# Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Nonrelativistic treatment in the NCQM symmetries

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**Abstract**

In this work, the approximate analytical solutions of both the modified Klein–Gordon equation and modified Schrödinger equation have been obtained with a newly proposed potential called modified screened cosine Kratzer potential (MSCKP) under the condition of equal scalar and vector potentials. The potential is a superposition of screened cosine Kratzer potential and some exponential radial terms. The aim of combining these potentials is to have an extensive application. The energy shift and the energy eigenvalue are calculated using the procedure of improved approximation of the centrifugal term, Bopp's shift method, and perturbation theory. We show that the new energy shift depends on the global parameters characterizing the noncommutativity space-space and the potential parameter ($D_e$, $r_e$, $\alpha$) in addition to the Gamma function and the discreet atomic quantum numbers ($j$, $l$, $s$, $m$). The present results are applied in calculating both the energy spectrum for a few heterogeneous (LiH, HCl, NO) and homogeneous (H2, I2, O2) diatomic molecules. Furthermore, we have applied our study to calculation the modified mass of the heavy quarkonium system such as the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ under MSCKP. We have also discussed some special cases of physical importance.

**Keywords**: Klein-Gordon equation, Schrödinger equation, the screened cosine Kratzer potential, Noncommutative quantum mechanics, star products.

**Resumen**

En este trabajo, las soluciones analíticas aproximadas tanto de la ecuación de Klein-Gordon modificada como de la ecuación de Schrödinger modificada se han obtenido con un potencial recientemente propuesto llamado potencial de Kratzer de coseno filtrado modificado (MSCKP) bajo la condición de potenciales escalares y vectoriales iguales. El potencial es una superposición del potencial de Kratzer de coseno filtrado y algunos términos radiales exponenciales. El objetivo de combinar estos potenciales es tener una amplia aplicación. El desplazamiento de energía y el valor propio de la energía se calculan mediante el procedimiento de aproximación mejorada del término centrífugo, el método de desplazamiento de Bopp y la teoría de la perturbación. Mostramos que el nuevo desplazamiento de energía depende de los parámetros globales que caracterizan el espacio-espacio de no conmutatividad y el parámetro de potencial ($D_e$, $r_e$, $\alpha$) además de la función Gamma y los números cuánticos atómicos discretos ($j$, $l$, $s$, $m$). Los presentes resultados se aplican en el cálculo del espectro de energía para unas pocas moléculas diatómicas heterogéneas (LiH, HCl, NO) y homogéneas (H2, I2, O2). Además, hemos aplicado nuestro estudio para calcular la masa modificada del sistema de quarkonium pesado como el charmonium $c\bar{c}$ y el bottomonium $b\bar{b}$bajo MSCKP. También hemos comentado algunos casos especiales de importancia física.

**Palabras clave:** ecuación de Klein-Gordon, ecuación de Schrödinger, potencial de Kratzer del coseno filtrado, mecánica cuántica no conmutativa, productos estelares.

## I. INTRODUCTION

The energy eigenvalues and corresponding wave functions give significant information in describing various quantum systems in both the relativistic and non-relativistic regime using both Schrödinger, Klein-Gordon, and Dirac equations with different potentials. The standard Kratzer potential (Kratzer potential) [1] is one of most interest by researchers in the fields of physics and chemistry; it has been presented and investigated by many researchers. This potential has played an important role in the history of molecular and quantum chemistry [2]. Furthermore, this potential approaches infinity as the inter-nuclear distance approaches zero, due to the repulsion that exists between the molecules of the potential and has a long-range attraction and a repulsive part [3]. The energy spectrum of the diatomic CO molecule with different quantum numbers can be successfully accounted for by applying the Kratzer potential [4]. Many techniques have been developed to obtain the solutions of relativistic and nonrelativistic wave equations under the Kratzer potential such as the asymptotic iteration method [5], the factorization method [6], the algebraic approach [7], the Nikiforov–Uvarov method [8] and others. Currently, there has been great interest in combination with two or more potentials in both the relativistic and non-relativistic regimes. The aim of combining these potentials is to have an extensive application. Purohit, K. R. *et al.* have obtained an approximate solution of the Klein-Gordon and Schrödinger equation for the screened cosine Kratzer potential in D dimensions, within the framework of Nikiforov–Uvarov method. They obtained bound state energy eigenvalues for a few heterogeneous (LiH, HCl, NO) and homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules [3]. This work, motivated by many various recent studies such as the non-renormalizable of the electroweak interaction, quantum gravity, string theory, the noncommutative relativistic and nonrelativistic quantum mechanics has attracted much attention of physical researchers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The noncommutativity in space-time is not a new and modern idea, it was proposed by W. Heisenberg in 1930 and then developed by H. Snyder in 1947. In 2017, we have studied the new modified Kratzer-type potential in the context of nonrelativistic two-dimensional noncommutative quantum mechanics [19]. The main objective of this work is to develop the study of Purohit *et al.* within the framework of the Klein Gordon equation and Schrödinger equation but in the context of symmetries of noncommutative quantum mechanics. For the purpose to get more investigation in the microscopic scales and from achieving more scientific knowledge of elementary particles and diatomic molecular in the field of nano-scales. The relativistic and nonrelativistic energy levels under the modified screened cosine Kratzer potential (MSCKP) have not been obtained yet in the context of the NCQM. Furthermore, we hope to find new applications and profound physical interpretations using a new, updated model of the MSCKP, this newly potential takes the form:

$$\begin{cases} V_{sc}(r) = -\left(\frac{a_v}{r} - \frac{b_v}{2r^2}\right) e^{-\alpha r} \cosh(\delta\alpha r) \\ S_{sc}(r) = -\left(\frac{a_s}{r} - \frac{b_s}{2r^2}\right) e^{-\alpha r} \cosh(\delta\alpha r) \end{cases}, \rightarrow \begin{cases} V_{sc}(\hat{r}) \equiv V_{sc}(r) - \frac{\partial V_{sc}(r)}{\partial r}\frac{\vec{L}\vec{\Theta}}{2r} + O\left(\Theta^2\right) \\ S_{sc}(\hat{r}) \equiv S_{sc}(r) - \frac{\partial S_{sc}(r)}{\partial r}\frac{\vec{L}\vec{\Theta}}{2r} + O\left(\Theta^2\right) \end{cases}. \quad (1)$$

Where $a_v = 2D_e r_e$, $b_v = 2D_e r_e^2$ ($r_e$ is the equilibrium bond length, $D_e$ is the dissociation energy, $\delta$ and $\alpha$ are the screening parameters while $r$ is the interatomic or particle distance). The new structure of NCQM based on new covariant noncommutative canonical commutations relations CNCCRs in Schrödinger, Heisenberg, and Interactions pictures (SP, HP, and IP), respectively, as follows [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]:

$$\begin{cases} [\hat{x}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = [\hat{x}^I_\mu(t) \overset{*}{,} \hat{p}^I_\nu(t)] = i\hbar_{eff}\delta_{\mu\nu}, \\ [\hat{x}_\mu \overset{*}{,} \hat{x}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)] = [\hat{x}^I_\mu(t) \overset{*}{,} \hat{x}^I_\nu(t)] = i\theta_{\mu\nu}. \end{cases} \quad (2)$$

We are generalized the CNCCRs to include HP and IP. It should be noted that, in our calculation, we have used the natural units $c = \hbar = 1$. Here $\hbar_{eff}$ is the effective Planck constant, $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ ($\theta$ is the non-commutative parameter), which are an infinitesimals parameter if it compared to the energy values and elements of antisymmetric 3 × 3 real matrix and $\delta_{\mu\nu}$ is the identity matrix. The symbol $(*)$ denotes to the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new modified form $\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv f(x) * g(x)$ in the symmetries of (RNC: 3D-RS) as follows [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]:

$$(fg)(x) \rightarrow (f * g)(x) = exp\left(i\theta\varepsilon^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}\right) f(x_\mu) g(x_\nu),$$

$$\cong fg(x) - \frac{i\varepsilon^{\mu\nu}}{2}\theta\, \partial^x_\mu f\, \partial^x_\nu g\Big|_{x_\mu = x_\nu} + O(\theta^2). \quad (3)$$

The indices $\mu, \nu \equiv \overline{1,3}$ and $O(\theta^2)$ stands for the second and higher-order terms of the NC parameter. Physically, the second term in Eq. (3) presents the effects of space-space noncommutativity properties. Furthermore, the new unified two operators $\hat{\xi}^H_\mu(t) = (\hat{x}_\mu \text{or } \hat{p}_\mu)(t)$ and $\hat{\xi}^I_\mu(t) = (\hat{x}^I_\mu \text{or } \hat{p}^I_\mu)(t)$ in HP and IP are depending on the corresponding new operators $\hat{\xi}^H_\mu \equiv \hat{x}_\mu \text{or } \hat{p}_\mu$ in SP from the following projections relations, respectively:

$$\begin{cases} \xi^H_\mu(t) = exp\left(i\hat{H}^{sc}_r T\right) \xi^S_\mu\, exp\left(-i\hat{H}^{sc}_r T\right), \\ \xi^I_\mu(t) = exp\left(i\hat{H}^{sc}_{or} T\right) \xi^S_\mu\, exp\left(-i\hat{H}^{sc}_{or} T\right), \end{cases} \Rightarrow$$
$$\begin{cases} \hat{\xi}^H_\mu(t) = exp\left(i\hat{H}^{sc}_{nc-r} T\right) * \hat{\xi}^S_\mu * exp\left(-i\hat{H}^{sc}_{nc-r} T\right), \\ \hat{\xi}^I_\mu(t) = exp\left(i\hat{H}^{sc}_{nc-or} T\right) * \hat{\xi}^S_\mu * exp\left(-i\hat{H}^{sc}_{nc-or} T\right). \end{cases} \quad (4)$$

Where $T = t - t_0$, the three unified coordinates $\xi^S_\mu \equiv (x_\mu \text{ or } p_\mu)$, $\xi^H_\mu(t) \equiv (x_\mu \text{or } p_\mu)(t)$ and $\xi^I_\mu(t) \equiv (x^I_\mu \text{or } p^I_\mu)(t)$ are represented in three relativistic quantum mechanics pictures, while the dynamics of new systems $\frac{d\hat{\xi}_H(t)}{dt}$ are described from the following motion equations in the modified Heisenberg picture as follows:

$$\frac{d\xi^H_\mu(t)}{dt} = \left[\xi^H_\mu(t), \hat{H}^{sc}_r\right] + \frac{\partial\xi^H_\mu(t)}{\partial t} \Rightarrow$$
$$\frac{d\hat{\xi}_H(t)}{dt} = \left[\hat{\xi}^H_\mu(t) \overset{*}{,} \hat{H}^{sc}_{nc-r}\right] + \frac{\partial\hat{\xi}^H_\mu(t)}{\partial t}. \quad (5)$$

The operators $\hat{H}^{sc}_{or}$ and $\hat{H}^{sc}_r$ are the free and global Hamiltonian for SCKP while $\hat{H}^{sc}_{nc-or}$ and $\hat{H}^{sc}_{nc-r}$ the corresponding Hamiltonians for the MSCKP. The present investigation aims at constructing a relativistic noncommutative effective scheme for the MSCKP. The outline of the paper organizes as follows: In the next section, we briefly review the Klein-Gordon equation KGE with SCKP. Sect. 3 is devoted to studying the modified Klein-Gordon equation MKGE by applying the ordinary Bopp's shift method and improved approximation of the centrifugal term to obtain the effective potential of MSCKP. Besides, via perturbation theory we find the expectation values of some radial terms to calculate the energy shift produced with the effect of the perturbed effective potential. Sect. 4 is devoted to present the global energy shift and the global energy spectra produced with MSCKP in the RNCQM symmetries. In Sect. 5, we treated the energy spectra for a few heterogeneous (LiH, HCl, NO) and homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules in three-dimensional noncommutative nonrelativistic (NC: 3D-RS) under MSCKP. The next section is devoted to determining the new masses of the heavy quarkonium system such as the charmonium$c\overline{c}$ and bottomonium $b\overline{b}$ under MSCKP. We generate the solutions of a few special potentials mainly found from our general form solution in Sect. 7. Eventually, in Sect. 8, we presented our conclusion of this paper.

## II. REVISED OF EIGENFUNCTIONS AND THE EIGENVALUES FOR THE SCREENED COSINE KRATZER POTENTIAL IN RQM

As already mentioned our objective is to obtain the spectrum of the MKGE with MSCKP in (RNC: 3D-RSP) symmetries, we need to revise the corresponding screened cosine Kratzer potential model in symmetries of ordinary relativistic quantum mechanics RQM [3]:

$$V_{sc}(r) = -\left(\frac{a_v}{r} - \frac{b_v}{2r^2}\right) e^{-\alpha r} \cosh(\delta \alpha r),$$
$$S_{sc}(r) = -\left(\frac{a_s}{r} - \frac{b_s}{2r^2}\right) e^{-\alpha r} \cosh(\delta \alpha r). \quad (6)$$

To achieve this goal of our current research it is useful to make a summary for the Klein–Gordon equation KGE with screened cosine Kratzer potential for a system of reduced mass$\mu$in three-dimensional relativistic quantum mechanics [3]:

$$\{-\Delta + (\mu + S_{sc}(r))^2 - (E_{nl} - V_{sc}(r))^2\}\Psi(r,\theta,\phi) = 0 \Rightarrow$$
$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (E_{nl}^2 - M^2) - 2(E_{nl}V_{sc}(r) + \mu S_{sc}(r)) + V_{sc}^2(r) - S_{sc}^2(r) - \frac{l(l+1)}{r^2}\right\} R_{nl}(r) = 0 \quad (7)$$

The vector potential $V_{sc}(r)$ due to the four-vector linear momentum operator $A^{\mu}\left(V_{sc}(r), \vec{A} = 0\right)$ and space-time scalar potential$S_{sc}(r)$ while $E_{nl}$ represents the relativistic energy eigenvalues in 3-dimensions and $l$ represents the principal and orbital quantum numbers, respectively. Since the screened cosine Kratzer, potential has spherical symmetry, allowing the solutions of the time-independent KGE of the known form$\Psi(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$ to separate the radial $R_{nl}(r)$ and angular $Y_l^m(\theta,\phi)$parts of the wave function and $\Delta$ is the ordinary 3-dimensional Laplacian operator. To eliminate the first-order derivative, we introduce a new radial wave function to the form$\chi_{nl}(r) = rR_{nl}(r)$, thus Eq. (7) becomes:

$$\left\{\frac{d^2}{dr^2} - (\mu^2 - E_{nl}^2) - 2\left(E_{nl}V_{sc}(r) + \mu S_{sc}(r)\right) + V_{sc}^2(r) - S_{sc}^2(r) - \frac{l(l+1)}{r^2}\right\}\chi_{nl}(r) = 0, \quad (8)$$

If we introduce the shorthand notation $V_{eff}^{sc}(r) \equiv 2\left(E_{nl}V_{sc}(r) + \mu S_{sc}(r)\right) - V_{sc}^2(r) + S_{sc}^2(r) + \frac{l(l+1)}{r^2}$ and $E_{eff}^{sc} \equiv \mu^2 - E_{nl}^2$, we obtain the following second-order Schrodinger-like equation:

$$\left\{\frac{d^2}{dr^2} - \left(E_{eff}^{sc} + V_{eff}^{sc}(r)\right)\right\}\chi_{nl}(r) = 0. \quad (9)$$

For the equal vector and scalar potential$V_{sc}(r) = S_{sc}(r)$, the effective potential reduces to the form:

$$V_{eff}^{sc}(r) \equiv 2(E_{nl} + \mu)V_{sc}(r) + \frac{l(l+1)}{r^2} \quad (10)$$

In the case of the nonrelativistic limit, Eq. (8) Becomes a Schrödinger equation with the interaction potential$2V_{sc}(r)$. To aim for $V_{sc}(r)$, not $2V_{sc}(r)$in the interaction potential under the nonrelativistic limit, (Kaushal R. Purohit *et al.*) and (Xiang-Jun Xie *et al.).* applied the scheme proposed by Alhaidari *et al*. [3, 42, 43] to rescale the vector potential $V_{sc}(r)$and scalar potential $S_{sc}(r)$, and rewrite Eq. (8) in the form of

$$\left\{\frac{d^2}{dr^2} - (\mu^2 - E_{nl}^2) + 2D_e(E_{nl} + \mu)\left(\frac{r_e}{r} - \frac{r_e^2}{2r^2}\right) e^{-\alpha r} \cosh(\delta \alpha r) - \frac{l(l+1)}{r^2}\right\}\chi_{nl}(r) = 0. \quad (11)$$

The Ref. [3] gives the energy eigenvalues $E_{nl}$of the KGE with equal scalar and vector potential $\frac{V_{sc}(r)}{2} = \frac{S_{sc}(r)}{2}$and the corresponding total wave function $\Psi(r,\theta,\phi)$ as follows:

$$E_{nl}^2-\mu^2=4\alpha^2 l(l+1)+2D_e(E_{nl}+\mu)(\alpha^2 r_e^2-\alpha r_e)-4\alpha^2\left[\frac{l(l+1)+D_e(E_{nl}+\mu)(\alpha^2 r_e^2-\alpha r_e)\left(\frac{r_e^2}{2}-\frac{r_e}{\alpha}\right)}{2\left(n+\frac{1}{2}+\sqrt{D_e r_e^2(E_{nl}+\mu)+\left(l+\frac{1}{2}\right)^2}\right)}\right]^2. \quad (12)$$

And

$$\Psi(r,\theta,\phi)=N_{nl}\frac{1}{r}u^{k_{nl}}(1-u)^{\zeta_{nl}}P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)Y_l^m(\theta,\phi)\,. \quad (13)$$

Here $\zeta_{nl}=1/2+\sqrt{D_e r_e^2(E_{nl}+\mu)+(l+1/2)^2}$, $u=exp(-2\alpha r)$, $k_{nl}=\sqrt{\frac{\mu^2-E_{nl}^2}{4\alpha^2}+l(l+1)+\frac{D_e(E_{nl}+\mu)(\alpha^2 r_e^2-\alpha r_e)}{2\alpha^2}}$ and $N_{nl}=\sqrt{\frac{\alpha n!(2\zeta_{nl}+1)\Gamma(2k_{nl}+2\zeta_{nl}+n+1)}{2^{2k_{nl}+2\zeta_{nl}-2}\Gamma(2k_{nl}+n+1)\Gamma(2\zeta_{nl}+n+1)}}$ is the normalization constant.

## III. SOLUTIONS OF MKGE UNDER MSCKP IN (RNC: 3D-RS) SYMMETRIES

At the beginning of this section, we shall give and to define a formula of MSCKP in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). To achieve this goal it is useful to write the MKGE by applying the notion of Weyl Moyal star product which have seen previously in Eq. (3), on the differential equation that satisfied by the radial wave function $\chi_{nl}(r)$ in Eq. (11), thus, the radial wave function $\chi_{nl}(r)$ in (RNC: 3D-RS) symmetries becomes as follows **[**44, 45, 46, 47, 48**]**:

$$\left\{\frac{d^2}{dr^2}-(\mu^2-E_{nl}^2)+2D_e(E_{nl}+\mu)\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)e^{-\alpha r}cosh(\delta\alpha r)-\frac{l(l+1)}{r^2}\right\}\chi_{nl}(r)=0\Rightarrow$$
$$\left\{\frac{d^2}{dr^2}-(\mu^2-E_{nl}^2)+2D_e(E_{nl}+\mu)\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)e^{-\alpha r}cosh(\delta\alpha r)-\frac{l(l+1)}{r^2}\right\}*\chi_{nl}(r)=0. \quad (14)$$

It is well known that the Bopp's shift method has been applied effectively and has succeeded in simplifying the three basic equations: modified Schrödinger equation MSE, MKGE and modified Dirac equation MDE with the notion of star product to the Schrödinger equation SE, KGE and Dirac equation DE with the notion of ordinary product, respectively. The results of the applications of this method were very useful and yielded promising results in many physical and chemical fields. The method reduced MSE, MKGE, and MDE, to the SE, KGE, and DE, respectively, under the simultaneous translation in space-phase. The CNCCRs with star product in Eq. (2) become new CCCRs without the notion of star product as follows [38, 39, 40, 41, 42, 43, 44, 45]:

$$\left[\hat{x}_\mu^S,\hat{x}_\nu^S\right]=\left[\hat{x}_\mu^H(t),\hat{x}_\nu^H(t)\right]=\left[\hat{x}_\mu^I(t),\hat{x}_\nu^I(t)\right]=i\theta_{\mu\nu}. \quad (15)$$

The generalized positions and momentum coordinates $\left(\hat{x}_\mu^S,\hat{p}_\mu^S\right)$, $\left(\hat{x}_\mu^H,\hat{p}_\mu^H\right)(t)$ and $\left(\hat{x}_\mu^I,\hat{p}_\mu^I\right)(t)$in the symmetries (RNC: 3D-RS) and (NC: 3D-RS) are defined in terms of the corresponding coordinates $\left(x_\mu^S,p_\mu^S\right)$, $\left(x_\mu^H,p_\mu^H\right)(t)$ and $\left(x_\mu^I,p_\mu^I\right)(t)$ in the symmetries of RQM And QM via, respectively [38, 39, 40, 41, 42, 43, 44]:

$$\left(x_\mu^S,p_\mu^S\right)\Rightarrow\left(\hat{x}_\mu^S=x_\mu^S-\frac{\theta_{\mu\nu}}{2}p_\nu^S,\hat{p}_\mu^S=p_\mu^S\right)$$
$$\left(x_\mu^H,p_\mu^H\right)(t)\Rightarrow\left(\hat{x}_\mu^H(t)=x_\mu^H(t)-\frac{\theta_{\mu\nu}}{2}p_\nu^H(t),\hat{p}_\mu^H=p_\mu^H(t)\right)$$
$$\left(x_\mu^I,p_\mu^I\right)(t)\Rightarrow\left(\hat{x}_\mu^I(t)=x_\mu^I(t)-\frac{\theta_{\mu\nu}}{2}p_\nu^I(t),\hat{p}_\mu^I(t)=p_\mu^H(t)\right) \quad (16)$$

This allows us to find the operator $r^2\Rightarrow r_{nc}^2=r^2-\vec{L}\vec{\Theta}$ in the symmetries of (RNC: 3D-RS) and (NC: 3D-RS) [47, 48, 49, 50]. It is convenient to introduce a shorthand notation which will save us a lot of writing $r_{nc}\rightarrow\hat{r}$ the previously relation reduced to the$r^2\Rightarrow\hat{r}^2=r^2-\vec{L}\vec{\Theta}$. The coupling $\vec{L}\vec{\Theta}$ equals$\left(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}\right)$ , here $L_x$,$L_y$and$L_z$ are present the usual components of angular momentum operator $\vec{L}$in RQM while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. According to Bopp's shift method, Eq. (14) becomes similar to the following like the Schrödinger equation (without the notions of star product):

$$\left\{\frac{d^2}{dr^2}-(\mu^2-E_{nl}^2)+2D_e(E_{nl}+\mu)\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)e^{-\alpha r}cosh(\delta\alpha r)-\frac{l(l+1)}{r^2}\right\}*\chi_{n_rl}(r)=0\Rightarrow$$
$$\left\{\frac{d^2}{dr^2}-(\mu^2-E_{nl}^2)+2D_e(E_{nl}+\mu)\left(\frac{r_e}{\hat{r}}-\frac{r_e^2}{2\hat{r}^2}\right)e^{-\alpha\hat{r}}cosh(\delta\alpha\hat{r})-\frac{l(l+1)}{\hat{r}^2}\right\}\chi_{n_rl}(r)=0. \quad (17)$$

For the simplifier we introduce the reduced potential $V_{sc}^r(r)=D_e\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)e^{-\alpha r}cosh(\delta\alpha r)$ and for$\delta=1$, we can obtain$V_{sc}^r(r)=D_e\left(\frac{r_e}{2r}-\frac{r_e^2}{4r^2}\right)(1+e^{-2\alpha r})$. At the first order of the parameter, $\Theta$we apply the Taylor expansion as follows:

$$V_{sc}^{r}(\hat{r})=V_{sc}^{r}(r)-\frac{\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}}{2r}\frac{\partial V_{sc}^{r}(r)}{\partial r}+O(\Theta^{2}) \ . \quad (18)$$

Now, it's easy to obtain the following results:

$$\frac{\partial V_{sc}^{r}(r)}{\partial r}=-D_e\left(\frac{r_e}{2r^2}-\frac{r_e^2}{2r^3}\right)(1+e^{-2\alpha r})-2\alpha D_e\left(\frac{r_e}{2r}-\frac{r_e^2}{4r^2}\right)e^{-2\alpha r} \text{ and } \frac{1}{\hat{r}^2}\approx\frac{1}{\hat{r}^2}+\frac{\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}}{r^4}+O(\Theta^2). \quad (19)$$

So far, we can rewrite the new modified radial part (new differential equation) of the MKGE in the symmetries of (RNC: 3D-RS) as follows:

$$\left\{\frac{d^2}{dr^2}-(\mu^2-E_{nl}^2)-2(E_{nl}+\mu)V_{sc}^{r}(r)-\frac{l(l+1)}{r^2}+2(E_{nl}+\mu)\frac{\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}}{2r}\frac{\partial V_{sc}^{r}(r)}{\partial r}-\frac{l(l+1)}{r^4}\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}\right\}\chi_{nl}(r)=0. \quad (20)$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol $V_{pert}^{sc}(r)$, thus the radial Eq. (20) becomes:

$$\left\{\frac{d^2}{dr^2}-\left[E_{eff}^{sc}+V_{nc-eff}^{sc}(r)\right]\right\}\chi_{nl}(r)=0, \quad (21)$$

with:

$$V_{nc-eff}^{sc}(r)=V_{r-eff}^{sc}(r)+V_{pert}^{sc}(r). \quad (22)$$

Moreover, $V_{r-eff}^{sc}(r)$ and $V_{pert}^{sc}(r)$ are given by the following relations:

$$V_{r-eff}^{sc}(r)\equiv 2(E_{nl}+\mu)V_{sc}^{r}(r)+\frac{l(l+1)}{r^2}. \quad (23)$$

And

$$V_{pert}^{sc}(r)=\left[\frac{l(l+1)}{r^4}-\frac{2(E_{nl}+\mu)}{2r}\frac{\partial V_{sc}^{r}(r)}{\partial r}\right]\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}. \quad (24)$$

Substituting Eq. (19) in (24) gives

$$V_{pert}^{sc}(r)=\left[\frac{l(l+1)}{r^4}-2(E_{nl}+\mu)D_e\left\{-\left(\frac{r_e}{4r^3}-\frac{r_e^2}{4r^4}\right)(1+e^{-2\alpha r})-\alpha\left(\frac{r_e}{2r^2}-\frac{r_e^2}{4r^3}\right)e^{-2\alpha r}\right\}^{-2\alpha r}\right]\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}. \quad (25)$$

A direct simplification gives

$$V_{pert}^{sc}(r)=\left[\frac{2l(l+1)+(E_{nl}+\mu)D_e r_e^2}{2r^4}+(E_{nl}+\mu)D_e\left\{-r_e^2\frac{e^{-2\alpha r}}{2r^4}+\frac{r_e}{2r^3}+(r_e-\alpha r_e^2)\frac{e^{-2\alpha r}}{2r^3}+\alpha r_e\frac{e^{-2\alpha r}}{r^2}\right\}\right]\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}. \quad (26)$$

Eq. (21) cannot be solved analytically for any state because of the centrifugal term and the studied potential itself. The effective perturbative potential is given in Eq. (23) has a strong singularity $r\rightarrow 0$; we need to use the improved approximation of the centrifugal term proposed by Badawi *et al*. [51], this method proved its power and efficiency when compared with Greene and Aldrich approximation [52]. The approximations type suggested by (Greene and Aldrich) and (Dong *et al*.) for a short-range potential that is an excellent approximation to the centrifugal term [3, 53, 54, 55, 56, 57]:

$$\frac{1}{r^2}\approx\frac{4\alpha^2}{(1-exp(-2\alpha r))^2} \ \Rightarrow \frac{1}{r}\approx\frac{2\alpha}{1-exp(-2\alpha r)}. \quad (27)$$

This allows us to rewrite the terms of the effective perturbative potential given in Eq. (26) as follows:

$$\frac{1}{r^2}\approx\frac{4\alpha^2}{(1-exp(-2\alpha r))^2}=\frac{4\alpha^2}{(1-u)^2}\Rightarrow\frac{1}{r^4}\approx\frac{16\alpha^4}{(1-u)^4}, \quad \frac{1}{r^2}\approx\frac{4\alpha^2}{(1-exp(-2\alpha r))^2}=\frac{4\alpha^2}{(1-u)^2}\Rightarrow\frac{e^{-2\alpha r}}{r^4}\approx\frac{16\alpha^4 u}{(1-u)^4}$$

$$\frac{1}{r}\approx\frac{2\alpha}{1-exp(-2\alpha r)}=\frac{2\alpha}{1-u}\Rightarrow\frac{1}{r^3}\approx\frac{8\alpha^3}{(1-u)^3}, \quad \frac{1}{r}\approx\frac{2\alpha}{1-exp(-2\alpha r)}=\frac{2\alpha}{1-u}\Rightarrow\frac{e^{-2\alpha r}}{r^3}\approx\frac{8\alpha^3 u}{(1-u)^3},$$

$$\text{and} \qquad \frac{1}{r^2}\approx\frac{4\alpha^2}{(1-exp(-2\alpha r))^2}=\frac{4\alpha^2}{(1-u)^2}\Rightarrow\frac{e^{-2\alpha r}}{r^2}\approx\frac{4\alpha^2 u}{(1-u)^2}. \quad (28)$$

Thus, the new form of the perturbative effective potential is given by:

$$V_{pert}^{sc}(u)=\left[\frac{8\left(2l(l+1)+(E_{nl}+\mu)D_e r_e^2\right)\alpha^4}{(1-u)^4}+(E_{nl}+\mu)D_e\left\{-\frac{8\alpha^4 r_e^2 u}{(1-u)^4}+\frac{4r_e\alpha^3}{(1-u)^3}+\frac{4\alpha^3(r_e-\alpha r_e^2)u}{(1-u)^3}+\frac{4r_e\alpha^3 u}{(1-u)^2}\right\}\right]\overrightarrow{\mathrm{L}}\overrightarrow{\Theta}. \quad (29)$$

The SCKP is extended by including new terms proportional with the radial terms $\frac{1}{(1-u)^4}$, $\frac{u}{(1-u)^4}$ , $\frac{1}{(1-u)^3}$, $\frac{u}{(1-u)^3}$ and $\frac{u}{(1-u)^2}$ to becomes MSCKP in (RNC-3D: RSP) symmetries. The additive part $V_{pert}^{sc}(r)$ of the new effective potential $V_{nc-eff}^{sc}(r)$ is also proportional to the infinitesimal vector $\overrightarrow{\Theta}=\Theta_{11}e_x+\Theta_{12}e_y+\Theta_{13}e_z$. This allows us to consider the additive effective potential $V_{pert}^{sc}(r)$ as a perturbation potential compared with the main potential (parent potential operator$V_{eff}^{sc}(r)$) in the symmetries of (RNC: 3D-RS), that is, the inequality $V_{pert}^{sc}(r) << V_{r-eff}^{sc}(r)$has become achieved. That is all the physical justifications for applying the time-independent perturbation theory to become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized $n^{th}$ excited states. Now, we apply the perturbative theory, in the case of RNCQM, we find the expectation values of the radial terms$\frac{1}{(1-u)^4}$, $\frac{u}{(1-u)^4}$, $\frac{1}{(1-u)^3}$, $\frac{u}{(1-u)^3}$ and $\frac{u}{(1-u)^2}$taking into account the wave function which we have seen previously in Eq. (13). Thus, after straightforward calculations, we obtain the following results:

$$\left\langle n,l,m\left|\frac{1}{(1-u)^4}\right|n,l,m\right\rangle = N_{nl}^2\int_0^{+\infty} u^{2k_{nl}}(1-u)^{2\zeta_{nl}}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)\right]^2\frac{1}{(1-u)^4}dr,$$

$$\left\langle n,l,m\left|\frac{u}{(1-u)^4}\right|n,l,m\right\rangle = N_{nl}^2\int_0^{+\infty} u^{2k_{nl}}(1-u)^{2\zeta_{nl}}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)\right]^2\frac{u}{(1-u)^4}dr,$$

$$\left\langle n,l,m\left|\frac{1}{(1-u)^3}\right|n,l,m\right\rangle = N_{nl}^2\int_0^{+\infty} u^{2k_{nl}}(1-u)^{2\zeta_{nl}}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)\right]^2\frac{1}{(1-u)^3}dr,$$

$$\left\langle n,l,m\left|\frac{u}{(1-u)^3}\right|n,l,m\right\rangle = N_{nl}^2\int_0^{+\infty} u^{2k_{nl}}(1-u)^{2\zeta_{nl}}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)\right]^2\frac{u}{(1-u)^3}dr,$$

$$\left\langle n,l,m\left|\frac{u}{(1-u)^2}\right|n,l,m\right\rangle = N_{nl}^2\int_0^{+\infty} u^{2k_{nl}}(1-u)^{2\zeta_{nl}}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(1-2u)\right]^2\frac{u}{(1-u)^2}dr. \quad (30)$$

We have applied the property of the spherical harmonics, which has the form$\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\sin(\theta)\,d\theta d\phi = \delta_{ll'}\delta_{mm'}$. We have$u=exp(-2\alpha r)$, this allows us to obtain $dr=-\frac{1}{2\delta}\frac{ds}{s}$. After introducing a new variable$z=1-2u$, we have$u=\frac{1-z}{2}$,$dr=\frac{1}{2\alpha}\frac{dz}{1-z}$ and$1-u=\frac{z+1}{2}$. From the asymptotic behavior of Eq. (21), when $r\to 0$ $(z\to -1)$ and$r\to+\infty$ $(z\to 1)$, this allows to reformulating Eq. (30) as follows:

$$A_{nlm} \equiv \left\langle n,l,m\left|\frac{1}{(1-u)^4}\right|n,l,m\right\rangle = \frac{N_{nl}^2}{2^{2\zeta_{nl}+2k_{nl}-3}\alpha}\int_{-1}^{+1}(1-z)^{2k_{nl}-1}(1+z)^{2\zeta_{nl}-4}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(z)\right]^2dz,$$

$$B_{nlm} \equiv \left\langle n,l,m\left|\frac{u}{(1-u)^4}\right|n,l,m\right\rangle = \frac{N_{nl}^2}{2^{2\zeta_{nl}+2k_{nl}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{nl}}(1+z)^{2\zeta_{nl}-4}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(z)\right]^2dz,$$

$$C_{nlm} \equiv \left\langle n,l,m\left|\frac{1}{(1-u)^3}\right|n,l,m\right\rangle = \frac{N_{nl}^2}{2^{2\zeta_{nl}+2k_{nl}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{nl}-1}(1+z)^{2\zeta_{nl}-3}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(z)\right]^2dz,$$

$$D_{nlm} \equiv \left\langle n,l,m\left|\frac{u}{(1-u)^3}\right|n,l,m\right\rangle = \frac{N_{nl}^2}{2^{2\zeta_{nl}+2k_{nl}-1}\alpha}\int_{-1}^{+1}(1-z)^{2k_{nl}}(1+z)^{2\zeta_{nl}-3}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(z)\right]^2dz,$$

$$E_{nlm} \equiv \left\langle n,l,m\left|\frac{u}{(1-u)^2}\right|n,l,m\right\rangle = \frac{N_{nl}^2}{2^{2\zeta_{nl}+2k_{nl}}\alpha}\int_{-1}^{+1}(1-z)^{2k_{nl}}(1+z)^{2\zeta_{nl}-2}\left[P_n^{(2k_{nl},2\zeta_{nl}-1)}(z)\right]^2dz. \quad (31)$$

For relieving the burden of writing, we will provide useful abbreviations$\langle n,l,m|A|n,l,m\rangle \equiv \langle A\rangle_{(n,l,m)} \equiv A_{nlm}$. For the ground state$n=0$, we have$P_0^{(2k_{0l},2\zeta_{0l}-1)}(z)$, thus the above expectation values in Eq. (31) reduce to the following simple form:

$$\left\langle\frac{1}{(1-u)^4}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2}{2^{2\zeta_{0l}+2k_{0l}-3}\alpha}\int_{-1}^{+1}(1-z)^{2k_{0l}-1}(1+z)^{2\zeta_{0l}-4}dz\,,$$

$$\left\langle\frac{u}{(1-u)^4}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2}{2^{2\zeta_{0l}+2k_{0l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{0l}}(1+z)^{2\zeta_{0l}-4}dz,$$

$$\left\langle\frac{1}{(1-u)^3}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2}{2^{2\zeta_{0l}+2k_{0l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{0l}-1}(1+z)^{2\zeta_{0l}-3}dz\,,$$

$$\left\langle\frac{u}{(1-u)^3}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2^{2\zeta_{0l}+2k_{0l}-1}\alpha}\int_{-1}^{+1}(1-z)^{2k_{0l}}(1+z)^{2\zeta_{0l}-3}dz,$$

$$\left\langle\frac{u}{(1-u)^2}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2^{2\zeta_{0l}+2k_{0l}}\alpha}\int_{-1}^{+1}(1-z)^{2k_{0l}}(1+z)^{2\zeta_{0l}-2}dz. \quad (32)$$

Where $\zeta_{0l}=1/2+\sqrt{D_e r_e^2(E_{0l}+\mu)+(l+1/2)^2}$, $k_{0l}=\sqrt{\frac{\mu^2-E_{0l}^2}{4\alpha^2}+l(l+1)+\frac{D_e(E_{0l}+\mu)(\alpha^2r_e^2-\alpha r_e)}{2\alpha^2}}$ and $N_{0l}=\sqrt{\frac{\alpha(2\zeta_{0l}+1)\Gamma(2k_{0l}+2\zeta_{0l}+1)}{2^{2k_{0l}+2\zeta_{0l}-2}\Gamma(2k_{0l}+1)\Gamma(2\zeta_{0l}+1)}}$. Comparing Eq. (32) with the integral of the form [58]:

$$\int_{-1}^{+1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn}\Rightarrow$$

$$\int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx=\frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)}\quad \text{for } (n=0,1,....). \quad (33)$$

A direct calculation gives the expectation values in Eq. (32) as follows:

$$B_{0lm}\equiv\left\langle\frac{u}{(1-u)^4}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2\alpha}\frac{\Gamma(2k_{0l}+1)\Gamma(2\zeta_{0l}-3)}{(2k_{0l}+2\zeta_{0l}-3)\Gamma(2k_{0l}+2\zeta_{0l}-3)},$$
$$C_{0lm}\equiv\left\langle\frac{1}{(1-u)^3}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2\alpha}\frac{\Gamma(2k_{0l})\Gamma(2\zeta_{0l}-2)}{(2k_{0l}+2\zeta_{0l}-3)\Gamma(2k_{0l}+2\zeta_{0l}-3)},$$
$$D_{0lm}\equiv\left\langle\frac{u}{(1-u)^3}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2\alpha}\frac{\Gamma(2k_{0l}+1)\Gamma(2\zeta_{0l}-2)}{(2k_{0l}+2\zeta_{0l}-2)\Gamma(2k_{0l}+2\zeta_{0l}-2)},$$
$$E_{0lm}\equiv\left\langle\frac{u}{(1-u)^2}\right\rangle_{(0,l,m)}=\frac{N_{0l}^2}{2\alpha}\frac{\Gamma(2k_{0l}+1)\Gamma(2\zeta_{0l}+1)}{(2k_{0l}+2\zeta_{0l}-1)\Gamma(2k_{0l}+2\zeta_{0l}-1)}. \quad (34)$$

Where $\beta_{0l}\equiv 2\varepsilon_{0l}+2k_l$. For the first excited state $n=1$, we have $P_1^{(2k_{1l},2\zeta_{1l}-1)}(z)=g_{1l}+h_{1l}(1-z)$, with $g_{1l}=2k_{1l}+1$, $h_{1l}=-(k_{1l}+\zeta_{1l}+1/2)$, the expectation values in Eq. (31) are reduced to the following simple form:

$$\left\langle\frac{1}{(1-u)^4}\right\rangle_{(1,l,m)}\equiv A_{1lm}\equiv T_{11}+T_{12}+T_{13},$$
$$\left\langle\frac{u}{(1-u)^4}\right\rangle_{(1,l,m)}\equiv B_{1lm}\equiv T_{21}+T_{22}+T_{23},$$
$$\left\langle\frac{1}{(1-u)^3}\right\rangle_{(1,l,m)}\equiv C_{1lm}\equiv T_{31}+T_{32}+T_{33},$$
$$\left\langle\frac{u}{(1-u)^3}\right\rangle_{(1,l,m)}\equiv D_{1lm}\equiv T_{41}+T_{42}+T_{43},$$
$$\left\langle\frac{u}{(1-u)^2}\right\rangle_{(1,l,m)}\equiv E_{1lm}\equiv T_{51}+T_{52}+T_{53}. \quad (35)$$

where the 15-elements $T_{ij}$ are given by:

$$\begin{pmatrix}T_{11}\\T_{12}\\T_{13}\end{pmatrix}=\begin{pmatrix}\frac{g_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-3}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}-1}(1+z)^{2\zeta_{1l}-4}dz\\\frac{h_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-3}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+1}(1+z)^{2\zeta_{1l}-4}dz\\\frac{2g_{1l}h_{1l}N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-3}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}}(1+z)^{2\zeta_{1l}-4}dz)\end{pmatrix},\begin{pmatrix}T_{21}\\T_{22}\\T_{23}\end{pmatrix}=\begin{pmatrix}\frac{g_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}}(1+z)^{2\zeta_{1l}-4}dz\\\frac{h_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+2}(1+z)^{2\zeta_{1l}-4}dz\\\frac{2g_{1l}h_{1l}N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+1}(1+z)^{2\zeta_{1l}-4}dz)\end{pmatrix},\begin{pmatrix}T_{31}\\T_{32}\\T_{33}\end{pmatrix}=$$
$$\begin{pmatrix}\frac{g_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}-1}(1+z)^{2\zeta_{1l}-3}dz\\\frac{h_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+1}(1+z)^{2\zeta_{1l}-3}dz\\\frac{2g_{1l}h_{1l}N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}}(1+z)^{2\zeta_{1l}-3}dz\end{pmatrix},$$
$$\begin{pmatrix}T_{41}\\T_{42}\\T_{43}\end{pmatrix}=\begin{pmatrix}\frac{g_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-1}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}}(1+z)^{2\zeta_{1l}-3}dz\\\frac{h_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-1}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+2}(1+z)^{2\zeta_{1l}-3}dz\\\frac{2g_{1l}h_{1l}N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}-1}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+1}(1+z)^{2\zeta_{1l}-3}dz)\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}T_{51}\\T_{52}\\T_{53}\end{pmatrix}=$$
$$\begin{pmatrix}\frac{g_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}}(1+z)^{2\zeta_{1l}-2}dz+\\\frac{h_{1l}^2N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+2}(1+z)^{2\zeta_{1l}-2}dz\\\frac{2g_{1l}h_{1l}N_{1l}^2}{2^{2\zeta_{1l}+2k_{1l}}\alpha}\int_{-1}^{+1}(1-z)^{2k_{1l}+1}(1+z)^{2\zeta_{1l}-2}dz)\end{pmatrix}. \quad (36)$$

We apply the integral in Eq. (33) to obtain the following results:

$$\begin{pmatrix}T_{11}\\T_{12}\\T_{13}\end{pmatrix}=\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l})\Gamma(2\zeta_{1l}-3)}{(X_{1l}-4)\Gamma(X_{1l}-4)}\\4h_{1l}^2\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\\4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-3)\Gamma(X_{1l}-3)}\end{pmatrix},\begin{pmatrix}T_{21}\\T_{22}\\T_{23}\end{pmatrix}=\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-3)\Gamma(X_{1l}-3)}\\4h_{1l}^2\frac{\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-1)\Gamma(X_{1l}-1)}\\4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\end{pmatrix},\begin{pmatrix}T_{31}\\T_{32}\\T_{33}\end{pmatrix}=\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l})\Gamma(2\zeta_{1l}-2)}{(X_{1l}-3)\Gamma(X_{1l}-3)}\\4h_{1l}^2\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-1)\Gamma(X_{1l}-1)}\\4_{1l}h_{1l}\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\end{pmatrix},$$

$$\begin{pmatrix}T_{41}\\T_{42}\\T_{43}\end{pmatrix}=\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\\4h_{1l}^2\frac{\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-2)}{X_{1l}\Gamma(X_{1l})}\\4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-1)\Gamma(X_{1l}-1)}\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}T_{51}\\T_{52}\\T_{53}\end{pmatrix}=\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-1)}{(X_{1l}-1)\Gamma(X_{1l}-1)}\\2h_{1l}^2\frac{2^{\alpha+\beta+1}\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-1)}{(X_{1l}+1)\Gamma(X_{1l}+1)}\\4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-1)}{X_{1l}\Gamma(X_{1l})}\end{pmatrix}. \quad (37)$$

With $X_{1l}\equiv 2k_{1l}+2\zeta_{1l}$, $N_{1l}^2\equiv\frac{\alpha(2\zeta_{1l}+1)\Gamma(X_{1l}+2)}{2^{X_{1l}-2}\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}+2)}$. This allows us to obtain the expectation values in the first excited state $(1,l,m)$as follows:

$$\begin{pmatrix}A_{1lm}\\B_{1lm}\\C_{1lm}\\D_{1lm}\\E_{1lm}\end{pmatrix}\equiv\frac{N_{1l}^2}{2\alpha}\begin{pmatrix}g_{1l}^2\frac{\Gamma(2k_{1l})\Gamma(2\zeta_{1l}-3)}{(X_{1l}-4)\Gamma(X_{1l}-4)}+4h_{1l}^2\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-2)\Gamma(X_{1l}-2)}+4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-3)\Gamma(X_{1l}-3)}\\g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-3)\Gamma(X_{1l}-3)}+4h_{1l}^2\frac{\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-1)\Gamma(X_{1l}-1)}+4g_{1l}h_{1l}\frac{2^{\alpha+\beta+1}\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-3)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\\g_{1l}^2\frac{\Gamma(2k_{1l})\Gamma(2\zeta_{1l}-2)}{(X_{1l}-3)\Gamma(X_{1l}-3)}+4h_{1l}^2\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-1)\Gamma(X_{1l}-1)}+4_{1l}h_{1l}\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-2)\Gamma(X_{1l}-2)}\\g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-2)\Gamma(X_{1l}-2)}+4h_{1l}^2\frac{\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-2)}{X_{1l}\Gamma(X_{1l})}+4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-2)}{(X_{1l}-1)\Gamma(X_{1l}-1)}\\g_{1l}^2\frac{\Gamma(2k_{1l}+1)\Gamma(2\zeta_{1l}-1)}{(X_{1l}-1)\Gamma(X_{1l}-1)}+2h_{1l}^2\frac{2^{\alpha+\beta+1}\Gamma(2k_{1l}+3)\Gamma(2\zeta_{1l}-1)}{(X_{1l}+1)\Gamma(X_{1l}+1)}+4g_{1l}h_{1l}\frac{\Gamma(2k_{1l}+2)\Gamma(2\zeta_{1l}-1)}{X_{1l}\Gamma(X_{1l})}\end{pmatrix}. \quad (38)$$

Our recent research is divided into two main physical parts. The first part is to correspond to replace the coupling of angular momentum operator with noncommutativity coupling $\vec{L}\vec{\Theta}$ by the new interest and significant equivalent coupling $\Theta\vec{L}\vec{S}$ (with$\Theta=(\Theta_{12}^2+\Theta_{23}^2+\Theta_{13}^2)^{1/2}$), we have chosen the arbitraries vector$\vec{\Theta}$parallel to the spin-s of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules under MSCKP and then we replace $\Theta\vec{L}\vec{S}$ by$\frac{\Theta}{2}\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$. Thus, the spin-orbit term containing $V_{pert}^{sc-so}(u)\equiv\Theta f(u)\vec{L}\vec{S}$, with

$$f(u)=\left[\frac{8\left(2l(l+1)+(E_{nl}+\mu)D_er_e^2\right)\alpha^4}{(1-u)^4}+(E_{nl}+\mu)D_e\left\{-\frac{8\alpha^4r_e^2u}{(1-u)^4}+\frac{4r_e\alpha^3}{(1-u)^3}+\frac{4\alpha^3(r_e-\alpha r_e^2)u}{(1-u)^3}+\frac{4r_e\alpha^3u}{(1-u)^2}\right\}\right]. \quad (39)$$

Furthermore, in the RQM and NRQM the operators ($\hat{H}_{nc-r}^{sc}$,$J^2$, $L^2$, $S^2$and $J_z$) forms a complete set of conserved physics quantities, the eigenvalues of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$ are equal the values $2k(l)\equiv j(j+1)-l(l+1)-s(s+1)$, with $|l-s|\leq j\leq|l+s|$. Consequently, the energy shift $\Delta E_{sc}(n=0,j,l,s)$ and $\Delta E_{sc}(n=1,j,l,s)$ due to the perturbed spin-orbit coupling which produced by the effect of the perturbed effective potential $V_{pert}^{sc}(r)$ for the ground state and the first excited state, respectively, in (RNC: 3D-RS) symmetries as follows:

$$\begin{aligned}\Delta E_{sc}(n=0,j,l,s)&=4k(l)\alpha^3(E_{0l}+\mu)\Theta\left\{2\alpha\left(\frac{2l(l+1)}{(E_{0l}+\mu)}+D_er_e^2\right)A_{0lm}-2\alpha D_er_e^2B_{0lm}+D_er_eC_{0lm}+D_e(r_e-\alpha r_e^2)D_{0lm}\right.\\&\left.+D_er_eE_{0lm}\right\},\\\Delta E_{sc}(n=1,j,l,s)&=4k(l)\alpha^3(E_{1l}+\mu)\Theta\left\{2\alpha\left(\frac{2l(l+1)}{(E_{1l}+\mu)}+D_er_e^2\right)A_{1lm}-2\alpha D_er_e^2B_{1lm}+D_er_eC_{1lm}+D_e(r_e-\alpha r_e^2)D_{1lm}+D_er_eE_{1lm}\right\}.\end{aligned} \quad (40)$$

Which allows us to generalize the above results to the case of $n^{th}$ excited states of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules under MSCKP in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{sc}(n,j,l,s) = 4k(l)\alpha^3(E_{nl}+\mu)\Theta\left\{2\alpha\left(\frac{2l(l+1)}{(E_{nl}+\mu)}+D_e r_e^2\right)A_{nlm}-2\alpha D_e r_e^2 B_{nlm}+D_e r_e C_{nlm}+D_e(r_e-\alpha r_e^2)D_{nlm}+D_e r_e E_{nlm}\right\}. \quad (41)$$

The second interest physical part is corresponding to replace both ($\vec{L}\vec{\Theta}$ and $\Theta_{12}$) by ($\sigma_{12}\aleph L_z$ and $\sigma_{12}\aleph$, respectively), we have also need to apply $\langle n,l,m|L_z|n',l',m'\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $-(l,l') \le (m,m') \le +(l,l')$). All of this data allows for the discovery of the new energy shift $\Delta E_{sc}(n=0,l,m)$ and $\Delta E_{sc}(n=1,l,m)$ due to the modified perturbed Zeeman effect which generated by the influence of the perturbed effective potential $V_{eff}^{sc}(r)$ for the ground state and the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{sc}(n=0,l,m) = 4\alpha^3(E_{0l}+\mu)\aleph\left\{2\alpha\left(\frac{2l(l+1)}{(E_{0l}+\mu)}+D_e r_e^2\right)A_{0lm}-2\alpha D_e r_e^2 B_{0lm}+D_e r_e C_{0lm}+D_e(r_e-\alpha r_e^2)D_{0lm} + D_e r_e E_{0lm}\right\}\sigma m,$$

$$\Delta E_{sc}(n=1,l,m) = 4\alpha^3(E_{1l}+\mu)\aleph\left\{2\alpha\left(\frac{2l(l+1)}{(E_{1l}+\mu)}+D_e r_e^2\right)A_{1lm}-2\alpha D_e r_e^2 B_{1lm}+D_e r_e C_{1lm}+D_e(r_e-\alpha r_e^2)D_{1lm}+D_e r_e E_{1lm}\right\}\sigma m. \quad (42)$$

Thus, we can generalize the previous results to the $n^{th}$ excited states of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules under MSCKP in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{sc}(n,l,m) = 4\alpha^3(E_{nl}+\mu)\aleph\left\{2\alpha\left(\frac{2l(l+1)}{(E_{nl}+\mu)}+D_e r_e^2\right)A_{nlm}-2\alpha D_e r_e^2 B_{nlm}+D_e r_e C_{nlm}+D_e(r_e-\alpha r_e^2)D_{nlm}+D_e r_e E_{nlm}\right\}\sigma m. \quad (43)$$

## IV. GLOBAL RELATIVISTIC SPECTRUM OF MSCKP

In this section, we report our results on based the superposition principle which permitted to deduce the additive energy shift, $\Delta E_{sc}(n=0,j,l,s,m)$ and $\Delta E_{sc}(n=1,j,l,s,m)$ for the ground state and the first excited state, respectively, in (RNC: 3D-RS) symmetries. Those energy shifts due to the spin-orbit coupling and modified Zeeman effect which are induced by $V_{eff}^{sc}(r)$ Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules under MSCKP as follows:

$$\Delta E_{sc}(n=0,j,l,s,m) = 4\alpha^3(E_{0l}+\mu)\left\{2\alpha\left(\frac{2l(l+1)}{(E_{0l}+\mu)}+D_e r_e^2\right)A_{0lm}-2\alpha D_e r_e^2 B_{0lm}+D_e r_e C_{0lm}+D_e(r_e-\alpha r_e^2)D_{0lm} + D_e r_e E_{0lm}\right\}\{k(l)\Theta+\aleph\sigma m\},$$

$$\Delta E_{sc}(n=1,j,l,s,m) = 4\alpha^3(E_{1l}+\mu)\left\{2\alpha\left(\frac{2l(l+1)}{(E_{1l}+\mu)}+D_e r_e^2\right)A_{1lm}-2\alpha D_e r_e^2 B_{1lm}+D_e r_e C_{1lm}+D_e(r_e-\alpha r_e^2)D_{1lm}+ D_e r_e E_{1lm}\right\}\{k(l)\Theta+\aleph\sigma m\}. \quad (44)$$

Which generalized easily to the $n^{th}$ excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{sc}(n,j,l,s,m) = 4\alpha^3(E_{nl}+\mu)\left\{2\alpha\left(\frac{2l(l+1)}{(E_{nl}+\mu)}+D_e r_e^2\right)A_{nlm}-2\alpha D_e r_e^2 B_{nlm}+D_e r_e C_{nlm}+D_e(r_e-\alpha r_e^2)D_{nlm}+ D_e r_e E_{nlm}\right\}\{k(l)\Theta+\aleph\sigma m\}. \quad (45)$$

The above results present the energy shift of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules under MSCKP in (RNC: 3D-RS) symmetries, which generated with the effect of noncommutativity properties of space-space; it depended explicitly with the noncommutativity parameters$(\Theta,\sigma)$. It is should be noted that the obtained effective energy $\Delta E_{sc}(n,j,l,s,m)$ under MSCKP has a carry unit of energy because it resulted from the perturbed effective energy $(\mu^2-E_{nl}^2)$ combined with the same energy value square and the mass square where we have the principle of equivalence between mass and energy at higher energy. This allows us to conclude the energy $E_{r-nc}^{sc}(D_e,r_e,\alpha,n,j,l,s,m)$ of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules, in the symmetries of (RNC: 3D-RS), corresponding the generalized $n^{th}$ excited states, as a function of the shift energy

$\Delta E_{sc}(n,j,l,s,m)$ and $E_{nl}$ due to the effect of SCKP in RQM, which obtained from Eq. (12), as follows:

$$E_{r-nc}^{sc}(D_e,r_e,\alpha,n,j,l,s,m)=\mu+E_{nl}+\left[4\alpha^3(E_{nl}+\mu)\left\{2\alpha\left(\frac{2l(l+1)}{(E_{nl}+\mu)}+D_e r_e^2\right)A_{nlm}-2\alpha D_e r_e^2 B_{nlm}+D_e r_e C_{elm}+D_e(r_e-\alpha r_e^2)D_{nlm}+D_e r_e E_{nlm}\right\}\{k(l)\Theta+\aleph\sigma m\}\right]^{1/2}. \quad (46)$$

## V. NONRELATIVISTIC SPECTRUM OF MSCKP

Now, in this section we trite the nonrelativistic effect of MSCKP on the Heterogeneous (LiH, HCl, and NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules. From the Eqs. (3.5) and (3.6), we can write MSCKP in the nonrelativistic noncommutative three-dimensional real space (NC: 3D-RS) symmetries as follows:

$$V_{sc}(\hat{r})=V_{sc}(r)+V_{sc}^{pert}(r)+O(\Theta^2). \quad (47)$$

Where $V_{sc}^{pert}(r)$ is the perturbative potential in the symmetries of nonrelativistic noncommutative QM:

$$V_{sc}^{pert}(r)=-D_e\left\{\begin{matrix}\left(\frac{r_e}{2r^3}-\frac{r_e^2}{2r^4}\right)(1+e^{-2\alpha r})\\ +2\alpha\left(\frac{r_e}{2r^2}-\frac{r_e^2}{4r^3}\right)e^{-2\alpha r}\end{matrix}\right\}\vec{L}\vec{\Theta}+O(\Theta^2). \quad (48)$$

A direct simplification gives:

$$V_{sc}^{pert}(r)=-D_e\left\{\begin{matrix}-\frac{r_e^2}{2r^4}-\frac{r_e^2}{2r^4}e^{-2\alpha r}+\frac{r_e}{2r^3}\\ +\frac{r_e-\alpha r_e^2}{2r^3}e^{-2\alpha r}+\frac{\alpha r_e}{r^2}e^{-2\alpha r}\end{matrix}\right\}\vec{L}\vec{\Theta}+O(\Theta^2). \quad (49)$$

The approximations type suggested by (Greene and Aldrich) and (Dong *et al*.) allows rewriting the perturbative potential $V_{sc}^{pert}(r)$as follows:

$$V_{sc}^{pert}(r)=-4D_e\left\{-\frac{2r_e^2\alpha^4}{(1-u)^4}-\frac{2\alpha^4 r_e^2 u}{(1-u)^4}+\frac{r_e\alpha^3}{(1-u)^3}+\frac{\alpha^3(r_e-\alpha r_e^2)u}{(1-u)^3}+\frac{\alpha^3 r_e u}{(1-u)^2}\right\}\vec{L}\vec{\Theta}+O(\Theta^2). \quad (50)$$

Thus, we need the expectation values of the radial terms$\frac{1}{(1-u)^4}$, $\frac{u}{(1-u)^4}$ , $\frac{1}{(1-u)^3}$ , $\frac{u}{(1-u)^3}$ and $\frac{u}{(1-u)^2}$to find the nonrelativistic energy corrections produced with the perturbative potential $V_{sc}^{pert}(r)$. We have calculated the expectation values for the ground state and the first excited state and we generalized the results to the $n^{th}$ excited state. To avoid the repetitions the previous calculations and to respect on considerations the corrections produced with the perturbed spin-orbit interaction and modified Zeeman effect, we need to generalize the obtained energy shift in the Eq. (45) to find the nonrelativistic global corrections under MSCKP for Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules as follows:

$$\Delta E_{sc}^{n\_re}(n,j,l,s,m)=-4D_e\alpha^3\{-2r_e^2\alpha A_{nlm}-2\alpha r_e^2 B_{nlm}+r_e C_{nlm}+(r_e-\alpha r_e^2)D_{nlm}+r_e E_{nlm}\}\{k(l)\Theta+\aleph\sigma m\}. \quad (51)$$

This allows us to find the nonrelativistic global energy $E_{nr-nc}^{sc}(D_e,r_e,\alpha,n,j,l,s,m)$ under MSCKP for Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules as follows in the nonrelativistic noncommutative three-dimensional real space (NC: 3D-RS) symmetries as follows:

$$E_{nr-nc}^{sc}(D_e,r_e,\alpha,n,j,l,s,m)=E_{nl}^{nr}-4D_e\alpha^3\{-2r_e^2\alpha A_{nlm}-2\alpha r_e^2 B_{nlm}+r_e C_{nlm}+(r_e-\alpha r_e^2)D_{nlm}+r_e E_{nlm}\}\{k(l)\Theta+\aleph\sigma m\}. \quad (52)$$

Where $E_{nl}^{nr}$is the nonrelativistic energy under for Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules [3]:

$$E_{nl}^{nr}=2\frac{\alpha^2}{\mu}l(l+1)+2D_e\alpha(\alpha r_e^2-r_e)-2\frac{\alpha^2}{\mu}\left[\frac{l(l+1)+D_e\mu\left(r_e^2-\frac{2r_e}{\alpha}\right)}{2\left(n+\frac{1}{2}+\sqrt{D_e r_e^2\mu+\left(l+\frac{1}{2}\right)^2}\right)}+\frac{1}{2}\left(n+\frac{1}{2}+\sqrt{D_e r_e^2\mu+\left(l+\frac{1}{2}\right)^2}\right)\right]^2. \quad (53)$$

## VI. QUARKONIUM MASS SPECTRO-SCOPY OF THE MSCKP

Quarkonium mass spectroscopy for the screened cosine Kratzer potential studied by K. R. Purohit *et al.* [3] in the symmetries of commutative quantum mechanics. We want to generalize this study to the case of NCQM. To achieve the goal we recall first for the mass formula of quarkonium in 3-dimensional space [59, 60, 61, 62, 63]:

$$M = m_q + m_{\overline{q}} + E_{nl} \rightarrow M = 2m_q + E_{nl} \qquad (54)$$

Here $m_q$ is the bare quark mass for quarkonium and $M$ denote to the mass the charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ in RQM under ordinary SCKP. To achieve this goal, we generalize the traditional formula to the new form:

$$M_{sc} = 2m_q + E_{nl}^{nr}(a', b', \alpha) \rightarrow M_{nc}^{sc} = 2m_q + E_{nr-nc}^{sc}(a', b', \alpha, n, j, l, s, m), \qquad (55)$$

where $E_{nr-nc}^{sc}(a', b', \alpha, n, j, l, s, m)$ and $E_{nl}^{nr}(a', b', \alpha)$ are obtained from Eqs. (50) and (51), respectively, by making the changed $D_e r_e \rightarrow a'$ and $D_e r_e^2 \rightarrow b'$ we obtain the modified energy and ordinary energy of quarkonium system such as the charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ in NCQM and QM under MSCKP ad SCKP, respectively, as follows:

$$E_{nr-nc}^{sc}(a', b', \alpha, n, j, l, s, m) = E_{nl}^{nr}(a', b', \alpha) - 4\alpha^3\{-2b'\alpha A_{nlm} - 2\alpha b' B_{nlm} + a' C_{nlm} + (a' - \alpha b')D_{nlm} + a' E_{nlm}\}\{k(l)\Theta + \aleph\sigma m\}. \qquad (56)$$

And

$$E_{nl}^{nr}(a', b', \alpha) = 2\frac{\alpha^2}{\mu} l(l+1) + 2\alpha(\alpha b' - a') - 2\frac{\alpha^2}{\mu}\left[\frac{l(l+1) + \mu\left(b' - \frac{2a'}{\alpha}\right)}{2\left(n + \frac{1}{2} + \sqrt{b'\mu + \left(l + \frac{1}{2}\right)^2}\right)} + \frac{1}{2}\left(n + \frac{1}{2} + \sqrt{b'\mu + \left(l + \frac{1}{2}\right)^2}\right)\right]^2. \qquad (57)$$

Thus, the modified mass of the heavy quarkonium system such as the charmonium $c\overline{c}$ and bottomonium becomes as follows:

$$M_{nc}^{sc} = M_{sc} + \delta M_{nc}^{hy}(a', b', \alpha, n, j, l, s, m). \qquad (58)$$

Where $M_{sc}$ are ordinary quarkonium mass spectroscopy in ordinary QM while $\delta M_{nc}^{hy}(a', b', \alpha, n, j, l, s, m)$ is the additive part of quarkonium mass spectroscopy in NCQM symmetries produced with the perturbative potential $V_{sc}^{pert}(r)$:

$$\delta M_{nc}^{hy}(a', b', \alpha, n, j, l, s, m) = 4\alpha^3\{2b'\alpha A_{nlm} 2\alpha b' B_{nlm} - a' C_{nlm} + (a' - \alpha b')D_{nlm} - a' E_{nlm}\}\{k(l)\Theta + \aleph\sigma m\}. \qquad (59)$$

Note that the additive part $\delta M_{nc}^{hy}(a', b', \alpha, n, j, l, s, m)$ is infinitesimal when it compared with the main part $M_{sc}$ because it is proportional to two infinitesimal parameters $(\Theta, \sigma)$.

## VII. STUDY OF IMPORTANT PARTICULAR CASES

After studying the bound states of arbitrary *l* state MKGE and MSE with the MSCKP, we consider some particular cases:

1- The first special case corresponds to the modified Coulomb plus inverse-square potential MCISP ($\delta = \alpha = 0, D_e r_e \rightarrow B$ and $D_e r_e^2 \rightarrow A$). The perturbative effective potential in Eq. (25) will be simplified to the form:

$$V_{pert}^{sc}(r) \rightarrow V_{pert}^{ic}(r) = \left[\frac{l(l+1)}{r^4} - (E_{nl} + \mu)\left(\frac{A}{r^4} + \frac{B}{r^3}\right)\right]\vec{L}\vec{\Theta}. \qquad (60)$$

Also, the energy shift in Eq. (56) was reduced to the corresponding values for MCISP with an additional condition $\alpha \rightarrow 1$ in corresponded terms in Eq. (26). Thus, we recover the results that we obtained in our previous reference [47].

2- The second special case corresponds to the new modified Kratzer-type interaction ($\delta = \alpha = 0, D_e r_e \rightarrow A = k r_e$ and $D_e r_e^2 \rightarrow B = k r_e^2$ in addition to imposing the special case $L_x = L_y = 0$ and $L_z = x p_y - y p_x \neq 0$) which corresponds to the translation into two-dimensional space from three-dimensional space. The perturbative potential in Eq. (5.4) will be simplified to the form:

$$V_{sc}^{pert}(r) \rightarrow V_{kp}^{pert}(r) = \left(\frac{A}{2r^3} - \frac{B}{r^4}\right) L_z \Theta. \qquad (61)$$

Moreover, the energy $E_{nr-nc}^{sc}(D_e, r_e, \alpha, n, j, l, s, m)$ in Eq. (52) was reduced to the corresponding values for the new modified Kratzer-type interaction with an additional condition $\alpha \rightarrow 1$ in corresponded terms in Eq. (26). Thus, we recover the results that we obtained in our previous reference [l9].

## VIII. CONCLUSIONS

This section of our paper gives a summary of the basic points in our work; we have presented the formulation of the noncommutative quantum mechanics based, and we have studied the effects of the noncommutative space-space on the MKGE and MSE under MSCKP in the relativistic and nonrelativistic noncommutative three-dimensional spaces.

The energy spectra ($E_{r-nc}^{sc}(D_e, r_e, \alpha, n, j, l, s, m)$ and $E_{nr-nc}^{sc}(D_e, r_e, \alpha, n, j, l, s, m)$) of Heterogeneous (LiH, HCl, NO) and Homogeneous ($H_2$, $I_2$, $O_2$) diatomic molecules due to the noncommutativity property corresponding to the generalized $n^{th}$ excited states in the symmetries of (RNC: 3D-RS) and (NC: 3D-RS) are calculated. The energy eigenvalues appear as a function of the shift energy $\Delta E_{hy}(n_r, j, l, s, m)$ and $E_{n_r l}$ of the SCKP is obtained via first-order perturbation theory and expressed by the Gamma function, the discreet atomic quantum numbers $(j, l, s, m)$, and the potential parameters $(D_e, r_e, \alpha)$ in addition to noncommutativity two parameters ($\Theta$and$\sigma$). This behavior is similar to the perturbed both modified Zeeman effect and modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V_{pert}^{sc}(r)$ in the symmetries of RNCQM and NRNCQM. Furthermore, we have calculated the new masses of the heavy quarkonium system such as the charmonium$c\overline{c}$and bottomonium$b\overline{b}$ with a spin-(0 or1) under the MSCKP. It is worth mentioning that, for all cases, when to make the two simultaneously limits$(\Theta, \sigma) \rightarrow (0,0)$, the ordinary physical quantities are recovered. The comparisons show that our theoretical results are in very good agreement with reported works.

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## REFERENCES

[1] Kratzer, A., *Die ultraroten Rotationsspektren der Halogenwasserstoffe*, Z. Physik **3**, 289–307 (1920). DOI: https://doi.org/10.1007/BF01327754.

[2] Aygun, M., Bayrak, O., Boztosun, I., and Sahin, Y., *The energy eigenvalues of the Kratzer potential in the presence of a magnetic field. The European Physical*, Journal D **66**(2), (2012). doi:10.1140/epjd/e2011-20319-5.

[3] Purohit, K. R., Parmar, R. H., and Rai, A. K., *Eigensolution and various properties of the screened cosine Kratzer potential in D dimensions via relativistic and non-relativistic treatment*, The European Physical Journal Plus **135**(3), (2020). doi:10.1140/epjp/s13360-020-00299-7.

[4] Onyenegecha, C. P., Onate, C. A., Echendu, O. K., Ibe, A. A. and Hassanabadi H., *Solutions of Schrodinger equation for the modified Mobius square plus Kratzer potential*, The European Physical Journal Plus **135**(3), (2020). doi:10.1140/epjp/s13360-020-00304-z.

[5] Bayrak, O., Boztosun, I., Ciftci, H., *Exact analytical solutions to the Kratzer potential by the asymptotic iteration method. Int. J. Quantum Chem*. **107**, 540 (2007). Doi: https://doi.org/10.1002/qua.21141.

[6] Sadeghi, J., *Factorization Method and Solution of the Non-Central Modified Kratzer Potential*, Acta Phys. Pol. **112**, 23-28 (2007).

[7] Setare, M. R., Karimi, E., *Algebraic approach to the Kratzer potential Phys*, Scr. **75**, 90 (2007), DOI: https://doi.org/10.1088/0031-8949/75/1/015.

[8] Berkdemir, C., Berkdemir, A., and Han, J., *Bound state solutions of the Schrödinger equation for modified Kratzer's molecular potential*, Chemical Physics Letters **417**, 326–329 (2006). doi:10.1016/j.cplett.2005.10.039. J. Sadeghi, Acta. Phys. Pol. **112**, 23 (2007).

[9] Capozziello, S., Lambiase, G., and Scarpetta, G., *Generalized uncertainty principle from quantum geometry*, Int. J. Theor. Phys. **39**, 15-22 (2000). doi:10.1023/A:1003634814685.

[10] Doplicher, S., Fredenhagen, K., and Roberts, J. E., *Spacetime quantization induced by classical gravity*, Physics Letters B **331**, 39–44 (1994). doi:10.1016/0370-2693(94)90940-7.

[11] Witten, E., *Refection on the fate spacetime*, Phys. Today **49**, 24 (1996). https://doi.org/10.1063/1.881493.

[12] Kempf, A., Mangano, G., and Mann, R. B., *Hilbert space representation of the minimal length uncertainty relation*, Physical Review D **52**, 1108–1118 (1995). doi:10.1103/physrevd.52.1108.

[13] Scardigli, F., *Some heuristic semi-classical derivations of the Planck length, the Hawking effect and the Unruh effect*, Il Nuovo Cimento B Series **11**, 1029–1034 (1995). doi:10.1007/bf02726152.

[14] Adler, R. J., and Santigo, D. I., *On gravity and the uncertainty principal*, Modern Physics Letters A **14**, 1371–1381 (1999). doi: 10.1142/s0217732399001462.

[15] Kanazawa, T., Lambiase, G., Vilasi, G. and Yoshioka, A., *Noncommutative Schwarzschild geometry and generalized uncertainty principle*, The European Physical Journal C **79**(2) (2019). doi:10.1140/epjc/s10052-019-6610-1.

[16] Scardigli, F., *Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment*, Physics Letters B **452**, 39–44 (1999). doi: 10.1016/s0370-2693(99)00167-7.

[17] Maireche, A., *Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics,* International Journal of Geometric Methods in Modern Physics (2020). doi: 10.1142/S021988782050067X

[18] Ho, P. M., and Kao, H. C., *Noncommutative Quantum Mechanics from Noncommutative Quantum Field Theory*, Physical Review Letters **88** (15), (2002). doi:10.1103/physrevlett.88.151602.

[19] Maireche, A., *Effects of Two-Dimensional Noncommutative Theories on Bound States Schrödinger Diatomic Molecules under New Modified Kratzer-Type Interactions*, International Letters of Chemistry, Physics and Astronomy **76**, 1-11 (2017).
doi: https://doi.org/10.18052/www.scipress.com/ILCPA.76.1

[20] Gnatenko, P., *Parameters of noncommutativity in Lie-*

*algebraic noncommutative space*, Physical Review D **99**, 026009-1 (2019). doi:10.1103/physrevd.99.026009.
[21] Maireche, A., *New nonrelativistic bound state solutions of the NGECSC potential involving GECSC potential and new central terms for hydrogen atom*, Afr. Rev Phys. **13**, 142-149 (2018).
http://aphysrev.ictp.it/index.php/aphysrev/article/view/1635/590.
[22] Bertolami, O., Rosa, J. G., De aragao, C. M. L., Castorina, P. and Zappala, D., *Scaling of varialbles and the relation between noncommutative parameters in noncommutative quantum mechanics*, Modern Physics Letters A **21**, 795–802 (2006). doi:10.1142/s0217732306019840.
[23] Maireche, A., *A Recent Study of Excited Energy Levels of Diatomics for modified more General Exponential Screened Coulomb Potential: Extended Quantum Mechanics*, J. Nano-Electron. Phys. **9**, 03021 (2017).
doi 10.21272/jnep.9(3).03021.
[24] Boumali, A., and Aounallah, H., *Exact solutions of vector bosons in the presence of the Aharonov-Bohm and Coulomb potentials in the gravitational field of topological defects in non-commutative space-time*, Revista Mexicana de Física **66**, 192–208 (2020).
doi: https://doi.org/10.31349/RevMexFis.66.192.
[25] Djemaï, E. F., and Smail, H., *On Quantum Mechanics on Noncommutative Quantum Phase Space*, Commun. Theor. Phys. (Beijing, China) **41**, 837–844 (2004). doi:10.1088/0253-6102/41/6/837.
[26] Yi Yuan, LI Kang, Wang Jian-Hua and Chen Chi-Yi, *Spin-1/2 relativistic particle in a magnetic field in NC phase space*, Chinese Physics C **34**, 543–547 (2010). doi:10.1088/1674-1137/34/5/005.
[27] Bertolami, O., and Leal, P., *Aspects of phase-space noncommutative quantum mechanics*, Physics Letters B **750**, 6–11 (2015). doi:10.1016/j.physletb.2015.08.024.
[28] Bastos, C., Bertolami, O., Dias, N. C. and Prata, J. N., *Weyl–Wigner formulation of noncommutative quantum mechanics*, Journal of Mathematical Physics **49**, 072101 (2008). doi:10.1063/1.2944996.
[29] Zhang, J., *Fractional angular momentum in non-commutative spaces*, Physics Letters B **584**, 204–209 (2004). doi:10.1016/j.physletb.2004.01.049.
[30] Gamboa, J., Loewe, M., and Rojas, J. C., *Noncommutative quantum mechanics*, Phys. Rev. D **64**, 067901 (2001). DOI : https://doi.org/10.1103/PhysRevD.64.067901.
[31] Chaichian, M., Sheikh-Jabbari and Tureanu, A., *Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED*, Physical Review Letters **86**, 2716–2719 (2001). doi:10.1103/physrevlett.86.2716.
[32] Maireche, A., *Nonrelativistic Bound State Solutions of the Modified 2D-Killingbeck Potential Involving 2D-Killingbeck Potential and Some Central Terms for Hydrogenic Atoms and Quarkonium System*, J. Nano- Electron. Phys. **11**, 04024 (2019). https://doi.org/10.21272/jnep.11(4).04024.
[33] Maireche, A., *New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank's Scales*, NanoWorld J. **1**, 122-129 (2016). doi:10.17756/nwj.2016-016.
[34] De Andrade, M. A., and Neves, C., *Noncommutative mapping from the symplectic formalism*, Journal of Mathematical Physics **59**, 012105 (2018).
doi:10.1063/1.4986964.
[35] Abreu, E. M. C., Neves, C., and Oliveira, W., *Noncommutativity from the symplectic point of view*, Int. J. Mod. Phys. A **21**, 5359 (2006).
doi: 10.1142/s0217751x06034094.
[36] Abreu, E. M. C., Neto, J. A., Mendes, A. C. R., Neves, C., Oliveira, W. and Marcial, M. V., *Lagrangian formulation for noncommutative nonlinear systems*, Int. J. Mod. Phys. A. **27**, 1250053 (2012), doi: 10.1142/s0217751x12500534.
[37] Mezincescu, L., *Star operation in quantum mechanics*, e-print arXiv: hep-th/0007046v2.
[38] Wang, J., and Li, K., *The HMW effect in noncommutative quantum mechanics*, Journal of Physics A: Mathematical and Theoretical **40**, 2197–2202 (2007). doi:10.1088/1751-8113/40/9/021.
[39] Li, K., and Wang, J., *The topological AC effect on non-commutative phase space*, The European Physical Journal C **50**, 1007–1011 (2007). doi:10.1140/epjc/s10052-007-0256-0.
[40] Maireche, A., *A New Nonrelativistic Investigation for the Lowest Excitations States of Interactions in One-Electron Atoms, Muonic, Hadronic and Rydberg Atoms with Modified Inverse Power Potential*, International Frontier Science Letters **9**, 33-46 (2016).
doi: https://doi.org/10.18052/www.scipress.com/IFSL.9.33
[41] Maireche, A., *New quantum atomic spectrum of Schrödinger equation with pseudo harmonic potential in both noncommutative three dimensional spaces and phases*, Lat. Am. J. Phys. Educ. **9**, 1301 (2015).
[42] Alhaidari, A. D., Bahlouli, H., and Al-Hasan, A, *Dirac and Klein–Gordon equations with equal scalar and vector potentials*, Physics Letters A, **349**, 87–97 (2006). doi:10.1016/j.physleta.2005.09.008.
[43] Xie, X.J. and Jia, C. S., *Solutions of the Klein–Gordon equation with the Morse potential energy model in higher spatial dimensions*, Physica Scripta **90**, 035207 (2015). doi:10.1088/0031-8949/90/3/035207.
[44] Motavalli, H. and Akbarieh, A. R., *Klein Gordon equation for the Coulomb potential in noncommutative space*, Modern Physics Letters A **25**, 2523–2528 (2010). doi:10.1142/s0217732310033529.
[45] Darroodi, M., Mehraban, H. and Hassanabadi, H., *The Klein–Gordon equation with the Kratzer potential in the noncommutative space*, Modern Physics Letters A **33**, 1850203 (2018). doi:10.1142/s0217732318502036.
[46] Saidi, A. and Sedra, M. B., *Spin-one (1 + 3)-dimensional DKP equation with modified Kratzer potential in the non-commutative space*, Modern Physics Letters A **35**, 2050014 (2019). doi:10.1142/s0217732320500145.
[47] Maireche, A., *The Klein–Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space*, Modern Physics Letters A **35**, 052050015 (2020). doi:10.1142/s0217732320500157.
[48] Maireche, A., *The Klein–Gordon Equation with Modified Coulomb Potential Plus Inverse-Square–Root Potential in Three-Dimensional Noncommutative Space,* To Physics Journal **3**, 186-196 (2019). Retrieved from
.https://purkh.com/index.php/tophy/article/view/489.

[49] Maireche, A., *A theoretical investigation of the Schrödinger equation for a modified two-dimensional purely sextic double-well potential*, YJES **16**, 41-54 (2018). www.yjes.org.sa

[50] Abdelmadjid Maireche, *Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics*, J. Nano- Electron. Phys. **10**, 06015-1 - 06015-7 (2018).
doi: https://doi.org/10.21272/jnep.10(6).06015

[51] Badawi, M., Bessis, N., and Bessis, G., *On the introduction of the rotation-vibration coupling in diatomic molecules and the factorization method*, Journal of Physics B: Atomic and Molecular Physics **5**, L157–L159 (1972). doi:10.1088/0022-3700/5/8/004.

[52] Stanek, J., *Approximate analytical solutions for arbitrary l-state of the Hulthén potential with an improved approximation of the centrifugal term*, Cent. Eur. J. Chem. **9**, 737–742 (2011). doi: 10.2478/s11532-011-0050-6.

[53] Greene, R. L. and Aldrich, C., *Variational wave functions for a screened Coulomb potential*, Physical Review A **14**, 2363–2366 (1976). doi:10.1103/physreva.14.2363.

[54] Dong, S. H., Qiang, W. C., Sun, G. H., and Bezerra, V. B., *Analytical approximations to the l-wave solutions of the Schrödinger equation with the Eckart potential*, Journal of Physics A: Mathematical and Theoretical **40**, 10535–10540 (2007). doi:10.1088/1751-8113/40/34/010.

[55] Onate, A. C., Ikot, A. N., Onyeaju, M. C. and Ebomwonyi, O., *Dirac Equation with Unequal Scalar and Vector Potentials under Spin and Pseudospin Symmetry*, Sri Lankan Journal of Physics **19**, 17–36 (2018). doi: http://doi.org/10.4038/sljp.v19i1.8043

[56] Hamzavi, M., Thylwe, K. E. and Rajabi, A. A., *Approximate bound states solution of the Hellmann potential*, Commun. Theor. Phys. **60**, 1-8 (2013). doi: http://doi.org/10.1088/0253-6102/60/1/01. 52.

[57] Oluwadare, O. J., Thylwe, K. E., and Oyewumi, K. J., *Non-relativistic phase shifts for Scattering on generalized radial Yukawa potential*, Commun. Theor. Phys. **65**, 434-440 (2016). doi: http://doi.org/10.1088/0253-6102/65/4/434. 53.

[58] Gradshteyn, S. and Ryzhik, I. M., *Table of Integrals, Series and Products* (7th. ed.; Elsevier, edited by Alan Jeffrey, University of Newcastle upon Tyne, England and Daniel Zwillinger, 2007).

[59] Kuchin, S. M., Maksimenko, N. V., *Theoretical Estimations of the Spin – Averaged Mass Spectra of Heavy Quarkonia and Bc Mesons*, Univ. J. Phys. Appl. **7**, 295-298 (2013). doi: 10.13189/ujpa.2013.010310.

[60] Abu-Shady, M., Mansour, H. M., Ahmadov, A. I., *Dissociation of Quarkonium in Hot and Dense Media in an Anisotropic Plasma in the Nonrelativistic Quark Model*, Adv. High Energy Phys. ID 4785615 (2019). doi: https://doi.org/10.1155/2019/4785615.

[61] Patel, B., Vinodkumar, P. C., *Properties of* $Q,\overline{Q}(Q\varepsilon b,c)$ *mesons in Coulomb plus power potential (CPPν)*, J. Phys. G: Nucl. Part. Phys. **36**, 035003 (2009).
https://doi.org/10.1088/0954-3899/36/3/035003.

[62] Maksimenko, N. V. and Kuchin, S.M., *Determination of the mass spectrum of quarkonia by the Nikiforov–Uvarov method*, Russ Phys J. **54**, 57 (2011).
https://doi.org/10.1007/s11182-011-9579-2.

[63] Rai, A. K., Parmar, R. H. and Vinodkumar, P. C., *Masses and decay constants of heavy–light flavour mesons in a variational scheme*, J. Phys. G: Nucl. Part. Phys. **28**, 2275–2282 (2002). https://doi.org/10.1088/0954-3899/28/8/313.