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ON THE ORBITAL STABILITY OF TRAVELING WAVES THAT BEHAVE SUCH AS PARTICLES

SOBRE LA ESTABILIDAD ORBITAL DE LAS ONDAS VIAJERAS QUE SE COMPORTAN COMO PARTÍCULAS

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RESUMEN

Las propiedades que las ondas solitarias comparten con las partículas han contribuido significativamente al desarrollo de nuevas teorías y avances tecnológicos en diferentes áreas del conocimiento. En este sentido, el estudio de la estabilidad orbital de las ondas solitarias es clave en la dinámica de las ondas solitarias. Aunque la definición de estabilidad orbital es relativamente simple, el análisis matemático necesario para verificarla es bastante complejo. Sin embargo, la teoría de Grillakis, Shatah y Strauss nos proporciona un criterio muy útil para verificar la estabilidad orbital. En este trabajo, presentamos su teoría y la aplicamos para analizar la estabilidad orbital de la ecuación generalizada de Korteweg-de Vries, la ecuación del fluido compresible y la ecuación unidimensional de Benney-Luke. Para las dos primeras ecuaciones, el criterio garantizaba la estabilidad orbital de las ondas solitarias. Para la tercera, se garantizaba sólo para ciertos rangos de sus parámetros

ABSTRACT

Properties that solitary waves share with particles have contributed significantly to the development of new theories and technological advances in different areas of knowledge. In this sense, the study of orbital stability of solitary waves is key in solitary wave dynamics. Although the definition of orbital stability is relatively simple, the mathematical analysis required to verify it is quite complex. However, the theory of Grillakis, Shatah and Strauss provides us with a very useful criterion to verify orbital stability. In this work, we present their theory and apply it to analyse the orbital

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stability of Generalized Korteweg-de Vries equation, Compressible fluid equation, and one-dimensional Benney-Luke equation. For the first two equations, the criterion guaranteed the orbital stability of the solitary waves. For the third one, it was guaranteed only for certain ranges of its parameters.

1. THE INTRODUCTION

In 1834, John Scott-Russell discovered solitary waves that propagate without deforming. These waves called solitons revolutionized particle physics because they show properties of particles, which led to very important advances in several areas of knowledge such as quantum mechanics, astronomy, optics, among others. Although mathematical models that describe the dynamics of solitons in different branches of science have been known since the 19th century, these could only be analyzed theoretically . With the invention of the computer in the mid-20th century, the situation changed radically. Processes could be simulated and contrasted with the results obtained from the mathematical analysis and empirical data.

A fundamental property of solitons is orbital stability, a solution ϕ is orbitally stable under a metric κ , if given $\epsilon > 0$,there exists $\delta > 0$ such that for any other solution φ satisfying $\kappa\big(\phi(0), \varphi(0)\big) < \delta$, implies $\kappa\big(\phi(t), \varphi(t)\big) < \delta$ for $t > 0$.

Although the concept is relatively straightforward, checking orbital stability is a complex process that is generally performed numerically. However, in 1987 M. Grillakis, J. Shatah and W. Strauss in [1,2] established a result (GSS theory) that characterizes the stability and orbital instability of solitary waves for problems framed as abstract Hamiltonian systems. This class of problems has special solutions of the form $u(x,t) = \phi_c(x - ct)$, where c > 0 is denominated speed-wave. Due to the translation invariance of the motion equation, it is possible to show the existence of a quantity Q that is conserved with respect to time [1,2]. This quantity is fundamental in the study of the stability of solitary waves since these are characterized as stationary points of the functional of energy $\Im = E + cQ$.

The criterion of Grillakis, Shatah and Strauss in [1] for orbital stability of a solitary wave requires the second variation of $\Im$ in ϕ_c.

Once all the hypotheses of the criteria have been verified we conclude that the solitary wave ϕ_c is orbitally stable if and only if the function $d(\cdot)$ defined as $d(c) = \Im(\phi_c)$ is strictly convex.

In this work, we analyze the orbital stability of three mathematical models by means of the GSS criterion. The paper is organized as follow, in second section is presented the Grillakis, Shatah and Strauss criterion, in the third section we applied the GSS theory to analyze the orbital stability of the Generalized Korteweg-de Vries equation, the one-dimensional Benney-Luke equation and the Compressible fluid equation, in fourth and fifth sections we present discussion and conclusion, respectively.

2. GRILLAKIS-SHATAH-STRAUSS CRITERION

In 1987, Manoussos Grillakis, Jalal Shatah and Walter Strauss developed their theory of stability of traveling waves solutions for the nonlinear evolution equations in the Hamiltonian forms

$$u_t = JE'(u(t)), \tag{1}$$

which are locally well-posed in a Hilbert space X with inner product (,) ; here E is a functional ("energy") and J is a skew-symmetric linear operator. Let be a one-parameter group of unitary operators on X. If X^* is its dual, there is a natural isomorphism $\bar{I}: X \to X^*$ defined by $\langle \bar{I}u, v\rangle = (u, v)$ where $\langle , \rangle$ denotes the pairing between X and X^*.

Let be a one-parameter group of unitary operators on X such that E is invariant under T and j "commutes" with T. Let $B: X \to X^*$ a bounded a bounded linear operator such that $B^*=B$ and the operator jB is an extension of T' (0).

Let the functional $Q: X \to \mathbb{R}$ defined by

$$Q(u) = \frac{1}{2}\langle Bu, u\rangle. \tag{2}$$

Considering the following assumptions

- *Assumption 1* (Existence of solutions) For each $u_0 \in X$ exists $t_0 > 0$ depending only on μ, where $\|u_0\| < \mu$

and there exists a solution u of equation (1) in the interval $I = [0, t_0)$ such that

a. $u(0) = u_0$ and

b. $E(u(t)) = E(u_0)$, $Q(u(t)) = Q(u_0)$ for $t \in I$.

- *Assumption 2* (Existence of Bound States) There exist real $\omega_1 < \omega_2$ and a mapping $c \to \phi_c$ from the open interval (ω_1, ω_2) into X which is C^1 such that for each $c \in (\omega_1, \omega_2)$

  a. $E'(\phi_c) = cE'(\phi_c)$

  b. $\phi_c \in D(T'(0)^3) \cap D(JIT'(0)^2)$,

  c. $T'(0)\phi_c \neq 0$.

- *Assumption 3* (Spectral structure For each $c \in (\omega_1, \omega_2)$, H_ω has exactly one negative simple eigenvalue and has its kernel spanned by $T'(0)\phi_c$ and the rest of its spectrum is positive and bounded away from zero

The main results of Grillakis, Shatah and Strauss is summarized in the following theorem

Theorem 1: *Given Assumptions 1, 2 and 3, let . $c \in (\omega_1, \omega_2)$. Then the $\phi_c -$ orbit is stable if and only if the function $d\,(\cdot)$ is convex in a neighborhood of ω, namely, $d''(c) > 0$.*

3. APPLICATION OF MAIN RESULT

In this section we will illustrate examples how this theory works in the case of solitary wave solutions and periodic traveling waves. A wide variety of applications of this theory have been obtained in the last twenty years to different equations or systems which appear in the physical description of phenomena, for example, in the dynamic of fluid, internal waves, nonlinear interactions in shallow-water ocean surface waves, optical, hydro dynamical systems and plasma physics [3].

3.1. STABILITY OF SOLITARY WAVE SOLUTIONS

Angulo J. in [3] study the stability of solitary wave solutions associated to the following class of equations

$$u_t + u^p u_x - M u_x = 0, \qquad (3)$$

where $u = u(x,t)$ is real valued, $x, t \in \mathbb{R}$, $p \in \mathbb{N}$, $p \geq 1$ and M is a linear operator defined as a Fourier multiplier operator by $\widehat{Mu}\,(\xi) = \alpha(\xi)\,\widehat{u}\,(\xi)$, , where the symbol $\alpha(\xi)$ is a measurable even function on $\mathbb{R}$ and satisfies

$$a_1|\xi|^{\beta_1} \leq \alpha(\xi) \leq a_2(1+|\xi|)^{\beta_2}, \quad \text{for} \quad \xi \in \mathbb{R} \; (4)$$

where the parameters defined in equation (4) satisfy $a_1, a_2 > 0$ and $\beta_1 \geq \beta_1 \geq 1$. By considering c > $inf_{\xi\in\mathbb{R}}\,\alpha(\xi)$ he verifies $M + c$ is a positive operator. The solitary wave solution $u(x,t) = \phi_c(x - ct)$ of (2) satisfies the equation

$$M\phi_c + c\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0, \qquad (5)$$

In this case, the linear space is defined by

$$X = \left\{ f \in L^2(\mathbb{R}) : \|f\|_X = \left(\int_{-\infty}^{\infty} |1+\alpha(\xi)| \left|\hat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}. \qquad (6)$$

Its dual X^* is the space of all tempered distributions Φ whose Fourier transform $\widehat{\Phi}$ i is given by a measurable function for which

$$\|\Phi\|_{X^*} = \left(\int_{-\infty}^{\infty} \frac{|\Phi(\xi)|^2}{1+\alpha(\xi)} d\xi \right)^{\frac{1}{2}} < \infty, \qquad (7)$$

where $\|.\|_{X^*}$ defined in equation (7) is the norm.

The pairing between X and X^* is determined for and $\Phi \in X^*$ as $\Phi(f)$, and it will be written as (Φ, f) If Φ is given by an $L^2(\mathbb{R})$ function φ, then $(\varphi, f) = \langle \varphi, f \rangle$ is the usual $L^2(\mathbb{R})$ inner product. On the other hand, from equation (3) on the symbol a of the operator M, the space X is continuously embedded in $H^{1/2}(\mathbb{R})$.By using the conservation law $E_M : X \to \mathbb{R}$ defined by

$$E_M(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2} u M u - \frac{1}{(p+2)(p+1)} u^{p+2} \right) dx, \quad (3)$$

He writes equation (2) in the Hamiltonian form

$$u_t(t) = -\frac{\partial}{\partial x} E'_M(u(t)), \qquad (9)$$

where E'_M defined in equation (9) is the derivative. Let the conservation law $Q: X \to \mathbb{R}$ defined by

$$Q(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx. \quad (10)$$

For equation (5), Pava verified that the assumptions of Grillakis et al. are given by

- *Assumption 1* (Existence of solutions)

There is a Banach space $(Y, \|\cdot\|_Y)$ continuously embedding in X such that for each $u_0 \in Y$ there exist $T = T(\|u_0\|_Y)$ and the unique solution $u \in C([-T, T], Y)$ of (2) satisfying

a. $u(0) = u_0$ and

b. $E_M(u(t)) = E_M(u_0)$, $Q(u(t)) = Q(u_0)$ for $t \in [0, T]$.

Assumption 2 (Existence of solitary wave solutions) There exist real $\omega_1 < \omega_2$ such that

a. The mapping $c \to \phi_c$ from the open interval (ω_1, ω_2) into $X \subseteq H^{\beta_1/2}(\mathbb{R})$ which is C^1.

b. $E'_M(\phi_c) + cQ'(\phi_c) = 0$ ($i.e.$ ϕ_c is a critical point of the functional $E_M(\phi_c) + cQ(\phi_c)$).

Assumption 3 (Spectral structure) For each $c \in (\omega_1, \omega_2)$, the self-adjunt, closed, unbounded linear operator $\mathcal{L}_c$, defined on the dense subspace of $L^2(\mathbb{R})$ as

$$\mathcal{L}_c \equiv M + c - \phi_c^p. \quad (11)$$

$\mathcal{L}_c$ defined in equation (11) satisfies the following spectral properties: it has a single negative eigenvalue which simple, the zero eigenvalue is simple with eigenfunction ϕ', and the remainder of the spectrum of $\mathcal{L}_c$ is positive and bounded away from zero.

In this case, the function $d(\cdot)$ is defined by

$$d(c) = E_M(\phi_c) + cQ(\phi_c), \quad (12)$$

where E_M and Q are defined in equations (8) and (10), respectively. From Assumption 2 it follow that $d'(c) = cQ(\phi_c)$, so the sufficient condition for obtaining the orbital stability given by Theorem 1 is reduced to

$$d''(c) = \frac{1}{2}\frac{d}{dc}\int_{-\infty}^{\infty} \phi_c^2(\xi)\, d\xi > 0. \quad (13)$$

The equations (12) and (13) characterize the convexity of the functional d.

3.2. STABILITY OF SOLITARY WAVE SOLUTIONS FOR ONE-DIMENSIONAL BENNEY-LUKE EQUATION

Quintero J R in [4] study the stability of solitons associated to the following nonlinear one-dimensional Boussinesq type equation or Benney–Luke equation.

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + \Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt} = 0, \quad (14)$$

where *a* and *b* are positive numbers such that $a - b = \sigma - 1/3$ (is named the Bond number). He studied the stability of traveling waves of lowest energy in the energy norm or simply solitary waves for equation (6) of the form $\Phi(x,t) = u(x - ct)$, where $c > 0$ satisfies $c^2 < min(1, a/b)$. where satisfies . By substituting in Φ in (14), we verifies the traveling wave profile *u* should satisfy the equation

$$(c^1 - 1)u_{xx} + (a - bc^2)u_{xxxx} - 3cu_{xx}u_x = 0. \quad (15)$$

By using the criterion of stability of Grillakis et al established in Theorem 1, Quintero showed that traveling wave solutions of equation (15) are orbitally stable when $0 < c < 1 < a/b$, which corresponds to the case Bond number $\sigma > 1/3$ and are orbitally unstable when 0 < *c* < *a/b* < 1, which corresponds to the case Bond number number $\sigma > 1/3$. Ibarguen-Mondragon E in [5] study the stability of solitons associated Benney–Luke equation (6) for the complementary case $c > 0$ *satisfies* $c^2 > min\ (1, a/b)$. He proved prove that the null solution is asymptotically stable due to the presence of a Hamiltonian structure and to the existence of invariant quantities with regard to time. In the case of a non–null solitary wave, he found the Hamiltonian structure but the verification that some quantities are conserved with respect to time has turned out to be a difficult numerical calculation. He showed that the criterion of stability and orbital instability of M. Grillakis J Shatah and W. Strauss is not applicable. Angulo and Quintero in [6] study the existence and orbital stability of cnoidal waves for Benney–Luke equation (6), they built periodic travelling-wave solutions with an arbitrary fundamental period T_0 by using Jacobian elliptic functions Stability (orbital) of these solutions by periodic disturbances with period T_0 will be a consequence of the general stability criteria given by M. Grillakis, J. Shatah, and W. Strauss. Quintero and Muñoz in [7] study the stability of solitons associated to generalized Benney–Luke equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + n\Phi_t(\Phi_x)^{n-1}\Phi_{xx} + 2(\Phi_x)^n\Phi_{xt} = 0. \quad (16)$$

They verified that criterion of Grillakis et al. result applied to (16) fails for to be applicable in this case. They implemented a finite difference numerical scheme which combines an explicit predictor and an implicit corrector step to compute solutions of the model equation.

3.3. STABILITY SOLITARY TRAVELING WAVE SOLUTIONS FOR COMPRESSIBLE FLUID EQUATIONS

Li et al. in [8] discuss the existence of traveling wave solutions for the following compressible fluid equations with capillarity term

$$v_t - u_x = 0,\ u_t - p(v)_x + \delta v_{xxx} = 0, \quad (17)$$

where x is the Lagrangian space variable, v is specific volume, u is velocity, $\delta > 0$ is the capillarity coefficient, and $p(v)$ is Van del Waals pressure. By applying the theory and method of planar dynamical system defined in equation (17), and obtain explicit expressions for all bounded traveling wave solutions by undetermined coefficient method, including kink and bell profile traveling wave solutions, as well as periodic wave solutions. They proved the kink profile solitary wave solution, both sides of which asymptotic values are not zero, is orbitally stable by the theory of Grillakis et al. stablished in the Theorem 1.

4. DISCUSSION

The physical relevance of solitary traveling waves in the form of pulses or solitons lies in the large number of scenarios in which they appear, solitons have been discovered in all states of matter and in various areas of knowledge such as basic science and engineering. At present, theoretical physicists suggest that solitons play a determining role in superconductivity, and also that through them information can be stored and transported in super-fast computers.

However, due to the nature of physical phenomena, the models formulated to describe the dynamics of solitons are quite complex. From physics and mathematics, efforts have been joined to advance in the understanding of the dynamics of solitons, so criteria such as that of Grillakis, Shatat and Strauss have contributed significantly to take giant steps in technological advances.

The GSS theory provides a way to use constrained energy as a Lyapunov function that allows us to analyze the stability or instability of solitary wave solutions. Since this type of functions must satisfy conservation laws, this makes it a great limitation for the application of the GSS theory. In references [10,11] we find other interesting applications of the GSS theory. As we could see in Section 3, the GSS criterion worked very well to verify the orbital stability of the solitons of the Generalized Korteweg-de Vries equation and the Compressible fluid equation. However, in the one-dimensional Benney-Luke equation, it worked for certain parameter ranges. It should be noted that although there are other methods to analyze orbital stability, the GSS Criterion is the most used to date, and the one that encompasses a broader range of differential equations [12]. The purpose of our work was focused on highlighting the importance of orbital stability in the dynamics of solitons and the usefulness of the Criterion of Grillakis, Shatah and Strauss in this process.

5. CONCLUSION

The dynamics of solitons still present many open questions. However, the properties that have been tested have contributed significantly to tectonological advances. Recent results on its role on issues such as superconductors are promising, however, they generate challenges in science that involve multidisciplinary work. In this sense, from mathematical physics it is vital that new theories about orbital stability arise or that existing ones such as the GSS theory are improved.

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