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STIRLING'S FORMULA VIA THE CHI-SQUARE DISTRIBUTION

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RESUMEN: Este artículo muestra una demostración de la fórmula de Stirling a partir de la distribución chi-cuadrado y del teorema central del límite. El propósito de este trabajo es presentar una demostración sencilla y asequible de dicha fórmula a partir de conceptos estadísticos ampliamente conocidos, sin necesidad de alargar dicha demostración de forma innecesaria.

Palabras Clave: Teorema central del límite, aproximación factorial, función Gamma, distribución normal.

ABSTRACT: In this article Stirling's formula is proved using the chi-square distribution and the central limit theorem. The purpose of this text is to present a short and simple demonstration of Stirling's formula from well-known probabilistic facts avoiding a long and tedious demonstration from purely mathematical arguments.

Keywords: Central limit theorem, factorial approximation, Gamma function, normal distribution.

1. Introduction

Stirling (1730)[1] was one of the first to demonstrate the famous formula that takes his name given in (1), although its demonstration is somewhat technical (using the formula of Euler-MacLaurin). Subsequently, many easier and more intuitive demonstrations were given by several authors. Moritz (1928)[2] used the Wallis' formula to obtain this result. Diaconis and Freedman (1986)[3] proof it with basic results about gamma function and a little of real analysis. Hu (1988)[4] used a sequence of iid random variables Poisson with parameter 1 and the Central Limit Theorem to proof Stirling's formula. Recently Pinsky (2007)[5] presented a demonstration of the Stirling's formula from the characteristic function of the Poisson distribution and properties of complex functions.

$$\lim_{n\to\infty} \sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n) = \lim_{n\to\infty} n! \tag{1}$$

The rationale for yet another demonstration of Stirling's formula is that the demonstrations presented so far still require good mathematical knowledge. Therefore, we will present a demonstration accessible to students who are new entrants to statistics courses that only require basic knowledge of probability and statistics. More specifically, knowledge about the mean and variance of the chi-square and normal distributions and the central limit theorem.

The purpose of Stirling's formula is to replace a "complicated" function, the factorial, with some expression which is "simpler". So you might object that $\sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n)$ is simpler than $n!$. But if I ask you the question whether e^n or n! grows faster when $n \to \infty$ you might appreciate Stirling's result. Or try to answer the question, how many digits $n!$ has when n is large. Also Stirling's formula happens to work just as well in the case when n is not integer, i.e. for computing the gamma function.

The Table 1 compares some factorials with values calculated using Stirling's Approximation. From the Table we can notice that as the value of n increases, the relative error decreases. For n greater than 10 the relative error is smaller than 1%. We have stated that as n tends to infinity, the formula tends to $n!$ However, even with small values of n, the approximations are quite close.

Table 1. Comparing Stirling's formula with the factorials.

n	Factorial	Stirling's formula	Relative Error
1	1	$9.2213 \cdot 10^{-1}$	0.0778
5	120	$1.1801 \cdot 10^{2}$	0.0165
10	$3.6288 \cdot 10^{6}$	$3.5986 \cdot 10^{6}$	0.0082
20	$2.4329 \cdot 10^{18}$	$2.4227 \cdot 10^{18}$	0.0041
30	$2.6525 \cdot 10^{32}$	$2.6451 \cdot 10^{32}$	0.0027
40	$8.1591 \cdot 10^{47}$	$8.1421 \cdot 10^{47}$	0.0020
50	$3.0414 \cdot 10^{64}$	$3.0363 \cdot 10^{64}$	0.0016
100	$9.3326 \cdot 10^{157}$	$9.3248 \cdot 10^{157}$	0.0008

2. Proof of Stirling's Formula

The proof was essentially based on Hu (1988)[4] and Diaconis and Freedman (1986)[3]. The chi-square and the normal density functions are given, respectively, by 2 and 3.

$$f(x) = x^{\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right), \ x, n > 0. \tag{2}$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x, \mu \in \mathbb{R} \text{ and } \sigma^2 \in \mathbb{R}^+. \tag{3}$$

Let $X_1, X_2, \ldots$, be a sequence of iid random variables with distribution χ_1^2, then defining

$$X_{2n} = \sum_{i=1}^{2n} X_i \tag{4}$$

and using the fact that sum of iid χ^2 has χ^2 distribution, $X_{2n} \sim \chi_{2n}^2$ and

$$\mathbb{E}[X_{2n}] = 2n \quad \text{and} \quad \operatorname{Var}(X_{2n}) = 4n. \tag{5}$$

Now using the Central Limit Theorem, see Inlow (2010)[6] for a accessible proof, for large n, $X_{2n} \approx N(2n, 4n)$, so

$$f(x) = \frac{x^{\frac{2n}{2}-1} \exp\left(-\frac{x}{2}\right)}{\Gamma(n) 2^{\frac{2n}{2}}} \approx \frac{1}{\sqrt{2\pi}\sqrt{4n}} \exp\left(-\frac{(x-2n)^2}{2*4n}\right), \tag{6}$$

when n is large.

Substituing $x = 2n$ in (6) and simplifying,

$$\frac{(2n)^{\frac{2n}{2}-1} \exp\left(-\frac{2n}{2}\right)}{\Gamma(n) 2^{\frac{2n}{2}}} \approx \frac{1}{\sqrt{2\pi}\sqrt{4n}} \exp\left(-\frac{(2n-2n)^2}{2*4n}\right) \tag{7}$$

$$\frac{(2n)^{n-1} \exp(-n)}{\Gamma(n) 2^n} \approx \frac{1}{\sqrt{2\pi}\sqrt{4n}}. \tag{8}$$

Isolating $\Gamma(n)$,

$$\Gamma(n) \approx \sqrt{2\pi}\sqrt{4n} \frac{(2n)^{n-1} \exp(-n)}{2^n} = \sqrt{2\pi}\sqrt{n} n^{n-1} \exp(-n). \tag{9}$$

Using the fact that $\Gamma(n+1) = n\Gamma(n)$ we have $\Gamma(n) = \Gamma(n+1)/n$, then

$$\Gamma(n+1) \approx n\sqrt{2\pi} n^{n-\frac{1}{2}} \exp(-n) = \sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n) \tag{10}$$

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n), \tag{11}$$

finally we have

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n)}{n!} = 1 \tag{12}$$

by central limit theorem and Stirling's formula is proved because the expression obtained is equivalent to (1).

3. Concluding remark

The probabilistic proof presented is just one example of how statistics and pure mathematics are strongly connected. Then in the future real problems from pure mathematics can be solved by rigorous probabilistic proofs, this point of view is very important because a problem can often be seen in several different ways and some ways are easier than others. Moreover, theoretical problems (such as the demonstration of Stirling's formula) favor the development of statistical reasoning, since it requires the application of statistical results in unusual situations.

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