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# On the correctness of problem solving in ancient mathematical procedure texts

## *Sobre la corrección de la resolución de problemas en textos antiguos de procedimientos matemáticos*

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**Abstract**

It has been argued concerning Old Babylonian mathematical problems that the validity or correctness of the procedures adopted to solve them is self-evident. One "sees" that a procedure is correct without it being accompanied by any explicit argument for its correctness. Even when agreeing with this view, one might ask how it is that the procedure turns out to be correct. In this work, we identify elements that are crucial for the correctness of ancient Egyptian and Old Babylonian mathematical procedures. We endeavor to make explicit how and why the procedures are reliable over and above the fact that their correctness is intuitive.

**Keywords**: mathematical problems, problem solving, mathematical procedure, correctness.

**Resumen**

Se ha argumentado respecto a problemas matemáticos de la antigua Babilonia que la validez o corrección de los procedimientos adoptados para su solución es evidente. Uno "ve" que el procedimiento es correcto sin que este vaya acompañado de un argumento explícito de su corrección. Incluso estando de acuerdo con este punto de vista, uno se puede llegar preguntar de qué manera el procedimiento es correcto. En este trabajo, identificamos elementos que son cruciales para la corrección de

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procedimientos matemáticos del antiguo Egipto y de la antigua Babilonia. Tratamos de hacer explícito cómo y por qué los procedimientos son fiables más allá del hecho de que es intuitiva su corrección.

**Palabras clave**: problemas matemáticos, solución de problemas, procedimientos matemáticos, corrección.

## 1. Introduction: "Seeing" the evident correctness of mathematical procedures

According to Jens Høyrup in the solution of problems in Old Babylonian mathematical texts, "the description of the procedure already makes its adequacy evident" (Høyrup 2012, 366).[1] To understand what this means we will consider Høyrup's presentation of a very simple Old Babylonian mathematical problem;[2] the first one inscribed in tablet BM 13901:[3]

1. The surfa[ce] and my confrontation I have accu[mulated]: 45´ is it. 1, the projection,
2. you posit. The moiety of 1 you break, [3]0´ and 30´ you make hold.
3. 15´ to 45´ you append: ⌊by] 1, 1 is equal. 30´ which you have made hold
4. in the inside of 1 you tear out: 30´ the confrontation.

The description of the procedure consists of a sequence of writing text with numerical signs[4] that the reader must follow/execute. According to Høyrup's reconstruction, we must consider that the reader has, e.g., a dustboard or a wax tablet in which he draws "geometrical" figures corresponding to the sequence of steps of the procedure (Høyrup 1990, 286; Høyrup 2002, 106-7; Robson 2008, 141).[5] The words have a specific meaning, which must be known to make sense of each step of the procedure and follow it.[6] In this

[1] Høyrup also speaks of the evident validity or self-evident validity of the procedures (Høyrup 2012, 364 & 378).

[2] We cannot do justice here to the importance of the mathematics of ancient Egypt and the ancient Near East. It will have to suffice to call attention to their relevance in the history of Calculus and the history of Algebra (see, e.g., Edwards 1979; Katz and Parshall 2014).

[3] See, e.g., Høyrup (2002, 50-2). Regarding the notation adopted in the reconstruction of lost or damaged passages, see, e.g., Høyrup (2002, 42).

[4] According to Høyrup, in the text it is adopted a place value system with base 60 and there is no indication of the absolute order of magnitude (Høyrup 2002, 12). Regarding the sexagesimal place value system, see, e.g., the treatments of this subject by Robson (2008, 75-78) or Proust (2016).

[5] It is unknown how the drawings were made. Robson mentions wooden boards with a waxed writing surface (a wax tablet) and ivory writing boards (Robson 2008, 141 & 145). Høyrup also mentions sand spread on an even surface, dustboards, and some other possibilities (Høyrup 1990, 286; Høyrup 2002, 106-107).

[6] In solving problems, Babylonians adopted what Høyrup called cut-and-paste procedures or manipulations

procedure text, we do not have an explicit statement of the problem at the beginning stating what we have to determine;[7] in this case, we want to determine the length of the side of a square.

A square is "conceptualized" (i.e. the way we think about it and use it in problems)[8] as the "confrontation" of two equal sides. A square, like any other figures, is "defined" by its boundary (Robson 2008, 64); as Høyrup puts it, "a Babylonian square (primary thought of as a square frame) "was" its side and "had" an area" (Høyrup 2012, 366, footnote 13).[9]

In the first step of the procedure, we are "adding" (accumulating)[10] a surface to the square (the confrontation). This is made by "projecting" (or sticking out) orthogonally to one side of the square a line of length 1 (60´) giving rise to a rectangle (see figure 1 left).

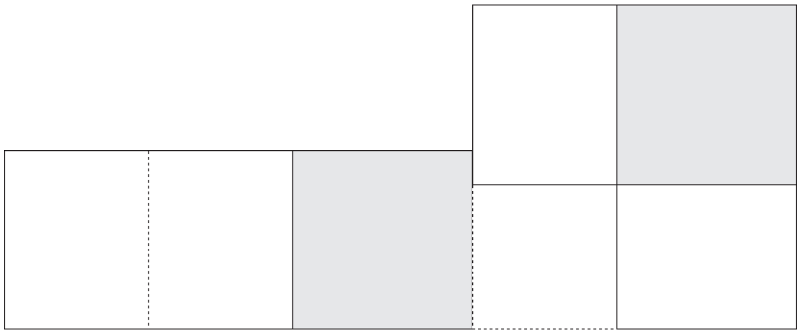

**Figure 1.** Cut-and-paste manipulations in the procedure of BM 13901 #1. The sides of the square are a bit larger in the drawing in relation to the rectangle when taking into account the calculated result.

(see, e.g., Høyrup 2002, 96-99).

[7] In the "standard format", the procedure texts start by stating the problem to be solved, after which comes the text of the procedure that solves the problem (Høyrup 1990, 59; Høyrup 2002, 32).

[8] Here, we adopt a notion of "concept" similar to that of (Høyrup 2004, 131). The key point for us is that a concept is only meaningful in a context ("network") of concepts and operations in which the way we think about the concept is intertwined with the way we apply it in relation to other concepts and operations.

[9] Høyrup adopts the term "confrontation" as corresponding to a Babylonian word used to denote a square (Høyrup 1990, 50; Høyrup 2002, 13 & 25).

[10] According to Høyrup, there are two kinds of additive operations: "accumulate" and "append". On this issue see, e.g., (Høyrup 2012, 367, footnote 17).

The total area of the square plus the rectangle is 45´ (3/4).[11, 12] In the next step, we break out the natural half of the rectangle (its outer moiety) and move it so to make a gnomon that maintains the total area of 45´ (a sort of inverted L-shape figure; see figure 1 right). The two "moieties" are made "hold" a square defined by their sides of 30´ and 30´ (1/2). In the next step, we "append" to the gnomon (with an area of 45´) this square (with an area 30´x 30´ = 15´), which results in a larger square (see figure 1 right). This square has an area of 45´+ 15´ = 1 (60´). It follows that a side of the larger square must have a length of 1. In the next step, we "tear out" from this length that of the side of the rectangle that we moved to form the gnomon, and we obtain the side of the original square, which we wanted to determine. We have for the side of the square 1 – 30´ = 30´; i.e. the side of the confrontation is 30´, as mentioned in the final step of the procedure.

There is no justification for the correctness of this procedure; we simply "see" that it is correct. However, there is some evidence that there was some worry about the correctness of the procedures. Several procedures end with a check, in which we confirm that the number obtained is correct (Høyrup 1986, 453-5). There is even a case were besides the check of the correctness of the numerical value there is, in part, a check of the method adopted (Høyrup 2012, 364-7). We could imagine a check of the procedure we just saw, in which we consider a square (confrontation) with side 30´ and make a "projection" of a line with length 1 so that we obtain figure 1. We then determine the area of the composed figure and check that its value is 45´.

Some other explicit elements regarding the correctness of a procedure deal with the strategy or method adopted and more meaningful use of the words. This last point can be seen, e.g., when instead of adopting the wording "append and tear out" one finds "from one tear out, to one append" (Høyrup 2012, 378-9). It only makes sense in a concrete practice to append a figure that we have previously torn out. Regarding the methods adopted to solve problems, there were found some texts in which all numbers are given. These are not problems, but what Høyrup called "didactic texts" (Høyrup 2002, 85; Høyrup 2012, 370-6). These texts present methods that can be applied in the solution of problems. Let us look at the example of TMS IX #1:[13]

1. The surface and 1 length accumulated, 4[0´. ¿30, the length,ˀ 20´ the width.]
2. As 1 length to 10´ ⌈the surface, has been appended,]

[11] The basic measure of distance is the nindan or rod (see, e.g., Robson 2008, 294-5). Usually, this unit is not written remaining implicit (Høyrup 2002, 17). Regarding areas, the unit is the square nindan or sar (Høyrup 2002, 17). This means that when one is "projecting" a line of length 1, one is drawing a line with the length of 1 rod. In the same way, the total area of the figure is 45´ sar.

[12] One must bear in mind that numbers are always determined within a metrological context, in which they signify a concrete quantity (Robson 2008, 52). However, during calculations, adopting the sexagesimal place value system, numbers could lose temporarily their metrological meaning (Robson 2008, 78).

[13] See, e.g., Høyrup (2002, 89-90).

3. or 1 (as) base to 20´, [the width, has been appended,]
4. or 1°20´ [¿is posited?] to the width which 40´ together ⌈with the length ¿holds?]
5. or 1°20´ toge<ther> with 30´ the length hol[ds], 40´ (is) [its] name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is appended: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. [30´ the length. T]hus the procedure.

In the procedure of BM 13901 #1, we had a square from which we "projected" a rectangle; we wanted to determine the sides of the square knowing the total area of the composite figure. Here, we have all the numbers. In this case, instead of a square, we have a rectangle of length 30´and width 20´.[14] We project a line of length 1, and are told that the total area is 40´. In the first part of the "demonstration" (from 1 to 5) we see that the complete figure has a length of 30´and a width of 1º20´, corresponding to an area of 40´. In the second part (from 6 to 9) we see that if we have a width of 1º20´ and an area of 40´ this corresponds to the rectangle having a length of 30´. That is, the "demonstration" shows us the relation between the different numbers (length, width, total area) in terms of a particular method, in which we "project" a line from the initial figure (a square or a rectangle). Learning a method by following didactic texts might help in "seeing" how it works in practice (when applied to a particular problem). On "intuitive" grounds, we can accept that the understanding of how a method works might help in making clearer the structure of a problem. In any case, even if we consider a mathematical procedure that ends with a check of the number obtained, the existence of some "auxiliary" texts with "demonstrations" (of the method adopted in the procedure) and improved use of the words adopted, these, on their own, do not suffice to establish the correctness of the mathematical procedure adopted to solve a problem.[15] This might not be problematic since "we do not argue explicitly for their correctness, but we "see" immediately that they are correct" (Høyrup 2010, 98). This is so because:

> The one who follows the procedure on the diagram and keeps the exact (geometrical) meaning and use of all terms in mind will feel no more need for an explicit demonstration than when confronted with a modern step-by-step solution of an

[14] Notice that this is a reconstruction by the editors of a part of the text that was destroyed.

[15] For example, the numerical value of the quantity being determined might be correct, but some steps of the procedure might be wrong (or some "transition" between steps), even if the method might be applied correctly in similar cases (this could be the case, e.g., when the method only "works" for particular combinations of numbers).

> algebraic equation, in particular because numbers are always concretely identified by their role … There should be no doubt that the solution *must be* correct. (Høyrup 2012, 367)

With the "seeing" of the correctness of a procedure comes an "intuitive" grasp of what it is for a procedure to be correct. However, we have not articulated the correctness of a procedure; that is, we have not determined/expressed/explained how and why a procedure for solving a mathematical problem turns out to be correct. Even if agreeing with Høyrup regarding the self-evidence of the correctness of a procedure, we might still try to go beyond "seeing" the procedure's correctness and articulate this correctness (or how a "didactic text" demonstrates). We address this issue in the next sections. We will make the point that what we take to be the intuitive grasp of the correctness of procedures depends on several related features of these procedures. The "seeing" of the correctness is grounded on these features, which are normally not explicitly taken into account by us. That is, we know that a procedure is correct in a great measure due to these features even if when asked what these features are, we would not be able to answer – we are not immediately aware of their role in our "intuitive" grasp of the correctness. We have to "dig" into a procedure text to bring these "mathematical" features into the light. We will see that the crucial elements that make a procedure correct are the following: the arithmetic operations on numbers and the numerical relations established between numbers, in the context of the definition and use of numbers in relation to metrological systems (section 2); also, keeping track of metrological units during arithmetic operations, and the area "conservation" in cut-and-paste manipulations (section 3).

## 2. Articulating the correctness of mathematical procedures: Egyptian mathematical problems

We will start the articulation of the correctness of mathematical procedures by considering James Ritter's views on mathematical problems from ancient Egypt and ancient Near East. Ritter proposes that "Egyptian mathematical problems work on three distinct levels" (Ritter 2000, 124). Let us look into an example to see what Ritter means. The following is a problem in which we have to determine a quantity[16] that when one adds to it its fourth-part we obtain 15:[17]

[16] In a way similar to Old Babylonian mathematics, a quantity is a number with a concrete meaning: a number of things or a number of a unit of a metrological system (like that for length, area, grain capacity, or weight; see Ritter (2000, 116), Imhausen (2016).

[17] See Ritter (2000, 124), Imhausen (2016, 70-73). Here, we adopt Otto Neugebauer's notation for fractions, and write $\overline{4}$ for ¼ (Neugebauer 1969, 74; Imhausen 2016, 52-54). The text follows Imhausen's presentation while taking into account some elements of Ritter's presentation. This is the problem 26 of the Rhind Mathematical Papyrus.

[1] A quantity, its $\bar{4}$ [is added] to it. It becomes 15.
Calculate with 4.
You shall calculate its $\bar{4}$ as 1. [Add 4 to 1.] Total: 5.
[2] Calculate starting from 5, to find 15.
[3] \ .    5
[4] \ 2    10
[5] 3 shall result.
Calculate starting from 3, 4 times.
[6] .    3
[7] 2    6
[8] \4    12
[9] 12 shall result.
[The procedure as it occurs:]
[10] .    12
[$\bar{2}$    6]
[11] $\bar{4}$    3
Total: 15.
[12] The quantity: 12,
[13] its $\bar{4}$ : 3,
Total: 15.

The most general level that Ritter identifies is that of what he calls the strategy or method to solve the problem (Ritter 2000, 125-6; Ritter 2004, 186).[18] As Ritter calls the attention to, "similar problems may call for quite different [procedures], depending on the specific values of their data" (Ritter 2000, 125). In this case, the method adopted consists in choosing a "false solution" from which we determine the corresponding sum with its fourth-part. We then determine the factor relating this sum to 15 and multiply the false solution with this factor to arrive at the correct number. In this case, since we are adding to a number its fourth-part we "start" with the trial number 4. Calculating its fourth-part (in this case 1) and adding it to the trial number 4 we obtain 5. We then determine the factor by which we must multiply 5 to obtain 15, which is 3. Finally, we multiply the false solution 4 by this factor 3 to obtain the correct result of 12. The final part of the procedure contains a numerical check that the number found is correct (Ritter 2000, 125; Imhausen 2016, 72).

[18] This has the same meaning as "method" with Høyrup (see, e.g., Høyrup 2002).

A “second level” identified by Ritter is that of the “operations necessary to carry out the [procedure]” (Ritter 2000, 125). In the case of the procedure being considered we start with a multiplication of 4 by its fourth-part (obtaining 1), we then add 4 to 1 (which is left implicit); after this step, we make a division of 15 by 5 (obtaining the “factor” 3), and then we multiply 4 (the trial number) by 3 to obtain the result 12.

A “third level” is that of the particular “techniques used to effectuate each operation [which] varies according to the specific values in play” (Ritter 2000, 126). In Egyptian procedure texts, the particular techniques adopted are included in the procedure, i.e. we see how exactly an operation is made. According to Ritter “that the level of techniques is really independent of that of operations can be seen from the fact that a technique can be utilized in any operation where it is needed” (Ritter 2000, 126).

Regarding Old Babylonian procedure texts, the main difference to the Egyptian’s is that they lack the “level” of technique (Ritter 2004, 185). In Ritter’s own words, “practically no Babylonian text speaks of calculational techniques” (Ritter 2004; see also Ritter 1989, 52-55).

Regarding the Egyptian procedure under consideration, is its correctness self-evidence in way similar to what Høyrup proposes in relation to Old Babylonian procedures? It seems so. In fact, the Egyptian procedure seems to consist of a sequence of arithmetic operations. We do not need any auxiliary drawing neither a special interpretation of the words. It seems quite clear from our perspective (after “learning” the Egyptian techniques to make operations). We have an unknown quantity to be determined and other numbers related to it and we make a series of arithmetic operations until we calculate this quantity.

However, it is important to consider the procedure in the context where it is employed. In this respect, we should take into account the notion of number being used. Egyptians adopted a number system of base 10 (a decimal number system), not place valued, and not having any symbol for our 0. They associated numbers to countable things. For example, in a drawing from predynastic times, there is a depiction in which we find a representation of a bull with numerical signs corresponding to 400000 (Imhausen 2016, 25). With the invention of the calendar, numbers are used to count “days” (Imhausen 2016, 33) and the division of these in “hours” (Neugebauer 1969, 81). With the notion of metrological systems, the use of number is extended to whatever concrete quantity that is measurable. In particular: “Egyptian units of length were generally derived (as in many other cultures) from parts of the human body: The basic measuring unit was the cubit (*mh*), which was derived from the length of a person’s forearm” (Imhausen 2016, 41).

Having, e.g., a cubit-rod[19] we can measure lengths in terms of the number of cubits; let us say that the length of one side of a table corresponds to three cubit-rods in sequence. We are extending the use of number from that of having three things (the rods) to that of a length of three cubits. From the unit of length, it is developed the metrological unit of area: "the smallest unit is again called "cubit," *mh*: it designates a square of 1 *mh* by 1 *mh*." (Imhausen 2016, 46). This metrological definition of unit of area also indicates how an area can be measured and has implicit the notion of square. For example, we can imagine constructing a wood square using cubit-rods for the measurement of its sides (a sort of unit square frame), which we can then use to measure larger areas. Other examples of metrological systems are developed in terms of a capacity unit used to measure the volume of grain (e.g. with a "standard" vessel corresponding to the standard unit) and a unit of weight, in which standard things (weights) are used to determined, by comparison, the weight of other things (Imhausen 2016, 48-51).

The procedure we are considering does not make any reference to metrological units. However, as Rittel called the attention to, the metrological units are not usually made explicit unless they are different from the basic unit (Ritter 1989, 44) and also, in the period to which this problem belong to, Egyptians had already adopted "inside" the procedure an "abstract number system, i.e., one that is independent of any particular metrological system" (Ritter 2000, 121).[20] This means that the (numerical) quantity we want to determine and numbers we find in the procedure can be related to things, time, or any other concrete quantity for which there is a standard unit and a measuring technique (which ultimately relies on things taken to represent the unit adopted).

In the procedure text under consideration, we find however another notion of number, which we now call fraction. The notion of fraction made its appearance after extending the use of number from things to other measurable quantities in the context of a metrological system. According to Imhausen, "fractions are first attested within the context of metrological systems" (Imhausen 2016, 53). Except for 2/3, Egyptian fractions were of the type of what we now would define as 1/n, where n is a positive integer (a natural number). The fractions were not conceptualized as a numerator being divided by a denominator but as the reciprocal of a natural number. Thus, e.g., the fraction 1/2 was the reciprocal of 2 and the numerical sign adopted was that of the numerical sign for 2 and a sign to designate it as the reciprocal of 2 (Imhausen 2016, 54).

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[19] A cubit-rod is a measuring instrument (a sort of yardstick) that indicates the length of one cubit and may include its subdivisions (Imhausen 2016, 168-169).

[20] One example of this, is problem 46 of the Rhind Mathematical Papyrus; In the problem, one starts with a volume given in a standard capacity unit and calculates a height in cubits. One does not maintain the metrological units throughout the calculation, and only recovers a concrete quantity with the final number with its adjacent metrological unit (Ritter 2000, 122-123).

As we have already seen briefly, the procedure text starts by stating the problem and giving the numerical data to work with: “a quantity, its $\bar{4}$ [is added] to it. It becomes 15”. We are given a (numerical) relation between an unknown quantity and known numbers (in this case $\bar{4}$ and 15). As mentioned, we adopt a tentative value for the (numerical) quantity to be determined. In this case 4. By definition of reciprocal, when multiplying a number by its reciprocal we obtain 1. In this way, the result of multiplying 4 by  is 1. Summing this result to the tentative quantity we have 4 + 1 = 5. The sum does not correspond to the value we want to obtain of 15. In the next part of the procedure we “calculate starting from 5, to find 15”. This results in finding the value 3. This is the number that we must multiply the tentative quantity of 4 by to arrive at the total 12. The final part of the procedure presents an explicit calculation showing that the numerical relation holds between all the numbers: we multiply the quantity 12 by $\bar{4}$ obtaining 3 and add this value to the quantity obtaining 15.

As mentioned, similar problems can be solved differently when adopting a different method due to the “specific values of their data” (Ritter 2000, 125). This might make the solution of the problem easier or more comprehensible but does not by itself make the procedure correct or incorrect. The crucial aspect of the method adopted is that it enables to determine the unknown quantity in terms of the given data.

This is not to say that the “level” of the method is not directly relevant to the issue of the correctness of the procedure. It is not simply a question of being easier or more comprehensible. The method must work out the numerical relation existing between quantity and numerical data, which, in this case, is given in the statement of the problem. For that, the method unfolds in a particular way this numerical relation, bringing to light the unknown quantity. Crucially, the operations and techniques adopted in the application of the method must be correct.

Where do we find then the correctness of the procedure? On one side, on the correctness of the individual arithmetic operations (as calculated using particular techniques). On the other side, on working out, with these operations, the numerical relation existing between the quantity and numerical data.

In what regards an arithmetic operation like the multiplication of 4 by 3, it is correct if we apply the multiplication rules or techniques correctly. This might seem like a circular statement, but it is not. Independently of how the rules of arithmetic operations came to be, if these are correctly applied in a step of a procedure, they will not introduce any error in that step of the procedure.

One might ask what grounds the multiplication rules making them correct? To Ludwig Wittgenstein, regarding the proposition “25 x 25 = 625” it can be true in two senses. One regarding objects of our experience, e.g. 25 identical objects weighing 25 grams, which we measure in a balance to weigh 625 grams. In another sense “the proposition is correct if calculation shows this – if it can be proved – if multiplication of 25 by 25 gives 625 according to certain rules" (Wittgenstein 1976, 41). As Wittgenstein calls the attention to, "we make its correctness or incorrectness independent of experience" (Wittgenstein 1976, 41). However, importantly, the proof that the proposition "25 x 25 = 625" is correct (by calculating using the adopted rules), "is only called proof because it gives results which are useful in experience" (Wittgenstein 1976, 42). This is so because according to Wittgenstein, "all the calculi in mathematics have been invented to suit experience and then made independent of experience" (Wittgenstein 1976, 43). This process of "dismotivation" was succinctly explained by Wagner as follows:

> [The] rules of multiplication are adopted because they provide a successful empirical description of practical counting, weighing and measuring. Only subsequently, because of its a posteriori success, does it become a rule that no experience can refute. (Wagner 2017, 60)

We will not pursue this line of approach in this work. Here, we do not engage in more strictly philosophical considerations. We adopt a detailed step-by-step analysis of the procedures, which enable us to make explicit and explain how and why they are reliable.

Regarding the numerical relation, we must notice that the numerical relation is taken to exist between a number that we do not know and other numbers that we know. Let us consider the numerical check at the end of the procedure from a different perspective. Let us say we are designing a problem to be solved by the method of "false position". We pick up a number that can be easily multiplied by a very common fraction, one of the oldest being employed in Egyptian mathematics, e.g. . We choose 12. When multiplying by  we obtain a natural number. This is not as simple as choosing 8 but not much larger than 4. In this way, the fourth-part of 12 is 3. To "hide" the number 12 a bit, we calculate the sum of 12 with its fourth-part obtaining 15. 12 is the number/quantity that added to its fourth-part gives 15. We can take this result to be a sort of "definition" or (numerical) "characterization" of the number 12. We have established a relation, through a sequence of arithmetic operations, between 12, , and 15. We now pretend to forget that we know the quantity and convert this "definition" of 12 into a problem with a supposedly unknown quantity: "a quantity, its  [is added] to it. It becomes 15". This statement of the problem refers to the relation existing between a number, its fourth-part (or the number ), and the number 15. We can see this numerical relation as established (or being establishable) by a sequence of arithmetic operations. Their "intrinsic" correctness brings about the relation. We can see that the correctness of the procedure arises in the (correct) application of arithmetic operations in a context of an established numerical relation between numbers (some known by us and another taken to be unknown).

Let us now consider another procedure text from ancient Egypt, in this case, related to a “geometrical figure”:[21]

> [1] Method of calculating a triangle.
> [2] If you are told:
> a triangle of 20 as its area.
> [3] Concerning that which you put as the length,
> you have put $\overline{3}$ $\overline{15}$ and it is the width.
> [4] You shall double 20.
> 40 shall result.
> [5] You shall divide 1 by $\overline{3}$ $\overline{15}$.
> 2 $\overline{2}$ times shall result.
> [6] You shall calculate 40 times 2 $\overline{2}$.
> 100 shall result.
> You shall calculate its square root.
> [7] 10 shall result.
> Behold, it is 10 as the length.
> [8] You shall calculate $\overline{3}$ $\overline{15}$ of 10.
> 4 shall result.
> Behold it is 4 as the width.
> What has been found by you is correct.

Here, we are given the area of a (rectangular) triangle. We have 20, an “abstract” number; no metrological unit is mentioned. We might consider that it is left implicit that we are working with cubits. We are also given the ratio $\overline{3}$ $\overline{15}$ (1/3 + 1/15) between two sides of the triangle (the ones that make a right angle between them): the “length” and the “width”. The width is $\overline{3}$ $\overline{15}$ of the length.

The procedure will enable to determine the value of the length and the width as numbers, since, as we have mentioned, the development of the metrological unit of length (and area), in which a thing, e.g. a cubit-rod, serves as our concrete instantiation of the unit (i.e. as the number 1), enables to think of lengths in terms of numbers. The numerical values of these quantities (unknown to us) are already determined by the numerical relation arising from being sides of a triangle with an area of 20 cubits and having a rate of $\overline{3}$ $\overline{15}$ between them. A procedure that uses this relation and arithmetic operations and arrives at the values of the length and width will be correct.

[21] This is the problem 17 of the Moscow mathematical papyrus. Here, we follow Imhausen (2016, 122).

To understand the adopted method, we must take into account that Egyptian mathematicians calculated the area of a triangle by calculating half the width (the base) and then multiplying it by the length (the height). In this way, the area of the triangle is seen as identical to "half the area of a rectangle with the base and height of the triangle as its sides" (Imhausen 2016, 121). In the procedure, one starts by doubling the area obtaining 40. We are calculating the area of the above-mentioned rectangle. Then, one determines $2\ \overline{2}$ (2 + 1/2), the reciprocal of the ratio. We multiply $2\ \overline{2}$ by 40 (the double of the area of the triangle, or the area of the rectangle). The result is 100. Here, we are so to speak extending the rectangle along the width (the base) until we have a square whose sides are equal to the length (the height) of the triangle, or saying it a bit differently, we are calculating the area of a square with sides equal to the length. We then calculate the square root of the area of the square obtaining the length of its sides, which is 10.[22] This is equal to the length of the triangle: "Behold, it is 10 as the length". Finally, we multiply the length by the ratio to obtain the width of the triangle: "You shall calculate $\overline{3}\ \overline{15}$ of 10". The result is 4.

To understand the procedure we must take into account several things: 1) our notion of number, in this case in relation to length and area (i.e. we have metrological systems that make meaningful to have the length and the area given as numbers); 2) the definition of fractions; 3) the arithmetic "definition" of the area of the triangle; 4) more generally, the numerical relations "inbuilt" in triangles and squares; 5) the numerical relation between the length and the width of the rectangular triangle given in terms of their ratio.

All this establishes a numerical relation between area, ratio, length, and width, as numbers. Contextualized in this way, the procedure's correctness becomes more "visible"; we can determine how and why it is correct.[23] This articulation of the correctness of mathematical procedures also makes clearer what is specifically mathematical about them: the definition and use of numbers in relation to metrological systems, the arithmetic operations on numbers, and the numerical relations established between numbers (that follows from a sequence of arithmetic operations and/or the definition of geometric figures with their inbuilt relations between several numbers, like the sides of a square having the same length).

[22] Here, we use the "fact" that, as defined, a square has an area equal to the multiplication of two of its sides.

[23] We can determine the "how" by realizing a detailed presentation and analyses of the procedure (which unfolds its correctness). This enables to determine the "why"; i.e., the crucial elements that make the procedure correct. In this case, the notion of number at play, the arithmetic operations, and the numerical relations between numbers.

## 3. Articulating the correctness of mathematical procedures: Old Babylonian mathematical problems

From the standpoint that we have arrived at in the previous section, let us look some further into Old Babylonian mathematical problems, of which we have seen an example in the first section. Let us consider the procedure to solve the eighth problem in the procedure text YBC 4663.[24] In this case, there is initially an explicit statement of the problem to be solved:

> 9 <shekels of> silver for a trench. The length exceeds the width by 3;30 (rods). Its depth is ½ rod, the work rate 10 shekels. Its wages are 6 grains. What are the length and width?
>
> You, when you proceed: solve the reciprocal of the wages, multiply by 0;09, the silver, so that it gives you 4 30. Multiply 4 30 by the work rate, so that it gives you 45. Solve the reciprocal of ½ rod, multiply by 45, so that it gives you 7;30.
>
> Break off ½ of that by which the length exceeds the width, so that it gives you 1;45. Combine 1;45, so that it gives you 3;03 45. Add 7;30 to 3;03 45, so that it gives <you> 10;33 45. Take its square-side, so that it gives you 3;15. Put down 3;15 twice. Add 1;45 to 1 (copy of 3;15), take away 1;45 from 1 (copy of 3;15), so that it gives you length and width.
>
> The length is 5 rods, the width 1 ½ rods. That is the procedure.

What notions of number are being used here? We find, like in the ancient Egyptian case, numbers of days, and numbers associated with metrological units: weight (shekel, mina, grain), length (rod), and area (sar). There are also numbers as "constant coefficients":[25] the wage of a worker (6 grains/day), and the work rate of a worker (10 shekels/day) (Robson 2008, 89).

We are asked to determine the length and width of a rectangular trench with a depth of ½ rods. The work of digging a trench was paid 9 shekels of silver, and we are told that the wage paid to a worker each day is 6 grains. We are also given the work rate which, in this case, gives an estimation of the total weight (or volume) dug in one day. The procedure unfolds as follows:

> A) First, we determine the reciprocal of the wage in terms of the unit of weight called mina (6 grains = 0;00 02 mina), then we multiply the reciprocal of the wage per day (i.e. in units of day/mina) by the total amount of silver that is equal to 0;09 mina. This results in the number of workdays paid for, which are 4 30 days (if we consider only one worker, this is the number of days that will take to dig the trench). Then

[24] We follow Robson's translation and notation (Robson 2008, 89).

[25] On constant coefficients see, e.g., (Robson 1999).

we multiply the number of workdays by the work rate given in terms of volume dug per day (0;10 sar/day, in which sar is a unit of volume). From this multiplication, we determine the volume of the trench, 45 sar (in which the volumetric sar = rod x rod x cubit). We then proceed to calculate the area of the trench. For this, we take into account that we are given the depth of the trench, ½ rods, which is equal to 6 cubits. We determine the reciprocal of 6 cubits and multiply it by the volume to obtain the area of 7;30 sar (in which sar = rod x rod is a unit of area with the same name as the above-mentioned unit of volume).

B) At this point we have the area of the trench (rectangle), which is 7;30 sar, and we know, from the statement of the problem, that the difference between the length and the width of the trench is 3;30 rods. We draw a rectangle representing the area of the trench and make a demarcation of the area corresponding to the part in which the length exceeds the width. We will "break off" half of this area, which corresponds to a length of 1;45 (see figure 2 left). The part we "break off" is "attached" to the part of the area corresponding to the square with sides given by the width, forming a gnomon. We pick the two "internal" sides of the gnomon, both with length 1;45, and "combine" them to form a square with an area of 3;03 45. This square is "added" to the gnomon forming a larger square (see figure 2 right). Since the gnomon has the same area as the initial figure (7;30 sar), the total area of the newly formed square is determined by adding 7;30 to 3;03 45, which is 10;33 45 sar. We determine the length of the side of the square by calculating the square root of this area, obtaining 3;15 rods. We then "put down" the number obtained twice. We add 1;45 (corresponding to half the difference between the length and the width) to one, obtaining the length of the trench; and we "take away" 1;45 from another, obtaining the width of the trench (in the unit of rod).

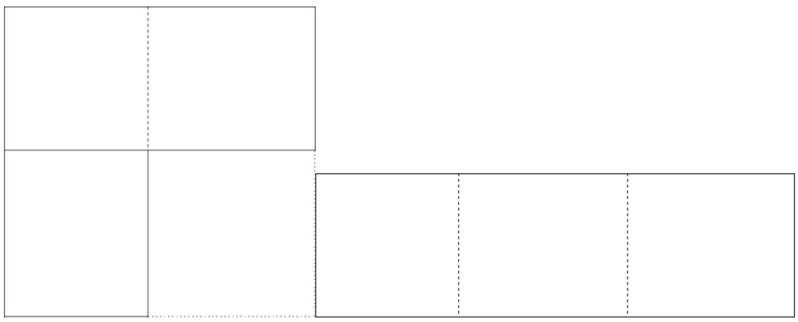

**Figure 2.** Cut-and-paste manipulations in the procedure of YBC 4663 #8.

Let us look in more detail into the steps of part A and B of the procedure. In part A the first step is in relation to the value of the wage to convert from grain to mina. This can be done simply by consulting a table that gives the relation between different metrological units and performing an arithmetic operation (a multiplication). We then calculate the reciprocal of the wage. In principle, this is done simply by consulting a table of reciprocals in which the result is given (having been previously calculated).[26] We can also imagine that at this point we have a "sub-procedure" in which we determine the value of the reciprocal (Robson 2008, 107-8). An important point in this step is that we are considering the wage of a day. In this way, its unit is not simply mina but mina/day. This means that when we determine the reciprocal of the wage its unit is day/mina.

In the procedure, we keep track of the units, since when in the next step we multiply the reciprocal of the wage by the total amount of silver (in which, previously, we converted its unit from shekel to mina), we obtain a quantity/number whose "unit" is "day", i.e. by doing the calculation we obtain a number corresponding to days: the total workdays.

In these steps, the wage is given as a number, metrologically defined in terms of a unit-weight (we also have a notion of wage per day). The number of silver is defined equivalently in terms of weight. We make different arithmetic operations corresponding to conversions between units (of the same metrological system), an "inversion operation" (which also entails inverting the unit from mina/day to day/mina), and a multiplication between numbers belonging to somewhat different metrological systems (we multiply a number in mina by a number in day/mina). This multiplication results in a number of days.

While the arithmetic operations if following the prescribed rules are correct (without any need for further justification), the meaning of the numbers changes according to the metrological units that result from the operations. This means that we must keep track of the units and how an operation with numbers, each with its metrological unit, results in a new number with a specific unit. By doing this, the result of the sequence of calculations is correct. In this way, some sort of intuitive and incipient dimensional analysis is at play in ancient mathematical procedures.[27]

In the next step of the procedure, we multiply the work rate, after converting it into the unit of sar/day, by the number of workdays (previously calculated). By keeping track of the meaning of the quantities and their units, the result of this multiplication is the total volume of the trench, as given in the volume unit of sar. Since we know the depth of the trench and its volume and, implicitly, its numerical relation to the (rectangular) area (the volume is given by the area multiplied by the depth), we can determine the value of the area of the trench. For that, we determine the reciprocal of the depth ½ rods = 6 cubits. We obtain the reciprocal in the unit of 1/cubit. By multiplying the reciprocal by the volume (in sar = rod x rod x cubit), we obtain the area in sar (rod x rod). Again, we keep track

[26] Regarding tables of reciprocals see, e.g., Proust (2016, 8-9).

[27] On the issue of dimensional analysis see, e.g., Gibbings (2011), Lemons (2017).

of the meaning of the numbers and their respective units in the arithmetic operations: we multiply the reciprocal of a length (in cubits) by a volume (in volumetric sar) and obtain an area (in sar).

This type of operation with numbers with different units can also be found in ancient Egyptian procedures (see, e.g., Ritter 2000, 122-123).[28] This, is another aspect to take into account in a more detailed articulation of the correctness of mathematical procedures. In fact, it is a basic feature of numbers in relation to metrological systems, the arithmetic operations on numbers, and the numerical relations established between numbers.

In part B of the procedure we move from an approach based mainly in arithmetic operations – applied in a metrological context in which the quantities have a concrete meaning –, into a more "geometric" approach, in which we apply cut-and-paste manipulations (similar to the ones we have seen in section 1), but still making use of arithmetic operations. Our unknowns are the length and width of a rectangle. These are in numerical relation between them (the difference between the length and width is given), and in numerical relation with the given area: by "definition" the area is given by the multiplication of the length and width (both in the unit of rod), and it has the unit of rod x rod that was called sar.

Our method consists in moving an area almost as an autonomous element, which we "cut" from the main figure and "past" again to (what was left of) the figure, forming a new figure – a gnomon –, which, afterwards, by "completion" we can take to be a square. Keeping track of the numerical information available (in this case the total area) we can determine the side of the square. From here we calculate the length and the width.

A key element in this method is the implicit idea of area "conservation". When moving area "elements" around making new configurations, in this case forming a gnomon, these have all the same total area. According to Peter Damerow, it is an assumption in Old Babylonian mathematics that "the size of a figure which consists of partial areas equals the sum of these partial areas" (Damerow 2016, 115). We can take this to be something evident, that we simply "see",[29] but also as arising from measurement practices: if we measure using rods the areas of the different configurations these have the same number.[30]

[28] This is not to say that we must always keep track of the units, step by step, in the procedure (see, e.g., Ritter 2000; Robson 2008, 78). However, one must keep track of how the sequence of operations bears on the units of resulting numbers even if just the final number, which must be given with its corresponding correct unit.

[29] According to Høyrup, we need no proof for the pertinence of the cut-and-paste manipulations; we simply "see" that what is done is correct (Høyrup 2017, 186).

[30] It is a fact that there was no notion of determining the value of a quantity within known limits of measurement error (Olesko 1995, 106); also, there was no constant standard like the metre whose constancy along the years was checked (Barrell 1962). Even if this is so, the cut-and-paste manipulations correspond to incipient measurement practices applied within the Old Babylonian metrology.

We then "add" a new area to form a "large" square. The area of the "total" figure is given by the sum of the numbers corresponding to the area of each "individual" figure (the gnomon and a "small" square). Again, we can take this result to be self-evident, but also as arising from measurement practices (this is a variation of the previous case of area "conservation").

When we "complete" the gnomon to obtain a square (a confrontation) we take advantage of the definition of square: we know that its area is given by the multiplication of the length of the two sides. We "reverse" the operation to calculate the area, and by calculating the square root of the number corresponding to the area (in units of sar = rod x rod) we "recover" the number corresponding to the length of one side (in the unit of rod). Again, we keep track of the meaning of the numbers and the associated metrological units. At this point, we have used the numerical relation between the sides of a square and its area. We now apply the numerical relations between the sides of the square and the length and width of the rectangle. These numerical relations are "derived" from the initially given relation between length and width, and we can keep track of them by reference to the drawings (see figure 2). The length of the rectangle corresponds to "summing" the length of the side of the square (3;15 rods) to half the difference between the length and the width (1;45 rods). In a similar way, the width of the rectangle is calculated by "subtracting" 1;45 rods to the side of the square.

The correctness of part B of the procedure is crucially dependent on cut-and-paste manipulations; these can be seen to rely on what we called the area "conservation". This is another feature to take into account to articulate the correctness of Old Babylonian mathematical procedures. The other elements at play have already been seen in part A of the procedure or in the procedures considered in section 2: numbers conceived in relation to metrological systems, arithmetic operations on numbers, keeping track of units during arithmetic operations, and numerical relations established between numbers.

## 4. Conclusions

While we agree with Høyrup that the correctness of Old Babylonian mathematical procedures is self-evident – and this view extends to ancient Egyptian procedures –, we still find that it is possible to articulate the correctness of a procedure. In this work, we identify elements that we consider to be crucial for the correctness of the procedures (we do not have the ambition to say that these are the only ones). Here, we have followed/reconstructed, step-by-step, procedures that we found helpful to bring to light elements that are relevant in the correctness of ancient mathematical procedures; we identify what we consider to be crucial elements that are general to ancient mathematical procedures (i.e. these elements are not particular to the procedures considered in this work but are of general effectivity regarding the correctness of Old Babylonian and ancient Egyptian mathematical procedures). The three procedures taken into account in detail in this work were chosen among the procedures available in the literature so that with only three problems we could present our views.

After some necessary introductory material in the first section, we considered, in section 2, two mathematical problems from ancient Egypt. The first one enabled to determine that the correctness of the procedure adopted to solve the problem arises from the (correct) application of arithmetic operations in a context of an established numerical relation between numbers (numbers that are defined in relation to metrological systems). A numerical relation can be seen, e.g., as established by a sequence of arithmetic operations. In the second mathematical procedure, we considered a "geometrical figure" (a rectangular triangle), which is the main difference in relation to the first procedure. We are given the area of the triangle and the ratio between two sides of the triangle (the ones making a right angle between them). One of the numerical relations taken into account in this procedure cannot be seen as resulting directly from arithmetic operations like in the previous case. It arises from numerical relations inbuilt in the definition of a triangle. In this case, the area is equal to half the multiplication of the sides. We have, so to speak, an extension of how numerical relations are established.

Next, in section 3, we considered an Old Babylonian mathematical problem. The procedure for solving this problem can be "divided" into two parts in which different elements are relevant for its correctness. The first part, besides the elements already identified in the two Egyptian procedures, enables us to notice the relevance of keeping track of the metrological units associated to each number. When making an arithmetic operation with numbers having associated metrological units, we need to take into account how the operation affects the units, in this way associating the correct unit to the number obtained in the arithmetic operation; i.e., we have an incipient form of dimensional analysis. The second part of the procedure unfolds through a more "geometric" approach, in which we apply cut-and-paste manipulations. Here, we find another element crucial for the correctness of the procedure – another "why" the procedure is correct. This element is what we have called the area "conservation". We can cut-out parts of figures, move them around, and paste them to the figures, making new ones. All the different configurations of figures have the same area. Area "conservation", implicit in the cut-and-paste manipulations, is another element given rise to the correctness of Old Babylonian mathematical procedures.

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