VALUATION OF FINANCIAL ASSETS USING MONTE CARLO: WHEN THE WORLD IS NOT SO NORMAL

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RESUMEN

Valorar activos financieros cuando el mundo no es tan normal como asumen muchos modelos financieros exige un método flexible para operar con diversas distribuciones, el cual, adicionalmente, pueda incorporar discontinuidades como las que se dan en procesos estocásticos de salto. El método Monte Carlo cumple con esos requisitos, además de generar una buena aproximación y ser eficiente, lo cual lo convierte en el más adecuado para aquellos casos en los cuales no se cumple el supuesto de normalidad. Este artículo explica como se aplica este método para la valoración de activos financieros, particularmente opciones financieras, cuando el activo subyacente sigue un proceso de volatilidad estocástica o de salto-difusión.

Palabras Clave: Método Monte Carlo, Valoración de opciones financieras, Procesos Estocásticos, Proceso de Salto-Difusión, Volatilidad Estocástica.
Clasificación JEL: C15, G12.

ABSTRACT

Valuing financial assets when the world is not as normal as assumed by many financial models requires a method flexible enough to function with different distributions which, at the same time, can incorporate discontinuities such as those that arise from jump processes. The Monte Carlo method fulfills all these requirements, in addition to being accurate and efficient, which makes this numerical method the most suitable one in those cases that do not conform to normality. This paper applies Monte Carlo to the valuation of financial assets, specifically financial options, when the underlying asset follows stochastic volatility or jump-diffusion processes.

Keywords: Monte Carlo Method, Financial Option Pricing, Stochastic Process, Jump-Diffusion, Stochastic Volatility.
JEL Classification: C15, G12.

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I. INTRODUCTION

Most financial models assume that financial prices follow a lognormal distribution, therefore, the percentage changes in prices –returns- follow a normal one. That is the case with the well known models of the Portfolio Selection by Markowitz (1959), the Capital Asset Pricing Model (CAPM) by Lintner (1965) and Sharpe (1964) and the Black-Scholes (1973) model for option pricing.

If returns really follow a normal distribution, we would only require information about the first two moments of the distribution, the mean and the variance, to properly price these assets and make our investment decisions. However, actual distributions of financial returns exhibit skewness and kurtosis, that is, there is more weight in the tails than is predicted by the normal distribution.

It does not matter if the series of returns correspond to developed or emerging economies. Neither does it matter what kind of asset returns is being studied, i.e. stocks, foreign exchange or commodities. There is, of course, a difference among these assets but it is only in the sense that more liquid markets exhibit a behavior close to normality, yet none of them can be described as a normal one. Possible explanations for non-normality are that assets follow some other stochastic processes such as jump-diffusion or stochastic volatility. Those can explain the existence of fatter tails in the distribution.

Valuing financial assets when the world is not so normal requires a method flexible enough to function with different distributions -even empirical distributions of the underlying variables- (Boyle, 1977), a method able to incorporate discontinuities such as those that arise from jump processes. The Monte Carlo method fulfills all these requirements in addition to being accurate and efficient which makes it the most suitable one in those cases that do not conform to normality.

Proving the previous conclusion is one of the aims of this paper after applying Monte Carlo to the valuation of financial assets, specifically financial options, when the underlying asset follows stochastic volatility and jump-diffusion processes. The other objective of this paper is to show how the use of common methods, v.gr. Black-Scholes for option pricing, trying to circumvent the use of the true distribution of the asset, leads to a wrong valuation.

II. OPTION VALUATION. THE CASE OF STOCHASTIC VOLATILITY

Returns on assets frequently have distributions different than normal. They present fatter tails and higher peaks than predicted by a Normal distribution (Wilmott, 1998). A possible explanation for this non-normality is the fact that volatility is not constant as assumed by the Black-Scholes model. Now, if volatility is not constant, what kind of process does it follow? It is commonly assumed –according to Wilmott (1995)- that asset prices and their variance follow these stochastic processes:

$$dS = \alpha S dt + \sigma S dW$$
$$dV = \mu V dt + \zeta V dZ \quad (1)$$

α and μ are the drift rates for *S* and *V*, respectively. σ and ζ represent the volatility of *S* and *V*. α may depend on *S*, σ, and t. μ and ζ may depend on σ and t, but it is assumed that they do not depend on *S*. *dW* and *dZ* are Wiener processes with correlation ρ between them.

The most commonly used methodology to value options with stochastic volatility is the one proposed by Hull and White (1987). In order to get an analytical solution, they make some simplifying assumptions such as:

- It is possible to use risk neutral valuation since neither the partial differential equation nor the boundary conditions depend on risk preferences. *V* is assumed to be uncorrelated with consumption.
- $\rho = 0$
- The value of the option is the risk-neutral expected terminal value of $Max\,[S_T - X, 0]$ discounted at the risk free rate. The distribution of S_T depends on both processes for *S* and *V*. The expectation is taken conditional on

$$\overline{V} = \frac{1}{\tau}\int_t^T \sigma_u^2 du \quad (2)$$

which is the average realized variance. If, additionally, $\rho = 0$, then

$$\ln(S_T / S_t \,|\, \overline{V}) \sim N\left(r\tau - \frac{\overline{V}\tau}{2}, \overline{V}\tau\right) \quad (3)$$

"If *V* is stochastic, there are an infinite number of paths that give the same mean variance, but all of these paths generate the same terminal distribution of stock price. From this, we may conclude that, even if *V* is stochastic, the terminal distribution of the stock price given the mean variance is lognormal" (Hull and White, 1987, p.285).

The option value that is obtained following the Hull and White method is the same Black-Scholes value conditional on the average realized variance. However, if *S* and *V* are correlated - $\rho \neq 0$ -, S_T does not follow a lognormal distribution, even conditional on the mean variance. It is in this case when numerical methods such as Monte Carlo should be used to assess the value of the option.

III. MONTE CARLO OPTION VALUATION WHEN THE UNDERLYING ASSET FOLLOWS A STOCHASTIC VOLATILITY PROCESS.

First of all, it is necessary to define the drift rate for the stochastic processes of both, *S* and *V*. According to the reasons argued above, risk neutrality may be assumed, thus the drift rate for the underlying asset's returns is the risk-free rate, *r*. For the variance's drift, "a mean reverting process appears to be a natural choice" (Wilmott, 1998) meaning that short-term deviations will tend to converge to the long-run level of the variance σ^* at a rate *a*. Additionally, since the volatility risk may not be perfectly hedged, the market should price it by demanding a risk premium Ψv. Then, the drift for the variance *V* is

$$\mu = a(\sigma^* - \sigma) - \Psi_v \tag{4}$$

where,
a is the rate of convergence of volatility to its long-run level,
σ^* is the long-run level of volatility,
σ is the short-run level of volatility, and
Ψv is the risk premium or price for volatility risk.

Having defined the processes for *S* and *V*, next I can generate *n* paths for the underlying asset price and variance using the following algorithm:

$$\begin{aligned} S_t &= S_{t-1} * e^{(r - \frac{1}{2}V_{t-1})dt + \varepsilon_t \sqrt{V_{t-1}dt}} \\ V_t &= V_{t-1} * e^{(\mu - \frac{1}{2}\zeta^2)dt + \zeta e_t \sqrt{dt}} \\ e_t &= \rho\varepsilon_t + \sqrt{1-\rho^2} * u_t \end{aligned} \tag{5}$$

ε_t, u_t are both standard normal innovations, and *e* and ε have correlation ρ. After *n* paths have been simulated, the option value i.e. an European call, can be computed as

$$C_0 = e^{-rT} * E_0^*\left[Max(S_T - X, 0\right] \tag{6}$$

E* being the risk-neutral expectation.

In order to improve the accuracy of the estimation I apply simultaneously two variance reduction techniques, antithetic variate and control variate (Maya, 2004). In order to check the accuracy of the estimations, I compare my results with the ones given in Hull and White (1987) when ρ is zero, ζ is one, and *a* is zero as well. As it can be seen in the second and third columns of Table 1, all the values estimated using Monte Carlo are very close to the Series Solution obtained analytically by Hull and White (1987).

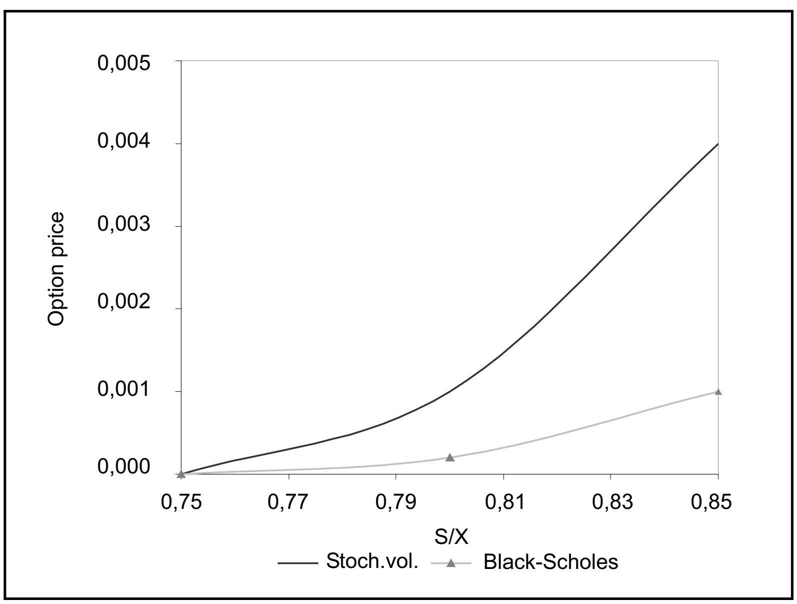
Table 1
Valuation of options with stochastic volatility
Comparison of Montecarlo Procedure and Series Sol.
Hull, White method (1987)
1000 trials, 90 time steps, sigmao=10%, T-t=180 days

Option Valuation with Stochastic Volatility
Different Time Span
1000 paths, 90 steps, r = 0%, long-run volatility = 10%; V-vol = 1

S/X	Monte Carlo Estimation*			Black-Scholes Option Value		
	1 month	3 months	6 months	1 month	3 months	6 months
0.80	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)			
0.90	0.0000	0.0004	0.0022	0.0000	0.0003	0.0020
	(0.0000)	(0.0000)	(0.0001)			
1.00	0.0114	0.0197	0.0277	0.0115	0.0199	0.0282
	(0.0000)	(0.0001)	(0.0001)			
1.10	0.1000	0.1007	0.1031	0.1000	0.1006	0.1030
	(0.0000)	(0.0001)	(0.0001)			
1.20	0.2000	0.2000	0.2002	0.2000	0.2000	0.2001
	(0.0000)	(0.0001)	(0.0001)			

The last column of Table 1 shows option prices obtained using Monte Carlo to approximate Black-Scholes, that is, keeping volatility constant. Compared to the Monte Carlo estimation when volatility is stochastic, Black-Scholes underestimates the price for out-of-the-money and deep in-the-money options and overestimates the price for at-the-money options as it is shown in Figures 1 – 3.

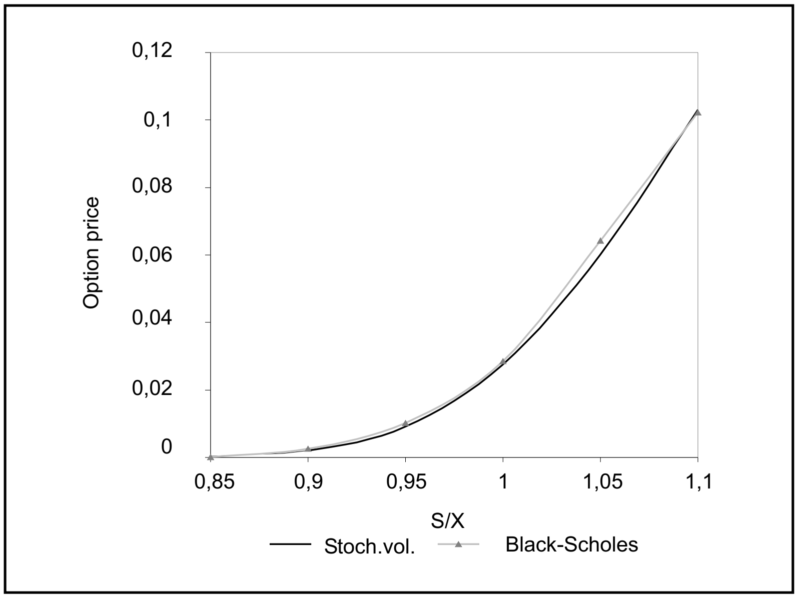
Figure 1
Out-of-the-Money Option Values: Stochastic



Vol. vs. Black-Scholes

Figure 2

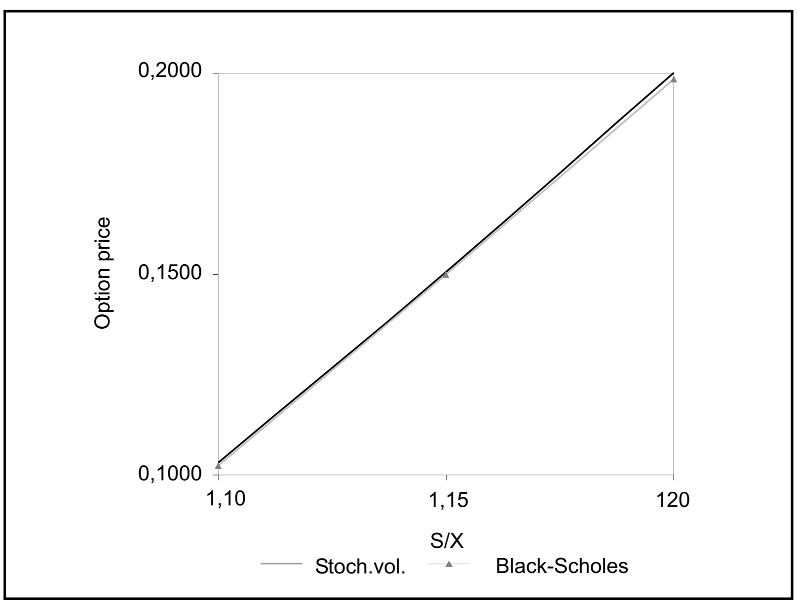
At-The-Money Option Values: Stochastic Vol. vs.



Black-Scholes

Figure 3

At-The-Money Option Values: Stochastic Vol. vs.



Black-Scholes

Since the variance and the asset price are independent, "the correct option price is the expected Black-Scholes price where the expectation is taken over the distribution of the mean variances" (Hull and White, 1987). If this value is compared to the Black-Scholes price with constant variance, i.e. E [$\bar{V}$], according to Jensen's inequality

$$C(.) convex \rightarrow E[C(\bar{V})] > C[E(\bar{V})] \quad (7)$$

Since *C* (.) is a convex function of the mean variance for high values of $\bar{V}$, that is, for "away from the money" options and a concave function for low values of $\bar{V}$ or "near the money" options, the result is that the constant volatility Black-Scholes underprices "away from the money" and "deep in the money" options and overprices "near the money" options. (Hull and White, 1987). Thus, there are cases when stochastic volatility can lower the option price compared with a non stochastic volatility. Although this result is a little surprising, Merton (1992) showed that if Black-Scholes is computed based on the expected variance for the case of a jump-diffusion process –the expectation taken over both jumps and continuous changes-, the price can be greater or less than the correct price.

An additional conclusion from the previous discussion is that $\delta C/\delta\zeta$ can be positive or negative. The greater the volatility of the variance, the greater the variance; however, as it was said before, a greater variance may increase or decrease the option price. Figure 4 shows the sensitivity of option prices to changes in the volatility of the variance, ζ, which based on estimations given by Hull and White (1987) has a value in the range of one to four. This figure shows how for out-of-the money and deep-in-the-money options, the greater the volatility, the higher the price of the option. For at-the-money options, prices move in the opposite direction.

On the other hand, a greater stock volatility translates into a higher option value, as expected. The difference among the prices for various volatility levels is larger when the option is at the money. Figure 5 shows these results.

Increasing time span also increases the value of the option. Table 2 shows the results of a set of simulations using Monte Carlo with stochastic volatility compared to the Black-Scholes option value which assumes constant volatility. The larger the time to expiration of the option the greater its value. More importantly, this table shows clearly how assuming a stochastic process different from the actual one leads to a wrong estimation, i.e. constant volatility, when this one is stochastic, undervalues options out and deep in the money, whereas it overvalues options in the money.

Figure 4
Option Values for Different Variance Volatitlity

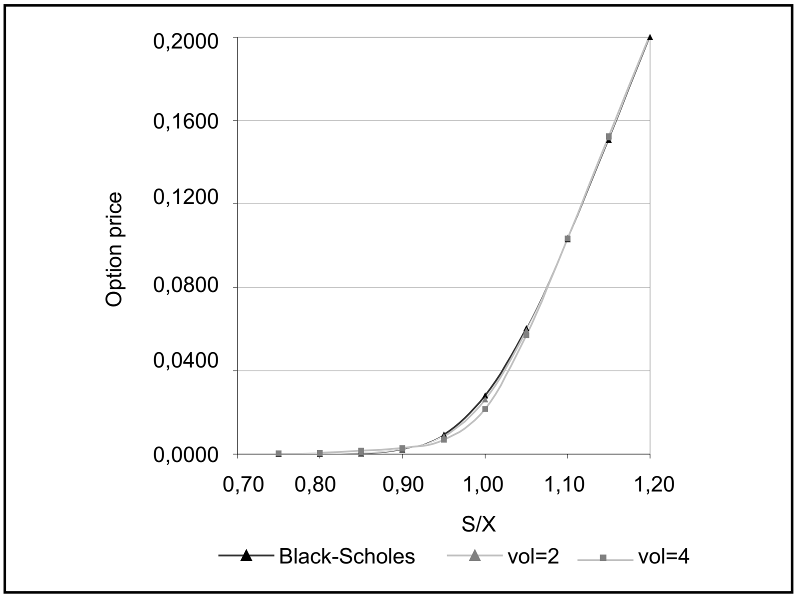


Figure 5
Option Values for Different Asset Price Volatility

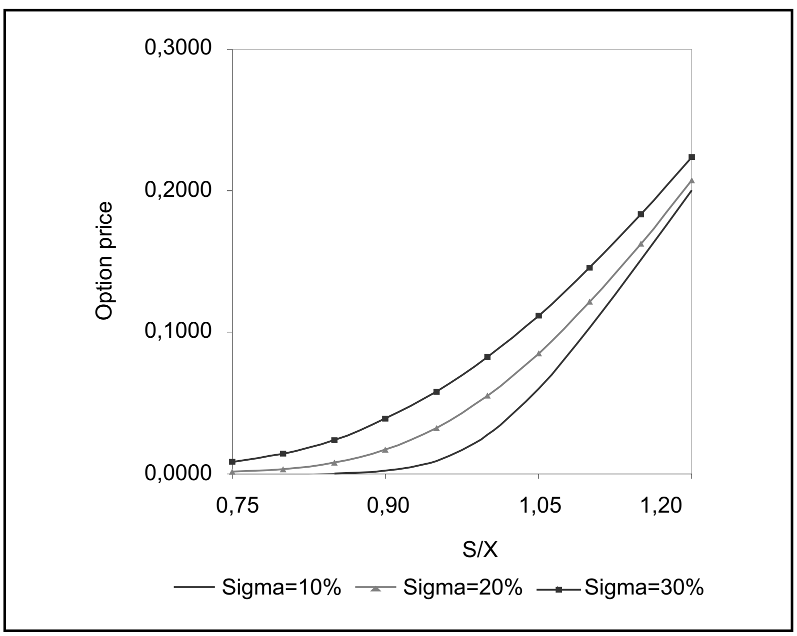


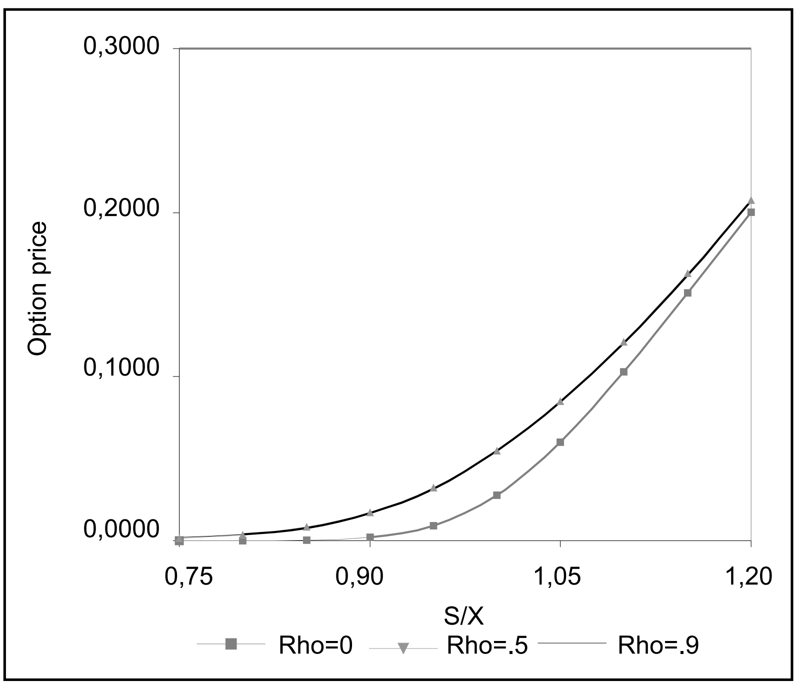
Table 2

Option Valuation with Stochastic Volatility Different Time Span 1000 paths, 90 steps, *r* = 0%, long-run volatility = 10%; V-vol = 1						
S/X	Monte Carlo Estimation*			Black-Scholes Option Value		
	1 month	3 months	6 months	1 month	3 months	6 months
0.80	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)			
0.90	0.0000	0.0004	0.0022	0.0000	0.0003	0.0020
	(0.0000)	(0.0000)	(0.0001)			
1.00	0.0114	0.0197	0.0277	0.0115	0.0199	0.0282
	(0.0000)	(0.0001)	(0.0001)			
1.10	0.1000	0.1007	0.1031	0.1000	0.1006	0.1030
	(0.0000)	(0.0001)	(0.0001)			
1.20	0.2000	0.2000	0.2002	0.2000	0.2000	0.2001
	(0.0000)	(0.0001)	(0.0001)			

* Estimation errors are in parenthesis

Lastly, simulations are run for different values of ρ, the correlation between the two Wiener processes, *dW* and *dZ*. For ρ = *0* the option price is lower than for a positive correlation. However, the differences between option values when correlation is positive are very small, as can be seen in Figure 6.

Figure 6
Option values for different correlations between
Wiener Processess dW-dZ



In conclusion, when volatility is stochastic, Black-Scholes underestimates the price of out-of-the-money and deep-in-the money options, and overestimates the price of at-the-money options. On the other hand, option values respond to changes in parameters such as time and asset price volatility in the same way they do when volatility is non-stochastic, that is, greater volatility and time span increase the price of the option. For changes in the volatility of the variance, option prices are higher the greater the volatility for out-of-the-money and deep-in-the-money options, and lower in the case of at-the money options. Finally, a higher correlation between the Wiener processes of stock prices and variance translate into higher option prices.

IV. OPTION VALUATION WITH JUMP-DIFFUSION PROCESSES

Another possible explanation for the fact that asset returns do not follow normal distributions –exhibiting fatter tails and higher peaks- is that assets may jump in value. If jumps happen, compared to a Normal distribution, there will be more weight in the tails farther than three standard deviations from the mean.[1] Now, portfolios are usually hedged against small movements, but large ones are difficult to hedge with the additional problem that they can affect the value of the portfolio greatly.

Modeling the possibility of jumps in asset prices requires the addition of a Poisson process to the diffusion process that is frequently assumed. Let the random variable n be the number of jumps that occur in a time interval dt. λdt is the expected number of jumps during this time interval. The cumulative probability of n events occurring in a period of time dt is given by a Poisson distribution:

$$f(n) = \frac{e^{-\lambda dt}(\lambda dt)^n}{n!} \tag{8}$$

For the case of small time intervals, those terms involving $(dt)^2$ and higher exponents can be ignored. Thus, in that case, we can say that the probability of zero events happening is (1- λdt), of one event is λdt, and of more than one event is zero (Marcus, 2001).

Being q, the Poisson process, the cumulative number of "jumps" in the asset price and based on the previous discussion, dq can be defined, approximately, as:

$$dq = \begin{cases} 0, \, probability(1-\lambda dt) \\ 1, \, probability(\lambda dt) \end{cases} \tag{9}$$

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[1] Some authors classify a jump as an event far five or more standard deviations from the mean (Mauboussin, 2002)

Thus, there is a probability λdt of a jump in S in a time-step dt. Call ξ the proportional change in the asset value when there is a jump, i.e. $\xi = (J-1)S$. Combining a Poisson and a Diffusion processes into a jump-diffusion one results in the following:

$$dS = \alpha S\, dt + \sigma S\, dW + \xi\, dq \tag{10}$$

where
$\alpha\ (S, t)$ is the continuous drift rate for S;
$\sigma\ (S, t)$ is the volatility of S;
dW is a Wiener process;
ξ is the proportional change in S conditional on a jump;
dq is a Poisson process;

The version of Ito's lemma for a jump-diffusion stochastic process is (Appendix A):

$$d\ ln\ S = (\alpha - ½\ \sigma^2)\ dt + \sigma\ dW + [ln\ J]dq \tag{11}$$

If, i.e. $\xi = (J-1)\ S$, when S jumps, it goes from S to SJ, J being a random variable itself, independent from S, with distribution $ln\ J \sim N(\mu,\ \sigma*)$.

Merton (1976) found an analytical solution for the jump-diffusion process when the risk of a jump is non-systematic and the only possible jump in the asset's value is to zero. In that case it will be necessary to hedge just the diffusion risk. The market should not price this non-systematic risk and the Black-Scholes formula can be used replacing r by $(r + \lambda)$. If such risk is systematic, there is no analytical solution and numerical methods should be used to approach the value of the option.

In this last case where risk is systematic, among the different numerical methods, Monte Carlo proves to be very useful thanks to its ability to function with different distributions -even empirical distributions of the underlying variables-. More importantly, this is a method able to incorporate discontinuities such as those that arise from jump processes, besides being accurate and efficient.

Next, I explain how a jump-diffusion process can be modeled using Monte Carlo. The final objective of this section is to show how the use of common methods, v.gr. Black-Scholes for option pricing, trying to circumvent the use of the true distribution of the asset, leads to a wrong valuation.

Firstly, n paths of asset prices are simulated as follows:

$$S_t = S_{t-1} * e^{(r-1/2\sigma^2)dt+\sigma\sqrt{dt}\varepsilon_t} * e^{(\mu-1/2\sigma^{*2}+\sigma^*\varepsilon_t^*)*Dummy} \tag{12}$$

where ε is a normally distributed random variable with a zero mean and unit variance. Dummy is one if a jump occurs and zero otherwise.

The annual rate of jumps is λ and per period is λΔt. Then, dummy is one when λΔt, the probability of a jump happening in a time period Δt, is greater than or equal to the value of a simulated uniformly distributed random number. If there is a jump, another normal random variable, ε*, is simulated, in order to compute

$$e^{(\mu - 1/2\sigma^{*2} + \sigma^{*}\varepsilon^{*})} \tag{13}$$

where μ and σ* are the mean and standard deviation of the jump, a process that is assumed to have a lognormal distribution. If dummy is zero, the process is just a diffusion process and only the following is computed:

$$S_t = S_{t-1} * e^{(r_f - 1/2\sigma^2)dt + \sigma\sqrt{dt}\varepsilon_t} \tag{14}$$

Figures 7 and 8 show prices following a diffusion process and a mixed jump-diffusion process, respectively.

Figure 7

Asset price simulation for a Diffusion process

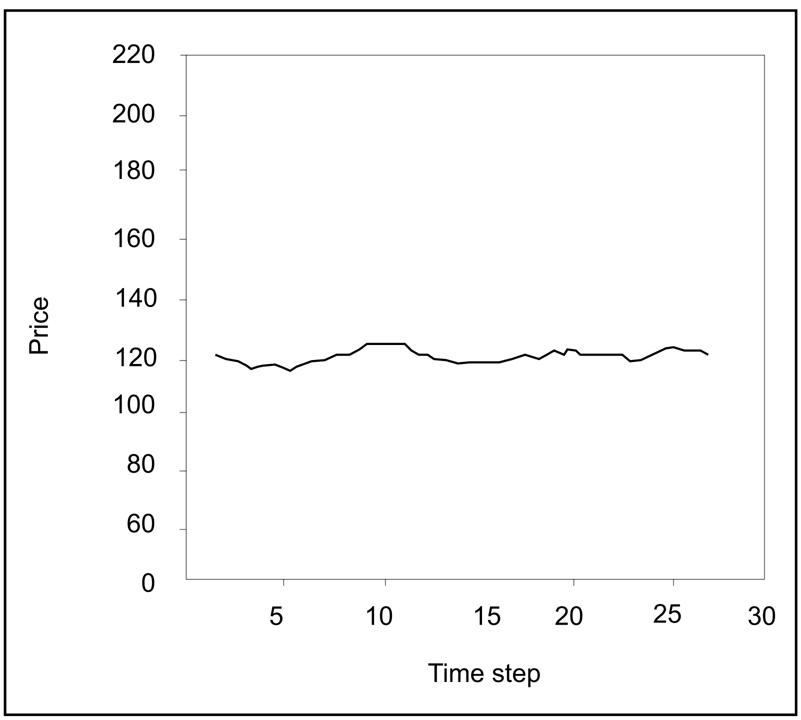
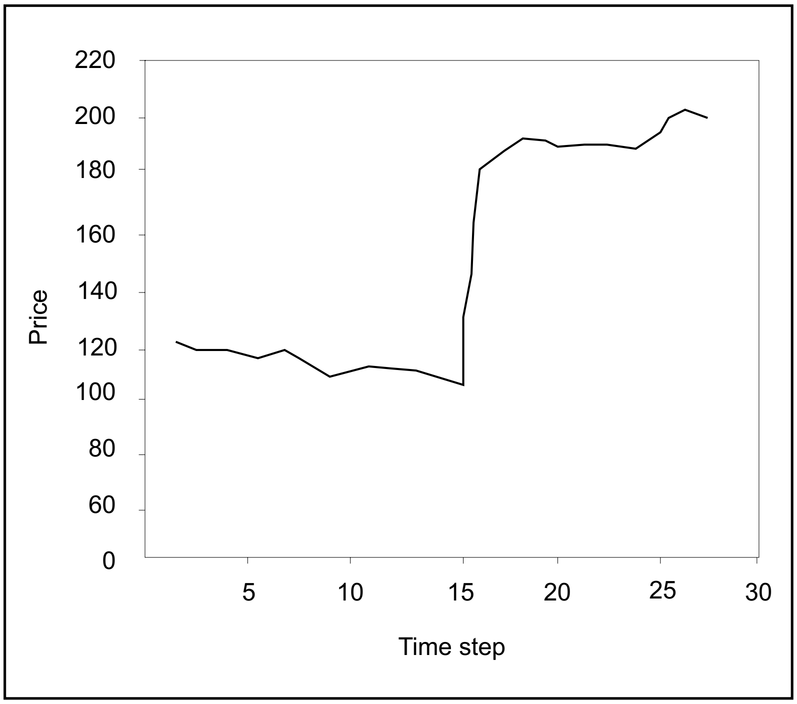


Figure 8

Asset price simulation for a Jump-diffusion process



If Δt is not small, the probability of one jump happening in the period of time Δt is:

$$f(1) = e^{-\lambda\Delta t}(\lambda\Delta t) \approx \lambda\Delta t - (\lambda\Delta t)^2 + \tfrac{1}{2}(\lambda\Delta t)^3 + \ldots \tag{15}$$

and the terms raised to the second or higher power will not fade away. Thus, instead of considering it fixed, λΔt should be simulated along with asset prices using the Poisson probability distribution function for *n* jumps, i.e. one.

If jumps in the asset price occur and Black-Scholes is used to price the option instead of assuming a jump-diffusion process, the price of out-of-the money and deep-in-the-money options is underestimated, and the price of at-the-money options is overestimated. This result is due to the fact that the probability of events on the right or left tail of the distribution is higher under a jump-diffusion process, something that is underestimated by Black-Scholes. Figures 9-11 show the comparison.

Figure 9

Black-Scholes vs. Jump-Diffusion: Out of the Money Options

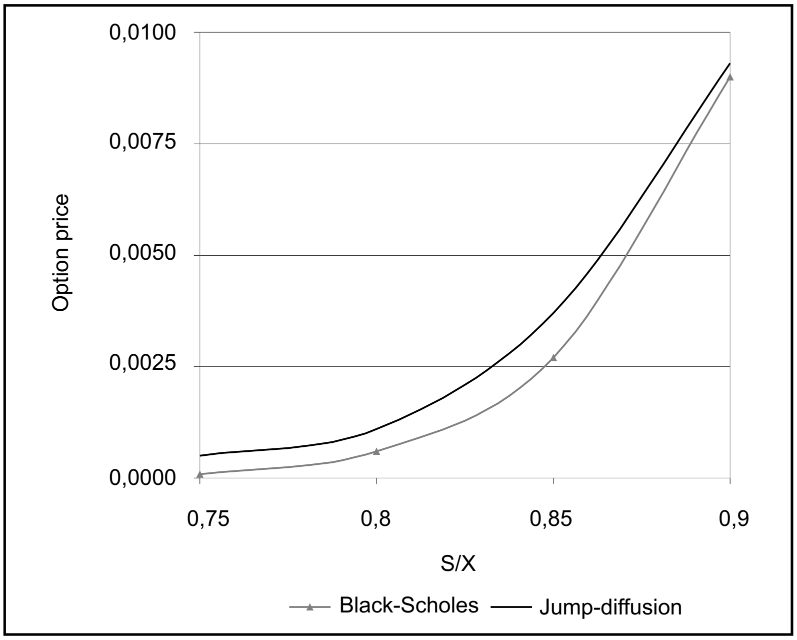


Figure 10

Black-Scholes and Jump-Diffusion: Deep-in-the-Money Options

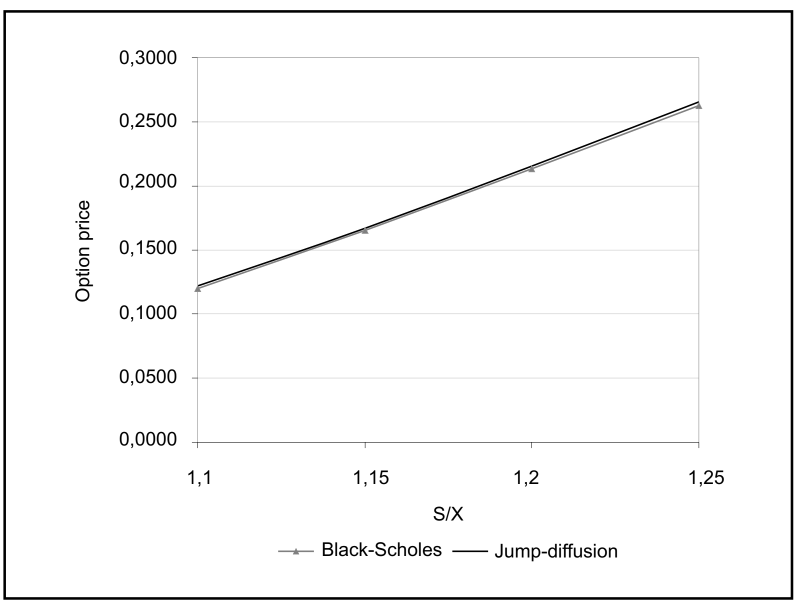
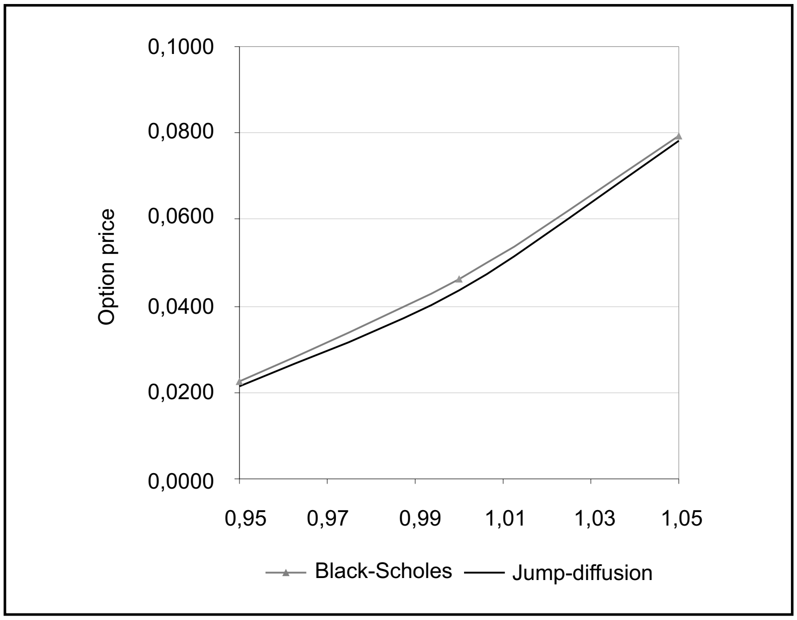


Figure 11

Black-Scholes and Jump-diffusion: At-the Money Options



In conclusion, the stochastic process followed by the underlying asset price should be identified properly before attempting to value an option. Particularly, in the case of an asset price that follows a jump or jump-diffusion process, assuming a simple diffusion process will lead to mispricing in every case.

Here, I presented the use of the Monte Carlo method to valuing financial options when the underlying asset price follows a stochastic volatility or a mixed jump-diffusion process. However, its use may be extended to the valuation of many other financial assets. In every case, a good approximation can be obtained as long as the stochastic process assumed to simulate the asset price paths is adequate to replicate the actual process the asset follows.

V. CONCLUSIONS

The Monte Carlo method offers a numerical approximation which is accurate enough compared to other numerical methods (Maya, 2004), besides having the appealing characteristics of being flexible to function with different distributions, even empirical distributions of the underlying variables. Additionally, it can incorporate discontinuities such as those that arise from jump processes. All these attributes make Monte Carlo the most suitable methodology to value financial assets when its price exhibits a behavior far from normality. This paper has shown how this method can be used to value financial assets in cases of stochastic volatility and jump-diffusion stochastic processes.

I also discuss the mispricing that would occur when the Black-Scholes method is used to value options instead of stochastic volatility or jump-diffusion. If volatility is stochastic, Black-Scholes underestimates the price of out-of-the-money and deep-in-the money options, and overestimates the price of at-the-money options. In the same way, if the underlying asset's price jumps and Black-Scholes is used instead, the price of out-of-the-money and deep-in-the-money assets is underestimated, and the price of at-the-money options is overestimated. Using Black-Scholes leads to mispricing in every case. As Cox and Ross (1976) said, determining the stochastic process of the underlying asset is essential to valuing options.

From the above I conclude that Monte Carlo is an adequate methodology to value options since it accurately approximates the value of the option and, at the same time, can properly handle option valuation when the world is not as normal as assumed by Black-Scholes. The use of the Monte Carlo method may be extended to the valuation of many other financial assets. In every case, a good approximation can be obtained as long as the stochastic process assumed to simulate the asset price paths is adequate to replicate the actual process the asset follows.

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APPENDIX 1

Ito's lemma for a jump-diffusion stochastic process.
If S, the asset price, follows a jump-diffusion process:

$$dS = \alpha S\, dt + sS\, dW + \xi\, dq \tag{1A}$$

where
$\alpha\ (S, t)$ is the continuous drift rate for S;
$\sigma\ (S, t)$ is the volatility of S;
dW is a Wiener process;
ξ is the proportional change in S conditional on a jump;
dq is a Poisson process;

And f is a function of S:

$$f = f(\mathrm{a}, \mathrm{s}, S, t, q)$$

According to Ito´s lemma:

$$df = fs\, dS + ft\, dt + ½\, fss\, (dS)^2 + [\,f\,(S+\xi) - f\,(S)]dq \tag{2A}$$

$(dS)^2 = \sigma^2(S, t)$, then

$$df = fs\ \mathrm{a}(S, t)dt + fs\ \sigma\ (S, t)\ dW + ft\ dt + ½\, fss\ \sigma^2(S, t) + [\,f\,(S+\xi) - f\,(S)]dq \tag{3A}$$

and

$$E\,[df] = fs\ \alpha(S, t)dt + ft\ dt + ½\, fss\ \sigma^2(S, t) + \lambda\,[\,f\,(S+\xi) - f\,(S)]dt \tag{4A}$$

since $E\,[dW] = 0$.

Also, $\mathrm{a}(S, t) = \alpha S;\ \sigma(S, t) = \sigma S;\ \xi = (J - 1)\,S;$

For instance, if $f = ln\ S$:

$$fs = 1 / S, ft = 0, fss = -1 / S^2,$$

then

$$d\ ln\ S = (\alpha - ½\, \sigma^2)\ dt + \sigma\ dW + [ln\ (SJ) - ln\ (S)]dq$$

$$d\ ln\ S = (\alpha - ½\, \sigma^2)\ dt + \sigma\ dW + [ln\ J]dq \tag{5A}$$