



## Analysis of physical systems using procedures analogous to gauge theory Análisis de sistemas físicos usando procedimientos análogos a la teoría gauge

Mauricio Rozo Clavijo<sup>1</sup>,  
Sandra Ávila Torres<sup>2</sup>,  
Javier A. Olarte T.<sup>3</sup>

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**Abstract:** In this article is described the motion of a particle in rotating in the complex plane using procedures similar to the construction of a local gauge theory. This analysis allows us to approach to the behavior of the particle, starting with the exigency of the gauge invariance of the equation of the position of the particle. When this invariance is demanded, it leads to the introduction of a compensatory parameter that is interpreted as the angular velocity of the particle. Finally, the dynamic effects which determine its movement are obtained, whose results are like to those are obtained from a description done with respect to reference systems in rotation using polar coordinates. This type of study acquires special importance when seeking to analyze a system in rotation from a different perspective to usually established.

**KeyWords:** Invariance gauge, dynamic effects, compensating parameter, transformation gauge.

**Resumen:** en el presente artículo se hace una descripción del movimiento de una partícula en rotación haciendo uso del plano complejo y utilizando procedimientos similares al de la construcción de una teoría gauge local. El análisis permite describir el comportamiento de la partícula partiendo de la exigencia de la invariancia gauge de

- 1 BSc. In Physics, Universidad Pedagógica Nacional, Colombia. Specialization in Physical Sciences, Universidad Nacional de Colombia, Colombia. MSc. In Physical Sciences Universidad Nacional de Colombia, Colombia. Current position: Professor Department of Physics Universidad Pedagógica Nacional, Colombia. E-mail: mclavijo@pedagogica.edu.co.
- 2 BSc. In Physics, Universidad Pedagógica Nacional, Colombia. Current position: Professor Gimnasio San Ángel, Colombia. E-mail: drasanita@hotmail.com.
- 3 BSc. In Physics, Universidad estatal de Tbilisi, URSS. MSc. And PhD. In Physical Sciences, Universidad Nacional de Colombia, Colombia. Current position: Professor Universidad Distrital Francisco José de Caldas. E-mail: jaolartet@udistrital.edu.co



la ecuación de posición. Invariancia que al ser exigida conduce a la introducción de un parámetro compensatorio que se interpreta como la velocidad angular de la partícula. A partir del análisis se obtienen los efectos dinámicos que determinan su movimiento, los cuales son análogos a los obtenidos desde una descripción hecha respecto a un sistema de referencia en rotación haciendo uso de coordenadas polares. Este tipo de estudio adquiere especial importancia cuando se busca hacer un análisis sobre un sistema en rotación desde una perspectiva diferente a la usualmente establecida.

**Palabras claves:** Invariancia gauge, efectos dinámicos, parámetro compensador, transformación gauge.

## 1. Introduction

The scheme of Newtonian mechanics allows a description of the motion of a particle with respect to inertial reference systems and non-inertial itself, along with the forces that are conditioned by the action exerted by other bodies are taken into account so-called inertial forces or pseudo forces. The Inertial forces are conditioned by the reference system, i.e. only appear when the movement of the particle with respect to a non-inertial reference system is analyzed.

In the description of the motion of a particle in rotation is generally used a system of polar coordinates, although you can use any coordinate system, this system simplifies the problem [1], [2]. Using the polar coordinate system and taking into account the so-called inertial forces, in addition to the forces exerted by other bodies, the dynamic effects of the particle are obtained as: centrifugal force  $\vec{F} = m\dot{r}\hat{r}$ , tangential force  $\vec{F} = mr\ddot{\theta}\hat{\theta}$ , the centripetal force  $\vec{F} = -mr\dot{\theta}^2\hat{r}$  and the force coriolis  $\vec{F} = 2m\dot{r}\dot{\theta}\hat{\theta}$  [3][4] which can be written as follows:

$$\vec{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \quad (1)$$

On the other hand, symmetries have played an important role in physics starting from the symmetries of space-time in relativity to the gauge symmetries are the motivation of this work. Generally, it found in the literature [5] that there is a symmetry  $\hat{A}$  if the Lagrangian of the system is invariant under a gauge transformation. This idea has no meaning or significance for students and neophytes in general and that does not refer to an experience known or treated by them. Many students analyze topics related to the gauge principle seems like a topic difficult to understand and little application.

It motivated by this problem is proposed to introduce the gauge idea and the use of the gauge transformation within the classical context, with the aim of showing the different procedures used in the construction of a gauge theory on one hand, and on the other, getting students They familiarize themselves with the new concepts used for the explanation of phenomena. Thus, it is intended that students acquire meaningful knowledge and meaning about the new looks that thinkers propose on the phenomena. In this perspective, bridges between the knowledge acquired by students and contemporary topics of physics are generated.

In this context, the motion of a particle in rotation using the complex plane to give the position of the particle and implement the gauge idea within the classical context is analyzed. This analysis leads to also get the dynamic effects such as: the Coriolis forces, centrifugal, centripetal and tangential [4], but in this case it is part of the requirement of gauge invariance of the equation position of the particle; invariance leads being required to obtain the dynamic effects. This analysis shows a different geometrical perspective to the usually established when a particle physics rotating addressed.

## 2. On the idea Gauge

The gauge idea how dynamic principle was first appreciated by Hermann Weyl motivated by finding a geometric base realize the gravitational and electromagnetic interactions. In his article “Gravitation and electricity” [6], first proposed the idea to gauge “guiding field” [6], [7] to unify the gravitational and electromagnetic force to then known (1900). His proposal is based on a geometry that explains not only the gravitational phenomena but also those related to electromagnetic fields within the context of a unified field theory of gravity and electromagnetism. Weyl proposed for this a generalization of the concept of measure, arguing that physical measurements are relative and what has physical meaning is the interval between them,

$$\Delta S^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad (2)$$

The interval measured by an observer between two points in a three-dimensional space, may be calculated differently by other observers, i.e., the initial and final position of a particle does not have a single way to be measured, but on the otherwise it can be recalibrated in different ways keeping interval

distance between the two points invariant for all observers. In this context, the events that occur at each point in space are determined by the coordinates  $\vec{r} = \vec{r}(x, y, z)$ . Two events occurring at two different points in space are specified by coordinates  $\vec{r}_i = \vec{r}_i(x_1, y_1, z_1)$   $\vec{r}_f = \vec{r}_f(x_2, y_2, z_2)$  respectively. There is, however, a way to recalibrate the start and end positions of the events by adding an arbitrary constant as follows:

$$\vec{r}_i \rightarrow \vec{r}'_i = \vec{r}_i + cte ; \vec{r}_f \rightarrow \vec{r}'_f = \vec{r}_f + cte. \quad (3)$$

This means that the initial and final position of the particle is not uniquely determined. Measuring the position of an event in each point of space can be arbitrary for each observer. A same distance interval may correspond different positions, start and end depending on the arbitrary constant which would be added. It is easy to see that the range does not change when replacing the transformation (3) (2). Coordinate transformation (3) does not change the distance interval between the two events. Which means, that ( $\Delta S$ ) is the difference of the magnitude considered, not its absolute value what has physical sense. This invariance is called gauge invariance overall interval between two events, since the constant added is the same for all observers.

There is another way to perform the recalibration of measuring the position of an event at a point in space. In this case, each observer can add a different constant in the initial and final position of the particle without the change distance interval ( $\Delta S$ ), i.e. the interval is invariant under transformation of a local measurement.

$$\begin{matrix} \vec{r}_i \rightarrow \vec{r}'_i = \vec{r}_i + c_1 ; \vec{r}_f \rightarrow \vec{r}'_f = \vec{r}_f + c_1, \\ \vec{r}_i \rightarrow \vec{r}'_i = \vec{r}_i + c_2 ; \vec{r}_f \rightarrow \vec{r}'_f = \vec{r}_f + c_2 \end{matrix} \quad (4)$$

This is called a local gauge invariance since the constant added up to each viewer to make their own recalibration.

### 3. Circular motion

The circular particle motion in the plane can be described from the gauge idea. In this case, the state of the particle at any instant is determined by its position and momentum.

The position of the particle can be set using a Cartesian plane specifying the angular position by the angle with a fixed direction of the plane as shown in the figure below,

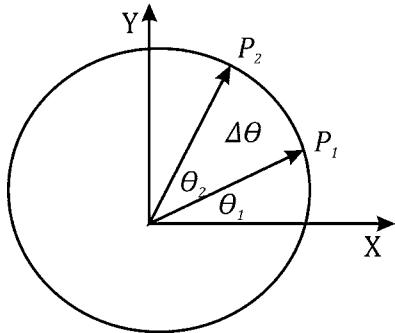


Figure 1. Angular position of a particle in a Cartesian plane. Source: own.

The initial angular position of the particle in ( $P_1$ ) shall be it described by the angle ( $\theta_1$ ) and end position ( $P_2$ ) is ( $\theta_2$ ) given with respect to the reference axis ( $x$ ), in other words, the angular position ( $\theta$ ) of the particle any time is described by the following equation of motion:

$$\theta(t) = 5t^2 \quad (5)$$

The reference angular position may be chosen arbitrarily without changing the equation

of motion of the particle. This means that the choice of corner reference point is unimportant since it has physical meaning is the difference of the magnitude considered in this case is the angular displacement ( $\Delta\theta$ ) during a time interval and not its absolute value in a given point of the plane.

Since there is arbitrariness in the definition of the angular position of the particle, one can introduce the notion gauge adding an arbitrary constant regarding the initial ( $\theta_1$ ) and final ( $\theta_2$ ) angular position given as:

$$\theta_1 \rightarrow \theta_1' = \theta_1 + cte ; \theta_2 \rightarrow \theta_2' = \theta_2 + cte. \quad (6)$$

In figure 2 show the global recalibration of the angular position of the particle respect to different reference.

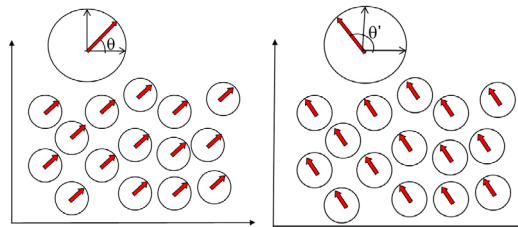


Figure 2. Overall calibration of the angular position of the particle relative to different observers. Source: own.

For the angular ( $\omega$ ) speed of the particle in an instant of time ( $t_0$ ), it is necessary to know the angular distance  $\theta_0 \rightarrow \theta' = \theta_0 + \Delta\theta$ , being  $\Delta\theta = cte$ , walking in the instant of time  $t_0 \rightarrow t' = t_0 + \Delta t$ , being very small to avoid changes in the speed of the particle. In other words, considering the time and the angular position of the particle as a recalibration given by

$$t_0 \rightarrow t' = t_0 + \Delta t ; \theta_0 \rightarrow \theta' = \theta_0 + \Delta \theta, \quad (7)$$

is obtained, the angular position of the particle a little further than it was before  $\theta_0(t_0) = 5t^2$ , i.e., the new position of the particle will be:

$$\theta' = 5t'^2$$

$$\theta_0 + \Delta\theta = 5(t_0 + \Delta t)^2$$

With respect to the previous  $\theta_0(t_0) = 5t_0^2$ . Doing algebra is obtained:

$$5t_0^2 + \Delta\theta = 5t_0^2 + 10t_0\Delta t + \Delta t^2$$

$$\Delta\theta = 10t_0\Delta t + \Delta t^2$$

The last equation corresponds to the additional angular distance traveled. In first approximation the angular speed of the particle is:

$$\omega = \frac{\Delta\theta}{\Delta t} = 10t_0 + \Delta t. \quad (8)$$

However, the instantaneous angular velocity is the value of this ratio when  $\Delta t$  made infinitely small

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = 10t_0. \quad (9)$$

The angular speed of the particle was obtained using transformations (7) corresponding to a recalibration of the angular position variables  $\theta = \theta(t)$  and time ( $t$ ). This leads to a global recalibration invariance of the angular speed of the particle as it is the same for all observers. The above analysis allows to characterize the role of the gauge idea formalization and explanation of the motion of a particle in rotation, which gives the ability to display a different set usually look. Moreover, obtaining the angular speed of the particle can be regarded from the equation derived from the angular position  $\theta_0(t_0) = 5t^2$  with respect to time. In this sense, the implementation of the idea in any gauge function can be consid-

ered as a way to obtain the derivative of the function.

Finally, the magnitude of a vector representing a physical measurement will not have an absolute value but rather depends on their difference. In other words, the measure of a physical quantity at each point of space-time can be arbitrary for each observer, since it has physical meaning are the differences of the magnitude considered.

#### 4. Movement of the particle in the complex plane

The analysis of the motion of a particle from rotating reference frame ( $0'$ ) is performed considering the pure rotation system without this having Translational respect to an inertial reference system ( $0$ ). The origins of the two reference systems and similarly match their axes  $z, z'$ . The angular velocity of the rotating system is considered along the common axis  $z, z'$ , and at the instant of time ( $t$ ) axes  $x, x'$  coincide. The position vector of the particle with respect to the reference systems ( $0'$ ) and ( $0$ ) is given by:

$$\vec{r} = x\hat{e}_1 + y\hat{e}_2 ; \vec{r} = x'\hat{e}'_1 + y'\hat{e}'_2 \quad (10)$$

Being  $\hat{e}_1, \hat{e}_2, \hat{e}'_1, \hat{e}'_2$ , the unit vectors of each coordinate base.

However, to describe the motion of the particle is convenient to use a rotary table relative to the coordinate axes defined by [8]:

$$\vec{r} = x + iy, \quad (11)$$

Whose components can be written as:

$$x = r \cos \theta ; y = r \sin \theta , \quad (12)$$

Without losing information. In general, you can do without vector notation by introducing a magnitude ( $z$ ) in the complex plane is identified with the  $\vec{r}$  definition given by,

$$\vec{r} \rightarrow z = x + iy , \quad (13)$$

As shown in Figure 3,

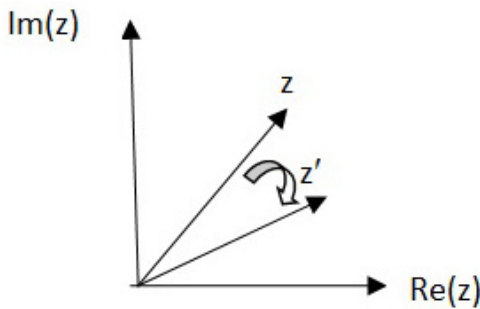


Figure 3. Complex vector representation of the position. Source: own.

Geometrically, equation (13) describes a continuous vector rotation ( $z$ ) about an axis. It is convenient to also introduce the function [8]

$$e^{i\theta} = \cos \theta + i \sin \theta , \quad (14)$$

It is representing a rotation value ( $\theta$ ) represented by vector ( $z$ ). This representation of a vector by a complex number, to formalize the motion of the particle in rotation from position vector:

$$z = r e^{i\theta}, \text{ or } z = e^{i\theta} \text{ con, } r = 1, \quad (15)$$

Where the numbers ( $1, e^{i\theta}, e^{i2\theta}, \dots$ ) form a group whose general structure is of the form  $\{e, a, a^2, \dots, a^{n-1}\}$  with ( $n$ ) any positive integer [9]. Write the position of the particle in terms of the ( $z$ ) complex plane, it allows one hand,

describe their behavior using the gauge invariance of the equation position, and secondly, to familiarize students with the concepts used in contemporary theories physics [10].

Since the angular position ( $\theta$ ) of the particle is not univocally defined, it is possible to choose the initial reference angle in an arbitrary manner as:

$$\theta \rightarrow \theta' = \theta - \alpha , \quad (16)$$

Which means making a redefinition or calibration [6] [10], of the angular position of the particle as shown in Figure 4,

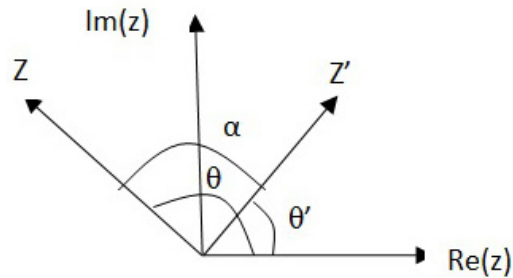


Figure 4. Calibrate the angular position of the vector represented by ( $z$ ). Source: own.

The equation of the particle does not change under the transformation,

$$z \rightarrow z' = e^{-i\alpha} z , \quad (17)$$

Being ( $\alpha$ ) a real and equal constant for all observers, with numbers ( $1, e^{i\alpha}, e^{i2\alpha}$ ) forming an overall group structure.

The derivative of the position of the particle transforms as:

$$\frac{dz'}{dt} = e^{-i\alpha} \frac{dz}{dt} , \quad (18)$$



Invariant being its magnitude,

$$\left| \frac{dz'}{dt} \right| = \left| \frac{dz}{dt} \right|.$$

Considering the constant speed of the particle, the acceleration of the particle is given by:

$$\frac{d^2 z'}{dt^2} = e^{-i\alpha} \frac{d^2 z}{dt^2}, \quad (19)$$

It is invariant. Therefore, it assuming the mass (m) of the particle is obtained invariant that the centripetal force will be equal for all inertial observers,

$$\vec{F}_c = \vec{F}'_c. \quad (20)$$

Furthermore, there is another way of performing recalibration of the position of the particle making the time constant ( $\alpha$ ) dependent. In this case, the position will depend on the moment of time  $\alpha = \alpha(t)$  it takes each observer to measure the initial position of the particle as shown in the figure below,

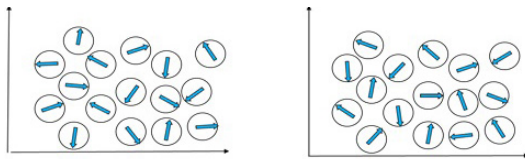


Figure 5. Local Calibration of the position of the particle relative to different observers. Source: own.

Using the time dependence of the phase  $\alpha = \alpha(t)$  the transformation (17) can be written as:

$$z \rightarrow z' = e^{-i\alpha(t)} z. \quad (21)$$

The equation derived from the position of the particle in this case becomes as,

$$\frac{dz'}{dt} = e^{-i\alpha(t)} \left( -iz \frac{d\alpha}{dt} + \frac{dz}{dt} \right), \quad (22)$$

Which does not transform as does (18) to obtain a non-invariance. Invariance is recovered by replacing the common derivative  $\frac{d}{dt}$  by the covariant derivative,

$$\frac{d}{dt} \rightarrow \frac{d'}{dt} = \frac{d}{dt} + iA(t), \quad (23)$$

or,

$$D \rightarrow D' = D + iA(t), \quad (24)$$

It is  $A = A(t)$  a time-dependent parameter compensator. Therefore, using the covariant derivative transforming the derivative of the position of the particle is given by:

$$\frac{d'z'}{dt} = e^{-i\alpha(t)} \frac{dz}{dt}, \quad (25)$$

if,

$$A(t) = \frac{d\alpha}{dt}, \quad (26)$$

With dimensions of angular velocity  $A(t) = \omega(t)$ . Thus, the requirement of invariance of the derivative of the equation leads to the introduction position of a compensating term  $A(t)$  through the covariant derivative; term that is interpreted as the angular velocity of the particle. Therefore, the use of the complex plane to determine the position of the particle, gives the possibility of introducing the idea gauge description motion of a particle in rotation. The particle velocity is then defined by:

$$v = Dz = \frac{dz}{dt} + i \frac{d\alpha}{dt} z, \quad (27)$$

the term being  $\text{Re}(v) = \frac{dz}{dt}$  the radial velocity component, and the term  $\text{Im}(v) = \frac{d\alpha}{dt} z$  the velocity component in the tangential direction of the particle. Differentiating acceleration is the velocity of the particle,

$$a = (Dv) = \left(\frac{d}{dt} + iA(t)\right) \left(\frac{dz}{dt} + i \frac{d\alpha}{dt} z\right), \quad (28)$$

or,

$$a = \ddot{z} - z\dot{\alpha}^2 + i(z\ddot{\alpha} + 2\dot{z}\dot{\alpha}). \quad (29)$$

In part  $\text{Re}(a) = \ddot{z} - z\dot{\alpha}^2$ , the term  $(\ddot{z})$  corresponds to the acceleration in the radial direction due to the change of the radial speed, and the term  $(-z\dot{\alpha}^2)$  corresponding to the centripetal acceleration due to the change of tangential velocity vector. In part  $\text{Im}(a) = z\ddot{\alpha} + 2\dot{z}\dot{\alpha}$ , the term  $(z\ddot{\alpha})$  gives

the particle acceleration in the tangential direction due to the change in the magnitude of the tangential velocity and the term  $(2\dot{z}\dot{\alpha})$  representing the Coriolis acceleration, which is partly due to the change in the radial speed and the tangential speed change.

In literature [11], [1], the acceleration of the particle written in polar coordinates as found:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}, \quad (30)$$

Whose shape is similar to (29). Finally, the dynamic effects that act on the particle are described by the forces,

$$F_{centrifuga} = m\ddot{z}; \quad F_{centripeta} = -mz\dot{\alpha}^2, \quad (31)$$

$$F_{tagencial} = mz\ddot{\alpha}; \quad F_{coriolis} = 2m\dot{z}\dot{\alpha} \quad (32)$$

or,

$$F = ma = m(\ddot{z} - z\dot{\alpha}^2) + im(z\ddot{\alpha} + 2\dot{z}\dot{\alpha}), \quad (33)$$

Polar coordinates	Complex plane
Position: $\vec{r} = r \hat{r}$	Position: $Z = x + iy$
Speed: $\vec{v} = \dot{r} \hat{r} + r\dot{\theta} \hat{\theta}$	Speed: $v = Dz = \frac{dz}{dt} + iz \frac{d\alpha}{dt}$
Acceleration: $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$	Acceleration: $a = \ddot{z} - z\dot{\alpha}^2 + i(z\ddot{\alpha} + 2\dot{z}\dot{\alpha})$
Force: $\vec{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$	Force: $F = ma = m(\ddot{z} - z\dot{\alpha}^2) + im(z\ddot{\alpha} + 2\dot{z}\dot{\alpha})$
$\vec{F}_{centrifuga} = m\ddot{r}\hat{r}; \quad \vec{F}_{centripeta} = -mr\dot{\theta}^2\hat{r}$	$F_{centrifuga} = m\ddot{z}; \quad F_{centripeta} = -mz\dot{\alpha}^2$
$\vec{F}_{tagencial} = mr\ddot{\theta}\hat{\theta}; \quad \vec{F}_{coriolis} = 2m\dot{r}\dot{\theta}\hat{\theta}$	$F_{tagencial} = mz\ddot{\alpha}; \quad F_{coriolis} = 2m\dot{z}\dot{\alpha}$

Table 1. Comparative table on the motion of a particle in rotation, using polar coordinates and complex plane. Source: own



The result (33) is similar to that given by (1), when an analysis of the motion of a particle in rotation using polar coordinates [1] is made. In the Table 1, it shows the comparative on the motion of a particle in rotation, using polar coordinates and complex plane.

## 5. Conclusion

- The classical scheme allows a description of the motion of a particle in rotation respect to arbitrary reference systems. In this perspective, the dynamic effects are obtained; Coriolis forces, centrifugal, centripetal and tangential. However, from a geometrical perspective you can be obtained the same results using the complex plane to determine the position of the particle, giving the possibility of introducing concepts used in the construction of a gauge theory.
- The idea put forward is a way to address physical reference systems in rotation within a geometric context, obtaining analogous to those obtained from a classical view results.
- The analysis has a pedagogical intention, as it seeks a way to address topics of physics in a different way characterized by new forms of readings and levels of explanation of phenomena. In this sense, a bridge between the knowledge acquired by students and new looks for the explanation of the phenomena is created.

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