STABILITY AND MINIMUM LATERAL BRACING FOR STEPPED COLUMNS WITH SEMI-RIGID CONNECTIONS INCLUDING SHEAR EFFECTS: II) VERIFICATION AND EXAMPLES

ESTABILIDAD Y ARRIOSTRAMIENTO MÍNIMO LATERAL PARA COLUMNAS ESCALONADAS CON CONEXIONES SEMI-RÍGIDAS INCLUYENDO LOS EFECTOS DE CORTANTE: II) VERIFICACIÓN Y EJEMPLOS

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ABSTRACT: The elastic stability analysis of single-stepped columns with sidesway totally inhibited, partially inhibited, and uninhibited subjected to concentrated axial loads located at the ends and at the intermediate joint including semi-rigid connections and shear force effects are presented using three different approaches in a companion paper. The first two approaches are those by Engesser and Haringx that include the shear component of the applied axial force proportional to the total slope (dy/dx) and to the angle of rotation of the cross section (ψ) along the member, respectively. The third approach is a simplified formulation based on the classical Euler theory that includes the effects of shear deformations but neglects the shear component of the applied axial force along the member. Definite criterion on minimum stiffness of lateral bracings for single-stepped columns is also presented. Five comprehensive examples are included that demonstrate the effectiveness of the proposed stability equations and minimum bracing stiffness criteria.

KEY WORDS: bracing, buckling, building codes, columns, construction, computer applications, frames, loads, semi-rigid connections, shear deformations, shoring, stability, stepped columns

RESUMEN: El análisis de estabilidad elástica de columnas con un cambio de sección con derivas laterales totalmente inhibidas, parcialmente inhibidas, y desinhibidas sometidas a cargas axiales concentradas en los extremos y en el nudo intermedio incluyendo los efectos de las conexiones y de las fuerzas a cortante utilizando tres modelos diferentes es presentado y discutido en una publicación adjunta. Los dos primeros modelos son de Engesser y de Haringx que incluyen la componente de la fuerza axial aplicada a cortante proporcional a la pendiente total (dy/dx) y al ángulo de giro de la sección transversal (ψ) a lo largo del miembro, respectivamente. El tercer modelo es una formulación simplificada basada en la teoría clásica de Euler, que incluye los efectos de las deformaciones por cortante, pero desprecia la componente a cortante de la fuerza axial aplicada a lo largo de la columna. Se presenta también criterios definidos para determinar la rigidez mínima de los arriostramientos laterales para columnas con un cambio de sección. Se incluye cinco ejemplos completos en esta publicación que muestran la efectividad de las ecuaciones de métodos propuesto en el análisis de estabilidad y en el cálculo de los arriostramientos mínimos para columnas con un cambio de sección.

PALABRAS CLAVE: arriostramiento, pandeo, códigos de construcción, columnas, construcción, aplicaciones informáticas, marcos, cargas, conexiones semi-rígidas, deformaciones por cortante, apuntalamiento, estabilidad, columnas con cambio de sección

1. INTRODUCTION

The elastic stability analysis and the corresponding equations for stepped columns subject to end and intermediate-axial loads including the effects of: 1) lateral and rotational restraints at the intermediate connection and column ends; and 2) shear deformations and shear force along the member caused by the applied axial loads are presented in a companion paper [1]. The main objective of this article is to present the use of the proposed stability equations and to demonstrate their effectiveness.
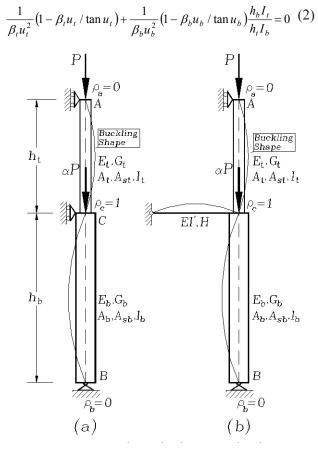
2. VERIFICATION STUDY AND EXAMPLES

2.1 Example 1. Consider the single-stepped column with hinged ends subjected to concentric axial loads P and αP at A and C shown in Fig. 1a (hinged at A and B: $\rho_a = \rho_b = 0$; and with rigid connection at C and no exterior rotational restraint: $\rho_c = 1$, $\kappa_c = 0$).

Solution: Using Eq. (10) listed in the companion paper and assuming that $\rho_a = \rho_b = 0$, and $k_c = 0$, the eigenvalue equation becomes

$$\frac{3\rho_c \beta_t u_t^2}{(1-\rho_c)\beta_t u_t^2 + 3\rho_c(1-\beta_t u_t / \tan u_t)} \frac{E_t I_t}{h_t} + \qquad (1)$$

$$\frac{\beta_b u_b^2}{1-\beta_b u_b / \tan u_b} \frac{E_b I_b}{h_b} = 0$$

When $\rho_c = 1$ and $E_t = E_b$, Eq. (1) can be reduced to Eq. (2) as follows:

$$\frac{1}{\beta_t u_t^2}(1-\beta_t u_t / \tan u_t) + \frac{1}{\beta_b u_b^2}(1-\beta_b u_b / \tan u_b)\frac{h_b I_t}{h_t I_b} = 0 \quad (2)$$

Figure1. Example1. Single-stepped column (according to Timoshenko & Gere [2], pp. 66-70): (a) on three supports with hinged ends; and (b) on end supports and rigidly connected at C to transverse beam with its far end hinged.

For the particular case of $\beta_t = \beta_b = 1$ (i.e., neglecting shear effects), Eq. (2) is identical to the solution reported by Timoshenko and Gere [1], Eq. (b), p.67.

An additional example was considered assuming that $\kappa_c = 3EI'/H$ (bending restraint provided by a transverse beam located at midspan with a hinged far end), $h_t =$

h_b, $I_t = I_b$, $\beta_t = \beta_b = 1$, $E_t = E_b = E$, and $\alpha = 0$, as shown in Fig. 1b. Then Eq. (2) becomes

$$\frac{u_t^2}{1-u_t / \tan u_t}\frac{E_t I_t}{h_t} + \frac{u_b^2}{1-u_b / \tan u_b}\frac{E_b I_b}{h_b} + \frac{3E'}{H} = 0 \ (3)$$

Since $u_t = u_b$, Eq. (3) is reduced to Eq. (4)

$$\frac{3}{u_t^2}(1-u_t / \tan u_t) = -\frac{2HI_t}{h_t I'} \qquad (4)$$

Expression (4) is identical to the solution reported by Timoshenko and Gere ([2], bottom of p. 69). Note that in Timoshenko's notation, $u_t = 2u_1$ and $u_b = 2u_2$.

Figure 2 shows the effects of shear deformations (using the Haringx approach) and semi-rigid connection at C (ρ_c) on the buckling loads of the column of Fig. 1(a) assuming that column segments are made of identical material, cross section, and equal length ($h_t = h_b = L/2$). Notice that: 1) shear deformations reduce the critical loads P and aP, while the effects of the semi-rigid connection at C are only significant when $GA_s/(EI/L^2)$ is very large and $aP \neq 0$; and 2) the proposed equation is capable of capturing the phenomena of buckling under tension when P or/and αP are negative

2.2. Example 2. The case under consideration is a single-stepped column with hinged supports at A and B subjected to concentric axial loads P_a and P_c at A and C as shown in Fig. 3 (hinged at A and B and rigidly connected at C: $\rho_a = \rho_b = 0$, and $\rho_c = 1$).

Using Eq. (10) listed in the companion paper and assuming that $\rho_a = \rho_b = 0$ and $r_c = 1$, the eigenvalue equation becomes

$$\left(\frac{\beta_t u_t^2}{1-\beta_t u_t / \tan u_t} + \frac{\beta_b u_b^2}{1-\beta_b u_b / \tan u_b}\frac{E_b I_b h_t}{E_t I_t h_b} + \frac{\kappa_c h_t}{E_t I_t}\right) \times$$

$$\left(\frac{\beta_t u_t^2}{\tan u_t / \beta_t u_t - 1} + \frac{\beta_b u_b^2}{\tan u_b / \beta_b u_b - 1}\frac{E_b I_b h_t^3}{E_t I_t h_b^3} + \frac{S_c h_t^3}{E_t I_t}\right) -$$
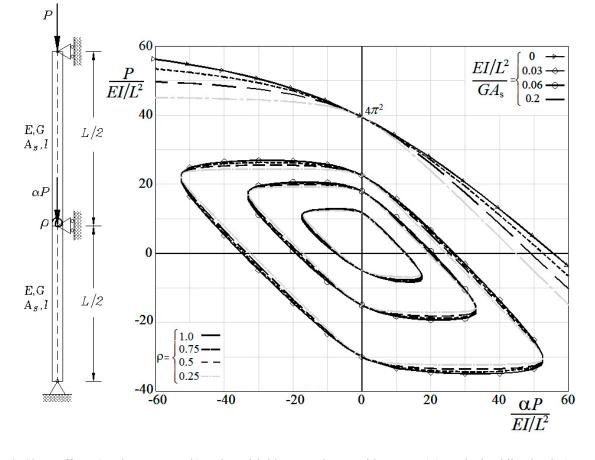
$$\left(\frac{\beta_t u_t^2}{1-\beta_t u_t / \tan u_t} + \frac{\beta_b u_b^2}{1-\beta_b u_b / \tan u_b}\frac{E_b I_b h_t^2}{E_t I_t h_b^2}\right)^2 = 0 \ (5)$$

Results from Eq. (5) were tested against tabulated values presented by Timoshenko and Gere ([2], Table 2-6, p. 100) for the particular case of $S_c = k_c = 0$, $\beta_t = \beta_b = 1$ and $E_t = E_b = E$. The values of the reduced length L

of the column calculated from Eq. (5) which are listed in Table 1 (denoted as m_{Cal}) are practically identical to those derived by Timoshenko and Gere [2] denoted as m_{Th}) where $(P_a + P_c)_{cr} = \pi^2 EI/L^2$.

Figure 4 show a series of interaction diagrams of $P/(EI/L^2)$-vs-$aP/(EI/L^2)$ for different values of $S/(EI/L^3)$

all obtained using Eq. (5) for the particular case of $h_t = h_b = L/2$, $G_t A_{st} = G_b A_{sb} = \infty$ and $E_t I_t = E_b I_b = EI$. Figure5 shows the variations of the compressive buckling load P_{cr} of a simply supported beam with the stiffness of the intermediate elastic support S located at midspan for different values of GA_s/P_e (notice that P_{cr} and S are normalized with respect to $P_e = \pi^2 EI/L^2$).

Figure 2. Shear effects (Haringx approach) and semirigid connection at mid-support (r_c) on the buckling loads (P and aP) of the single-stepped column ABC of Problem 1 [assuming that:$r_a = r_b = 0$, $E_t I_t = E_b I_b = EI$, $GA_{st} = G_b A_{sb} = GA_s$, and $h_t = h_b = L/2$ in the model shown in Fig. 1(a)].

Notice also that: 1) the values indicated in Fig. 5 corresponding to the curve with $S = 0$ are identical to those presented by Timoshenko and Gere ([2], p. 100 Table 2-6 for n = 1); 2) the buckling load capacity of the beam-column is affected significantly not only by the location and stiffness of the mid-support but also by the magnitude of the shear stiffness of the members as shown by the curves in Fig. 5 and 3) the curve corresponding to $GA_s/P_e = \infty$ in Fig. 5 shows that the compressive critical load increases in approximately

the same proportion as S. This curve is identical to that presented by Timoshenko and Gere ([2], p. 73).

2.3 Example 3. The validity of Eq. (12) was tested against the solutions obtained utilizing the classical method of slope-deflection (Salmon and Johnson [3], p. 840) for a stepped column with hinged ends subjected to concentric axial loads P_a and P_c at A and C as shown in Fig. 6 (free at A: $\rho_a = 0$, $S_a = 0$; braced at C with no exterior rotational restraint: $\rho_c = 1$, $\kappa_c = 0$; and hinged at B: $\rho_b = 0$).
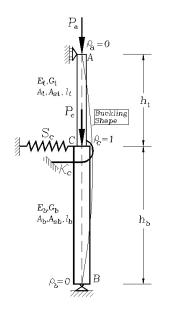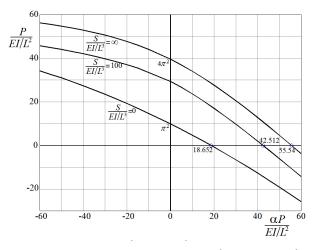
Figure 3. Example2: Single-stepped column simply supported at A and B

Figure 4. Interaction diagrams $P/(EI/L^2)$-vs-$aP/(EI/L^2)$ for the column of Example 2 [for the particular case of $h_t = h_b = L/2$, $k_c = 0$, $G_t A_{st} = G_b A_{sb} = \infty$ and $E_t I_t = E_b I_b = EI$] assuming that $P_a = P$ and $P_c = \alpha P$ with an intermediate elastic support with $S_c/(EI/L^3) = \infty$, 100, and 0
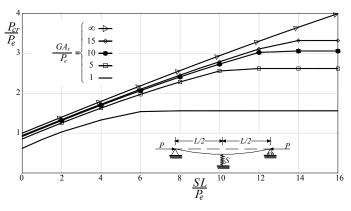

Figure 5. Variations of the critical load of a simply supported column with the stiffness of the intermediate elastic support S and its shear stiffness GA_s ($P_e = p^2 EI/L^2$) using the Haringx approach

Table 1. Calculated vs. theoretical values of L/l for a single-stepped column (Problem 2: according to Timoshenko & Gere [2], Table 2-6, p. 100)

$\dfrac{P_a + P_b}{P_a}$		1.00		1.25		1.50		1.75		2.00		3.00
I_2/I_1	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}
1.00	1.00	1.0000	0.95	0.94904	0.91	0.91397	0.89	0.88847	0.87	0.86892	0.82	0.82257
1.25	1.06	1.06229	1.005	1.00505	0.97	0.96553	0.94	0.93662	0.91	0.91455	na	0.86187
1.50	1.12	1.12354	1.06	1.06045	1.02	1.01675	0.99	0.98470	0.96	0.96019	na	0.90149
1.75	1.18	1.18321	1.11	1.11467	1.07	1.06707	1.04	1.03208	1.01	1.00529	na	0.94088
2.00	1.24	1.23810	1.16	1.16745	1.12	1.11620	1.08	1.07846	1.05	1.04951	na	0.97976

Substituting into Eq. (12) $\rho_a = \rho_b = 0$, $\rho_c = 1$, and $\kappa_c = 0$, the eigenvalue equation becomes

$$\left[\frac{\beta_t u_t^2}{1-u_t/\tan u_t}\frac{E_t I_t}{h_t}+\frac{\beta_b u_b^2}{1-u_b/\tan u_b}\frac{E_b I_b}{h_b}\right]\times$$ (6a)

$$\left[\frac{1}{1-u_t/\tan u_t}-1\right]\beta_t u_t^2\frac{E_t I_t}{h_t^3}-\left[\frac{\beta_t u_t^2}{1-u_t/\tan u_t}\frac{E_t I_t}{h_t^2}\right]^2=0$$

or

$$\frac{u_t}{\tan u_t}+\frac{(1-u_t/\tan u_t)\beta_b u_b^2}{(1-u_b/\tan u_b)\beta_t u_t \tan u_t}\frac{E_b I_b}{E_t I_t}\frac{h_t}{h_b}-1=0 \quad (6b)$$
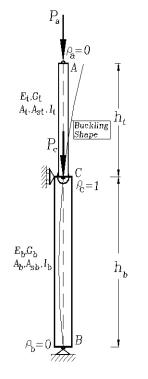

Figure 6. Example 3: Single-stepped column simple supported at C and B and top sidesway

For the particular case: $P_c = 0$, $h_t = h_b = h$, $E_t I_t = E_b I_b = EI$, and $\beta_t = \beta_b = 1$, the stability equation is reduced to $2u_t = \tan u_t$, whose solution is $u_t = 1.16556$ or $(P_a)_{cr} = \pi^2 EI/(2.6953h^2)$. Using the slope-deflection method (Salmon and Johnson [3], pp. 840-842) a stability equation identical to (6b) can be obtained.

2.4 Example 4. Results from Eq. (13) were tested against tabulated solutions presented by Timoshenko and Gere ([2], Table 2-10, p. 115) for a stepped cantilever Euler-Bernoulli column ($\beta_t = \beta_b = 1$) shown in Fig. 7 free at A ($\rho_a = S_a = 0$), rigidly connected and

unrestrained at C ($\rho_c = 1$, $S_c = \kappa_c = 0$), and perfectly fixed at B ($\rho_b = 1$).

Table 2 shows that the values of m calculated using Eq. (13), which are listed with five significant figures, are practically identical to those by Timoshenko. Note that m is for the hinged-hinged column, and $m/4$ for the cantilever column, where m is used by Timoshenko in $P_{cr} = mEI_2/(h_1+h_2)^2$.

2.5 Example 5: *Minimum lateral bracing for a bent-up frame*

Utilizing the minimum bracing criteria and the steps described in the companion paper, determine the lateral bracings required to convert the bent frame shown in Fig. 8a into a braced frame. The bracings that need to be analyzed are: 1) S_a along the top level (i.e., frame braced at A and A' only); 2) S_c at the intermediate joint (i.e., frame braced at C and C' only); and 3) S_a and S_c along the top and intermediate joints (i.e., frame braced at A and C). Assume that $\rho_b = \rho_c = 1$ in both columns and neglect the effects of shear deformations (i.e., $b = 1$ in all members).

Solution: Since the frame is symmetrical, columns AB and A'B' are both identical with the same loads and boundary conditions. Therefore, the stability analysis can be reduced to that of a single column.

The first step is to find the fixity factors at the column ends for unbraced and braced conditions. Relationships between the rotational restraints and the fixity factors in framed structures are presented in the Appendix of the companion paper. For this particular frame, the fixity factors are as follows:

i) For *unbraced conditions* along AA', the frame would buckle in a anti-symmetric shape (Figs. 8b-c) with the beam providing rotational restraints at both ends A and A' of magnitude $6EI/L = 1.5EI/h$ (since $L = 4h$). Therefore, $\rho_a = 1/(1+3/1.5) = 1/3$, and $\rho_b = \rho_c = 1$.

ii) For *braced conditions* along AA', the frame would buckle in a symmetric shape (Fig. 8d) with the beam providing rotational restraints at both ends A and A' of magnitude $2EI/L = 0.5EI/h$. Therefore, $\rho_a = 1/(1+3/0.5) = 1/7$, and $\rho_b = \rho_c = 1$.

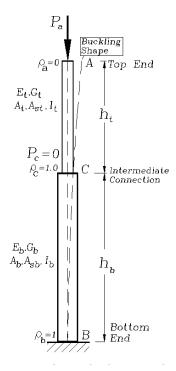The required bracings for column AB for each one of the requested cases are calculated as follows:

Figure 7. Example 4: Single-stepped cantilever column (Timoshenko & Gere [2], p. 115).

1.) Braced at top end A only. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_t = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$, the solution for braced conditions obtained from Eq. (11) are: $(P_a)_{cr} = 0.19072(p^2 EI/h^2)$ and $(P_a + P_c)_{cr} = 0.76288(\pi^2 EI/h^2)$ (or $u_t = \pi\sqrt{0.19072} = 1.37198$ and $u_b = \pi\sqrt{1.52576} = 3.88055$). Now, the magnitude of S_a can obtained directly from Eq. (11) by substituting the values of $u_t = 1.37198$ and $u_b = 3.88055$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$. The result is $S_a = 3.3818 EI/h^3$ per column.

2.) Braced at the intermediate joint C only. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_t = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$, the solution for braced conditions obtained from Eq. (12) are: $(P_a)_{cr} = 0.20909(\pi^2 EI/h^2)$ and $(P_a + P_c)_{cr} = 0.83637(\pi^2 EI/h^2)$ (or $u_t = \pi\sqrt{0.20909} = 1.43654$ and $u_b = \pi\sqrt{1.67273} = 4.06315$). Now, the magnitude of S_c can obtained directly from Eq. (12) by substituting the values of $u_t = 1.43654$, $u_b = 4.06315$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$. The result is $S_c = 2.77322 EI/h^3$ per column.

3.) Braced at end A and at the intermediate joint C simultaneously. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_t = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$, the solution for braced conditions obtained from Eq. (10) are: $(P_a)_{cr} = 0.329338(\pi^2 EI/h^2)$ and $(P_a + P_c)_{cr} = 1.31735(\pi^2 EI/h^2)$ (or $u_t = \pi\sqrt{0.329338} = 1.802898$ and $u_b = \pi\sqrt{2.634707} = 5.09936$).

Now, the magnitude of S_a can obtained directly from the characteristic equation $\|[K]\| = 0$ by deleting the second row and column (or $K_{11}K_{33} - K_{31}^2 = 0$) and substituting the values of $u_t = 1.802898$, $u_b = 5.09936$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$. The result is $S_a = 47.3354 EI/h^3$ per column.

Similarly, S_c can obtained directly from the characteristic equation $\|[K]\| = 0$ by deleting the third row and column (or $K_{11}K_{22} - K_{21}^2 = 0$) and substituting the values of $u_t = 1.802898$, $u_b = 5.09936$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$. The result is $S_c = 1.096677 EI/h^3$ per column.

It is interesting to note that: 1) the trend for this particular frame is indicated by Eq. (18) as follows: $(P_a + P_c)_{cr} = 1.31735(\pi^2 EI/h^2) > 0.83637(\pi^2 EI/h^2) > 0.76288(\pi^2 EI/h^2) > 0.16074(\pi^2 EI/h^2)$ (this last value corresponds to the unbraced frame); and 2) the required lateral bracing at A is relatively small compared to that required at the intermediate joint C.

Table 2. Calculated vs. theoretical m-factor for a single-stepped column
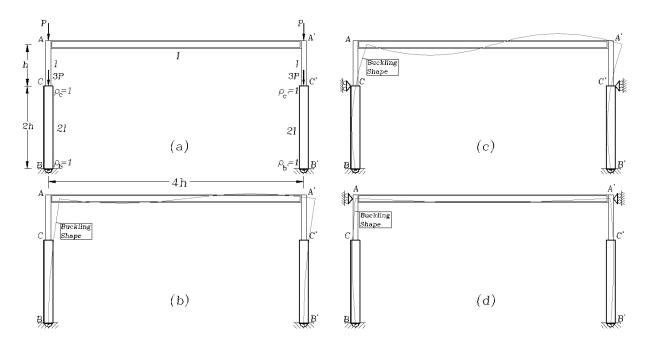(Problem 4: according to Timoshenko and Gere [2], Table 2-10, p. 115)

$\dfrac{h_t + h_b}{h_t}$	0.2		0.4		0.6		0.8	
I_2/I_1	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}
0.01	0.15	0.15344	0.27	0.27052	0.60	0.59843	2.26	2.25706
0.1	1.47	1.46750	2.40	2.40063	4.50	4.49778	8.59	8.58799
0.2	2.80	2.79651	4.22	4.22180	6.69	6.69418	9.33	9.33015
0.4	5.09	5.08844	6.68	6.67739	8.51	8.50980	9.67	9.67421
0.6	6.98	6.97941	8.19	8.18500	9.24	9.24378	9.78	9.78394
0.8	8.55	8.55122	9.18	9.17672	9.63	9.63146	9.84	9.83755

Figure 8. Example 5: Minimum lateral bracings for a simple bent frame with stepped columns: (a) structural model; (b) buckling mode with sidesway between A and B uninhibited; (c) buckling mode with sidesway between C and B totally inhibited; and (d) buckling mode with sidesway between A and B totally inhibited

3. SUMMARY AND CONCLUSIONS

A complete column classification and the corresponding stability equations for single-stepped columns with sidesway totally inhibited, partially inhibited, and uninhibited subjected to concentrated axial loads located at the ends and at the intermediate joint including the shear force effects are presented using three different approaches in a companion paper.

To understand the four-way classification of single-stepped columns and post shores, the corresponding stability equations, five examples are used in this paper and the results are compared to those using other methods. A verification study indicates that the calculated elastic buckling loads are theoretically exact.

Analytical studies indicate that: 1) the stability of a single-stepped column increases substantially with the magnitude of the lateral restraints and the fixity at the base and at the intermediate connection; 2) shear deformations and the shear forces induced by the applied axial loads reduce the buckling loads; 3) the degree of fixity at the top end has less influence on the overall stability of stepped columns and post

shores in frames with sidesway inhibited than in frames with sidesway uninhibited; 4) the critical axial loads using the Engesser approach is lower than that using the Haringx and simplified Euler approaches; 5) the critical loads in tension and in compression predicted by the Haringx approach are very sensitive to the shear stiffness GA_s; 6) the Haringx approach is the only one among the three approaches capable of capturing the phenomena of tension buckling; and 7)as expected, the critical axial loads in compression are highly affected by the degree of flexural fixity at the supports, but the critical axial loads in tension are not affected as much. It is important to emphasize that shear effects are not of significance and can be neglected for slender columns of solid cross sections like rectangular and I-cross sections.

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NOTATION

A_t and A_b = cross area of column segments AC and CB, respectively;

I_t and I_b = moments of inertia of segment column segments AC and CB, respectively;

ht and hb = spans of columns segments AC and CB, respectively;

Kj = stiffness coefficients for the 3-DOF stepped column given by (1);

κ_{ij} = stiffness coefficient for a single prismatic element given by Eqs. (22)-(27);

$(P_a)_{cr}$ = critical load at A;

$(P_c)_{cr}$ = critical load at C;

$(P_a + P_c)_{cr}$ = total critical load;

P_a = applied axial load to column segment AC at A;

P_c = applied axial load to column segment CB at C;

ρ_a and ρ_b = rotational fixity factors of column AB at top A and bottom B, respectively;

ρ_c = rotational fixity factor of column segment AC at the intermediate joint C;

S_a = lateral stiffness restraining column AB against sidesway at top A;

S_c = lateral stiffness restraining column AB against sidesway at intermediate joint C;

κ_c = rotational stiffness restraining column AB externally at intermediate joint C;

r and s = "modified" stability stiffness coefficients for a column under axial load

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