\rho = (a_0|0\rangle + a_1|1\rangle)(a_0^*\langle 0| + a_1^*\langle 1|) = \begin{pmatrix} a_0a_0^* & a_0\, a_1^* \\ a_1a_0^* & a_1\, a_1^* \end{pmatrix}$ where $Tr\,\rho = |a_0|^2 + |a_1|^2 = 1$. On the other hand, the main achievement of the present work is the formulation of the *purity* of a one-qubit state in terms of the Wigner discrete transform W through Eq. (14). It is worth to mention that such a formulation is absent in the literature. With Eqs. (14) and (17) we can state a criteria for concluding when a one-qubit is in a *pure state*.

Prospective analysis

The formulation of a one-qubit state in terms of W is elegant and allows to geometrize the probabilities $p(_{i,j})$ of Eq. (11). Indeed, the constrain of Eq. (12) on the probabilities $p_{i,j}$ implies that they can be represented in a Bloch sphere.

On the other hand, the understanding of the formulation of the *purity* of a one-qubit state in terms of the discrete Wigner transform as stated from Eqs. (15)-(17) could help in the future to understand intriguing properties of qubits such as the relation between W and the speed of quantum information processing.

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