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Some properties for matrices that commute with their transpose
Algunas propiedades para matrices que conmutan con su traspuesta

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Abstract: The purpose of this article is to present some conditions in the general $n \times n$, and the particular 2×2 and 3×3 cases, that characterize the matrices set:

$$T_n = \{A \in M_n(\mathbb{C}) \mid AA^T = A^T A\}$$

Where $M_n(\mathbb{C})$ denotes the squared matrices set of nth order, $\mathbb{C}$ the set of complex numbers and T the transposition operator.

Keywords: squared Matrix, transposition operator, commutativity, complex numbers.

Resumen: el propósito de este artículo es presentar algunas condiciones que caracterizan el conjunto de matrices $n \times n$ -y el caso particular de 2×2 y 3×3-:

$$T_n = \{A \in M_n(\mathbb{C}) \mid AA^T = A^T A\}$$

Donde $M_n(\mathbb{C})$ denota el conjunto de las matrices cuadradas de orden n, $\mathbb{C}$ el conjunto de los n**úmeros complejos y** el operador de trasposici**ón.**

Palabras clave: Matriz cuadrada, operador de transposición, conmutatividad, números complejos.

1. Introduction

For some time now, people have been studying the properties and characterizations of the matrices sets that commute, see [1] and [2]. In this field of investigation the set of normal matrices is of great importance and up to the date, we know around 100 different characterizations, see [3], [4] and [5], that reflect normality from different points of view, as for instance: eigenvalues and singular values, invariant spaces, decomposition in terms of Hermitian and skew-Hermitian matrices, polar decomposition, commutativity of the conjugate and commutativity with its transpose, amongst others. Motivated by this work, we begin the study of the T_n set.

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Note that T_n contains the sets of symmetric, skew-symmetric, orthogonal, and normal matrices with real entries and the Hadamard matrices. Of this topic there are few results, see [6] page 475. That is why, our work in this second phase consisted in presenting a description of some geometric and topological characteristics of the general case, and particularities of the 2×2 and 3×3 cases. This characterizations reflect the commutativity of a matrix with its transpose evidencing its centralizer, which is done through non-derogatory matrices, the decomposition in terms of symmetric and skew-symmetric matrices and its eigenvalues.

In Some Cuantum Mechanics problems Cardy's differential equation [7] is cited:

$$u(u-1)\frac{d^2\phi}{du^2}+\frac{2}{3}(1-2u)\frac{d\phi}{du}=0, with\, \phi(0)=0, \phi'(1)=1$$

Which is deduced in a heuristically manner. Some algebraic aspects in the equation are viewed when a symmetry property is used: the orthonormal complete solution to this equation is given in commutative algebra, i.e. as normal matrices or matrices that commute with its transpose [8]. The following notations are used throughout this article and are useful in its understanding.

$\mathbb{C}$: Set of complex numbers

$M_n(\mathbb{C})$ $n\times n$: matrices with complex entries

$AS_n(\mathbb{C})$: Skew-symmetric matrices with complex entries

$S_n(\mathbb{C})$: Symmetric matrices space with complex entries

A_s: Symmetric part of $A\in M_n(\mathbb{C})$

A_{as}: Skew-symmetric part of $A\in M_n(\mathbb{C})$

$||A||F$: The Frobenius norm $A\in M_n(\mathbb{C})$

$\sigma(A)$: The set of the eigenvalues of $A\in M_n(\mathbb{C})$

$\partial(p)$: The degree of the polynomial p

I_n Identity matrix of order n

2. Geometric and Topological Characteristics of T_n

Regarding the matrices set $M_n(\mathbb{C})$ we will consider the inner product given by $\langle A,B\rangle=Tr(B^*A)$, this inner product induces the norm over the matrices space

$$||A||_F=\left(T_r\left(A^*A\right)\right)^{1/2}$$

The inducted metric by this norm is

$$d(A,B)=||A-B||_F$$

Lastly, regarding $M_n(\mathbb{C})$ we will consider the topology inherited by this metric.

The following results are basic in the development of the next section which can be found in [9] and [1].

Definition 2.1.

Let $A\in M_n(\mathbb{C})$, the centralizer of A which we will denote $C(A)$ and define by

$$C(A)=\{B\in Mn(\mathbb{C})|AB=BA\}$$

Theorem 2.2.

$A\in M_n(\mathbb{C})$ is a non derogatory matrix, if and only if, the characteristic polynomial and the minimal of are equal.

Theorem 2.3.

$A\in M_n(\mathbb{C})$ is diagonalizable, if and only if, for each eigenvalue of its algebraic and geometric multiplicity match up.

Theorem 2.4.

Let $A \in M_n(\mathbb{C})$ be symmetric. A is diagonalizable, if and only if, the matrix that diagonalizes is an orthogonal complex matrix.

For a given field P, $P[x]$ is denoted the ring of all polynomials $p(x)$ over the field P. In a similar way, for a square matrix A with elements of P, we denote as $P[A]$ the ring of all matrices that can be written in the form $p(A)$, where $p(x) \in P[x]$.

Theorem 2.5.

If A is a non-derogatory matrix of order n, then

$$C(A) = \{p(A) \mid \partial(p) \leq n-1\}$$

Theorem 2.6.

If A is matrix of the orden n, then the minimal and characteristic polynomial of A match up, if and only if, $C(A) = P[A]$.

As an immediate consequence of the theorems given previously we have:

Proposition 2.7.

If A is non-derogatory and $A \in T_n$, then $C(A) \subseteq T_n$.

Proposition 2.8.

Let $A \in S_n(\mathbb{C})$. If A is non-derogatory, then $C(A) \subseteq S_n(\mathbb{C})$.

Corollary 2.9.

For each symmetric non-derogatory matrix, an skew-symmetric not null matrix that commutes with it does not exist.

Corollary 2.10.

Let $A \in M_n(\mathbb{C})$, where A is written in the form $A = A_s + A_{as}$. If A_{sn} is non-derogatory and $A_{sn} \neq 0$, then $A \notin T_n$.

2.1.1. Case n × n

This section will present topological characteristics of the T_n set in the case n × n.

Proposition 2.11.

T_n satisfies the following conditions:

I. T_n is closed.

II. T_n is whole.

III. T_n is not compact.

IV. T_n is connected by pathways in star form.

V. $T_n - \{0\}$ is connected by pathways.

VI. if $H \in C(A^T)$, then $f'(A)(H) = 0$ for each $A \in T_n$, where

$$f: M_n(\mathbb{C}) \to S_n(\mathbb{C})$$

$$X \to f(X) = XX^T - X^T X$$

Proff.

I. First. f is differentiable in each point, for which we find

$$\begin{aligned} f(X+H) &= (X+H)(X+H)^T - (X+H)^T(X+H) \\ &= XX^T + XH^T + HX^T + HH^T - (XX^T + X^TH + H^TX + H^TH) \\ &= XX^T - XX^T + XH^T + HX^T - XX^T - H^TX + HH^T - H^TH \\ &= f(X) + (XH^T + HX^T - X^TH - H^TX) + (HH^T - H^TH) \end{aligned}$$

Where $(XH^T + HX^T - X^TH - H^TX) = f'(X)(H)$, $(HH^T - H^TH) = R(H)$

This shows that f' exists for each point and also

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$$f': M_n(\mathbb{C}) \to S_n(\mathbb{C})$$

$$H \to f'(X)(H) = XH^T + HX^T - X^T H - H^T X$$

This implies the continuity of f, which in consequence proofs that T_n is closed because $T_n = f'^{-1}(0)$.

II. Since T_n is closed and $M_n(\mathbb{C})$ is a whole space, then T_n is whole.

III. T_n is not compact, since is not bounded, since $\alpha I \in T_n$ for $\alpha \in \mathbb{C}$ and $\| \alpha I_n \|_F = |\alpha|\sqrt{n}$.

IV. Let $A_1, A_2 \in T_n$ and

$$\alpha(t) \begin{cases} 2tI + (1-2t)A_1, if\ 0 \le t \le \frac{1}{2} \\ (2-2t)I + (2t-1)A_2, if \frac{1}{2} \le t \le 1 \end{cases}$$

Note that α is a continuous path, $\alpha(o) = A_1$ and $\alpha(1) = A_2$. Besides, if $A \in T_n, k_1 I + k_2 A \in T_n$, for k_1 and k_2 in $\mathbb{R}$, which implies that T_n is connected in star form around $\langle I \rangle$.

V. Case 1: If $A_1, A_2 \notin \langle \mathrm{I} \rangle$, then the path is taken as in (iv).

Case 2: If $A_2 \in \langle \mathrm{I} \rangle$, and $A_1 \notin \langle \mathrm{I} \rangle$, then $A_2 = kI$ with $k \in \mathbb{C}$. The path is defined as

$\alpha(t) = kI + (1-t)A_1$, which is continuous and different from zero for each $0 \le t \le 1$

Case 3: If $A_1, A_2 \in \langle \mathrm{I} \rangle$, we take $A \in T_n$ with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus [0]_{n-2}$, where $\oplus$ denotes direct adition and $[0]_{n-2}$ the square null matrix of order $n-2$ and α the path given by

$$\alpha(t) \begin{cases} 2tIA + (1-2t)A_1, if\ 0 \le t \le \frac{1}{2} \\ (2-2t)A + (2t-1)A_2, if \frac{1}{2} \le t \le 1 \end{cases}$$

Note that α is continous path, $\alpha(0) = A_1$ and $\alpha(1) = A_2$, and also that, $[0] \notin \alpha$.

VI. Is obtained directly from f' found in (i).

2.1.2. Case 2×2

This section presents a complete description of the T_2 set.

Proposition 2.12.

For each matrix in $A \in M_2(\mathbb{C})$, A is either a scalar multiple of the identity matrix or a non-derogatory matrix.

Proof. Case1: If A is diagonalizable then A, either has two equal eigenvalues and in this case is a multiple of the identity, or its eigenvalues are different and is non-derogatory. Case 2: If A can't be diagonalized, then its canonic Jordan form is

$$\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} or \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix} \ with\ \alpha \neq \beta$$

In either of these cases A is non-derogatory.

Corollary 2.13.

For each $A \in M_2(\mathbb{C})$ where $A \notin \langle \mathrm{I}_2 \rangle$, $C(A) = \{p(A) \mid \partial(p) \le 1\}$

By direct calculation we have that T_2 is given by:

$$T_2 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{C} \right\}$$

Proposition 2.14.

T_2 satisfies the following conditions:

I. $T_2 - \langle I_2 \rangle$ is a set with two connected components.

II. $T_2 - C(A)$ is a connected set if A is skew-symmetric and disconnected if A is symmetric.

III. Locally T_2 is a two or three dimensional manifold except for each point in $\langle I_2 \rangle$.

Prof.

I. Note that form (2.1)

$$T_2 = \langle I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle \cup S_2(\mathbb{C})$$

Also,

$$S_2(\mathbb{C}) = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, I_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$$

From these we have that

$$T_2 - \langle I_2 \rangle = \left(\langle I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle - \langle I_2 \rangle \right) \cup (S_2(\mathbb{C}) - \langle I_2 \rangle) = \widehat{T_2} \cup \widetilde{T_2}$$

Where $\widehat{T_2} = \langle I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle - \langle I_2 \rangle$ and

$$\widetilde{T_2} = S_2(\mathbb{C}) - \langle I_2 \rangle$$

Now let's see how $\widehat{T_2}$ is connected:

If

$$A_1 = k_1 I_2 + k_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \; and$$

$$A_2 = r_1 I_2 + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \widehat{T_2}$$

Where $k_1, k_2, r_1, r_2 \in \mathbb{C}$ with $k_2, r_2 \neq 0$ and

$$\alpha_1(t) = \begin{cases} 2t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (1-2t) A_1, & if \; 0 \leq t \leq \frac{1}{2} \\ (2-2t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (2t - I) A_2, & if \; \frac{1}{2} \leq t \leq 1 \end{cases}$$

From this definition we conclude that α_1 is a continuous path and that $\langle I_2 \rangle \notin \alpha_1(t)$ for each $0 \leq t \leq 1$.

In a similar form, for $\widetilde{T_2}$ we have:

$$A_1 = k_1 I_2 + k_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \; and$$

$$A_2 = r_1 I_2 + r_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \widetilde{T_2}$$

Where $k_1, k_2, k_3, r_1, r_2, r_3 \in \mathbb{C}$ with $k_2, k_3, r_2, r_3 \neq 0$ and

$$\alpha_2(t) = \begin{cases} 2t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (1-2t) A_1, & if \; 0 \leq t \leq \frac{1}{2} \\ (2-2t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (2t - I) A_2, & if \; \frac{1}{2} \leq t \leq 1 \end{cases}$$

From this definition we conclude that α_2 is a continuous path and that $\langle I_2 \rangle \notin \alpha_2(t)$ for each $0 \leq t \leq 1$.

II. If A is skew-symmetric $T_2 - C(A) = \widetilde{T_2}$, which is connected by (i). If A is symmetric $T_2 - C(A) \subseteq$ and $T_2 - \langle I_2 \rangle = \widehat{T_2} \cup \widetilde{T_2}$, which is disconnected because of

$$T_2 - C(A) \cap \widehat{T_2} \neq \emptyset$$

And

$$T_2 - C(A) \cap \widetilde{T_2} \neq \emptyset$$

This last one is satisfied if we take $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ matrix, which only commute with multiples of the identity.

III $T_2 = \langle I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle \cup \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, I_2, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$that and $\langle I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle$ is a two manifold and $\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, I_2, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle$ is a three manifold.

2.1.3. Case 3 x 3

This section presents a description of the matrices with complex entries that are inside the T_3 set.

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Propositon 2.15.

If A is an skew-symmetric non null matrix, then A is non-derogatory.

Proof.

Let

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

With a, b and c complex non null scalars. The eigenvalues of A are: 0, $\sqrt{a^2+b^2+c^2}\, i$ and $-\sqrt{a^2+b^2+c^2}\, i$. If $a^2+b^2+c^2 \neq 0$, then the matrix has all its eigenvalues different and as a consequence its characteristic and minimal polynomial are equal; from this and from theorem 2.2 we conclude that A is non-derogatory. On the other hand, if $a^2+b^2+c^2=0$, then the characteristic polynomial of A is $p(x)=x^3$ and since

$$A^2 = \begin{bmatrix} -a^2-b^2 & -bc & ac \\ -bc & -a^2-c^2 & -ab \\ ac & -ab & -b^2-c^2 \end{bmatrix} \neq 0$$

And

$$A^3 = \begin{bmatrix} 0 & -a(a^2+b^2+c^2) & -b(a^2+b^2+c^2) \\ a(a^2+b^2+c^2) & 0 & -c(a^2+b^2+c^2) \\ b(a^2+b^2+c^2) & c(a^2+b^2+c^2) & 0 \end{bmatrix} = 0$$

From these the characteristic and minimal polynomials of A coincide, from where we conclude that by theorem 2.2 A is non-derogatory.

Corollary 2.16.

If A is an skew-symmetric not null matrix, then

$$C(A_{as}) = \{p(A_{as}) \mid \partial(p) \leq 2\}$$

Proof.

This result is obtained from proposition 2.15 and from theorem 2.5.

Proposition 2.17.

Let A be a matrix such as that its skew-symmetric part A_{as} is not null, then $A \in T_3$, if and only if, $A = \alpha_0 I_3 + \alpha_1 A_{as} + \alpha_2 A_{as}^2$.

Proof.

Let's suppose that $A \in T_3$ and write A as $A = A_s + A_{as}$, then $A_s A_{as} = A_{as} A_s$. From here and corollary 2.16 we have that $A \in C(A_{as})$, and because of these $A \in C(A_{as})$. Reciprocally, if $A \in C(A_{as})$ then there are complex scalars α_0, α_1 and α_2 such as,

$A = \alpha_0 I_3 + \alpha_1 A_{as} + \alpha_2 {A_{as}}^2$, from where A commutes with transpose.

As a direct consequence of the last proposition we have:

Corollary 2.18.

$$T_3 = S_3(\mathbb{C}) \cup \left(\cup_{A_{as} \neq 0} C(A_{as}) \right)$$

Proposition 2.19.

If B is a symmetric matrix that commutes with an skew-symmetric matrix $A \neq 0$, then:

I. $B = \alpha I_3 + \beta A^2$ *with* $\alpha, \beta \in \mathbb{C}$

II. $B \in \langle I_3 \rangle$ *or* B *has only two different eigenvalues, one of the wit geometric multiplicity of two.*

Proof.

If B commutes with A then from corollary 2.16 we have that

$$B = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2$$

with $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$. Since $B = B^T$, then $\alpha_1 = 0$ and $B = \alpha_0 I_3 + \alpha^2 A^2$.

Let's consider

$$B = \alpha I_3 + \beta A^2 \qquad (2.2)$$

with $\alpha, \beta \in \mathbb{C}$. As A is skew-symmetric of 3×3 order, its eigenvalues are: $0, \lambda$ and $-\lambda$ for some $\lambda \in \mathbb{C}$, as a consequence, the eigenvalues of A^2 are: 0 y λ^2 with algebraic multiplicity of 2. From the Schur triangularization theorem there is an unitary matrix U that upper triangularizes A^2 and from the equality (2.2) such unitary matrix will also upper triangularizate B; from this last remark, we conclude that the eigenvalues of B are: α and $\alpha + \beta\lambda^2$. Now, if $\beta = 0$ then $B \in \langle I_3 \rangle$, on the other hand B has only two different eigenvalues , one of them with geometrical multiplicity 2, if that isn't the case, the geometrical multiplicity would be 1, and B will be non-derogatory, and from theorem 2.5

$$C(B) = \{p(B) \mid \partial(p) \leq 2\} \subseteq S_3(\mathbb{C})$$

in this case A will be a symmetric not null matrix, which would be a contradiction.

Corollary 2.20.

Each symmetric matrix B that commutes with an skew-symmetric matrix $A \neq 0$ is diagonalizable.

Proof.

From proposition 2.19 we have that $B \in \langle I_3 \rangle$, and in this case, B is clearly diagonalizable or B has only two different eigenvalues, and one of them has geometric multiplicity of 2. From this remark, we conclude that the geometric multiplicity of the eigenvalues of B are equal, which implies, from theorem 2.3, that the matrix is diagonalizable.

Proposition 2.21.

If A and B are non null skew-symmetric matrices with $A^2 \neq \alpha B^2$ for each $\alpha \in \mathbb{C}$, then

$$C(A) \cap C(B) = \langle I_3 \rangle$$

Proof.

Let $X \in C(A) \cap C(B)$ and let's write

$$X = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2 = \beta_0 I_3 + \beta_1 B + \beta_2 B^2$$

With $\alpha_i, \beta_i \in \mathbb{C}$, then

$$\alpha_1 A = \beta_1 B \qquad (2.3)$$

And

$$\alpha_0 I_3 + \alpha_2 A^2 = \beta_0 I_3 + \beta_2 B^2$$

Note that X is a symmetric matrix, if $\alpha_1, \beta_1 = 0$, then by (2,3)

$$A^2 = \alpha B^2$$

For some $\alpha \in \mathbb{C}$ which is a contradiction. Therefore X is symmetric matrix of the form

$$X = \alpha_0 I_3 + \alpha^2 A^2 = \beta_0 I_3 + \beta_2 B^2$$

Also, lets observe that by corollary 2.20 X is diagonalizable. On the other hand, if $\alpha_2 = 0$ or $\beta_2 = 0$, then $X \in \langle I_3 \rangle$, on the opposite case

$$\frac{X - \alpha_0 I_3}{\alpha_2} = A^2 \qquad (2.4)$$

And

$$\frac{X - \beta_0 I_3}{\beta_2} = B^2 \qquad (2.5)$$

Since X is diagonalizable, then A^2 and B^2 are also diagonalizable, therefore, both

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matrices have a null eigenvalue. Let $\sigma(X) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 = \lambda_2$, let's asume, without a loss of generality that λ_2 has algebraic multiplicity 2, then by (2.4) and (2.5)

$$\frac{\lambda 1-\alpha_0}{\alpha_2} = 0 \; y \; \frac{\lambda 1-\beta_0}{\beta_2} = 0 \tag{2.6}$$

Or

$$\frac{\lambda 2-\alpha_0}{\alpha_2} = 0 \; y \; \frac{\lambda 2-\beta_0}{\beta_2} = 0 \tag{2.7}$$

Or

$$\frac{\lambda 1-\alpha_0}{\alpha_2} = 0 \; y \; \frac{\lambda 2-\beta_0}{\beta_2} = 0 \tag{2.8}$$

Note that (2.6) and (2.7) can't be give, since in that case by (2.4) and (2.5) $A^2 = \alpha B^2$ for some $\alpha \in \mathbb{C}$. Like that $\alpha_0 = \lambda_1$ *and* $\beta_0 = \lambda_2$. Given that the algebraic multiplicity of λ_2 is 2, then 0 is an eigenvalue of B^2 with algebraic multiplicity of 2, because of $\sigma(B^2) = \{0\}$ and since B^2 is diagonalizable, its necessarily that $B^2=0_{3x3}$ therefore $X \in \langle I3 \rangle$.

Lemma 2.22.

I. If B is a non null 3×3 symmetric matrix, we have:

1. B is non-derogatory for the following cases:

2. B has 3 different eigenvalues.

3. B has 2 eigenvalues both with geomet ric multiplicity of one.

in any of these cases $C(B) \subseteq S3(\mathbb{C})$.

II. B is derogatory for the following cases:

1. B has only one eigenvalue with geometric multiplicity of three, and therefore $B \in \langle I3 \rangle$.

2. B has only one eigenvalue with geometric multiplicity of two, and therefore it won't exist a non null skew-symmetric matrix that commute with B.

3. B has two eigenvalues, one of them with geometric multiplicity of two, and therefore $B \in C(A)$ one for some non null skew-symmetric matrix A.

Proof.

I. a.b.c Are an immediate consequence of proposition 2.8.

II. a Is a consequence of theorem 2.3.

II. b Is a consequence of proposition 2.19.

II. Let $\sigma(B) = \{\lambda_1, \lambda_2\}$ *with* $\lambda 1 \neq \lambda 2$ without loss of generality let's assume that $\lambda 2$ has geometric multiplicity of 2. From there and by theorem 2.3 B is diagonalizable, therefore by theorem 2.4 there will exist a complex orthogonal matrix Q that will diagonalize through similarity B. If there is a non null skew-symmetric matrix X such that $B \in C(X)$. Is enough to proof that the equation

$$X^2 + \lambda_1 I_3 = B$$

has a solution inside the non null skew-symmetric matrices set. In effect, one such skew-symmetric matrix that is a solution of the equation (2.9) is given by:

$$X = i\sqrt{\lambda 2 - \lambda 1}\, Q^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} Q$$

Where Q is the ortogonal matrix that diagonalices B.

Next there is a list of properties of non null skew-symmetric matrix centralizers.

Proposition 2.23.

Let A be a non null skew-symmetric matrix, then:

I. $dim(C(A)) = 3$.

II. $dim(C(A) \cap S_3(C)) = 2$.

III. $C(A) - \langle I_3 \rangle$ *is a connect set.*

IV. $UC(A)$ *is a connect set*

V. $UC(A)$ *is not a convex set.*

Proof.

I. Let α_0, α_1 and α_2 be complex scalars. Considering the linear combination

$$\alpha_0 I_3 + \alpha_1 A + \alpha_2 A_2 = 0 \tag{2.10}$$

For this to be true the scalars will necessarily be null. From (2.10) we obtain:

$$\alpha_1 A = 0 \tag{2.11}$$

$$\alpha_0 I_3 + \alpha_2 A^2 = 0 \tag{2.12}$$

Then by (2.11) $\alpha_1 = 0$. Now, if $\alpha_0 \neq 0$ from (2.12)

$$I_3 = \frac{-\alpha_2}{\alpha_0} A^2$$

Which necessarily implies $\alpha_2 \neq 0$ and $\sigma(A^2) = \{\frac{-\alpha_0}{\alpha_2}\}$ therefore $0 \notin \sigma(A^2)$, which is a contradiction. Similarly the condition $\alpha_2 \neq 0$ leads to a contradiction.

II. Let $X, Y \in C(A) - \langle I_3 \rangle$, where

$$X = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2$$

$$Y = \beta_0 I_3 + \beta_1 A + \beta_2 A^2$$

Note that $(\alpha_1, \alpha_2), (\beta_1, \beta_2) = (0,0)$.

Let's consider the following cases:

Case 1:

If $\alpha_1 \neq 0$ we take the continuous path $\psi_1(t) = tX + (1-t)A^2$ for some

$0 \leq t \leq 1$, because in the contrary case we will have that

$$t(\alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2) + (1-t)A^2 = kI_3$$

for some $k \in \mathbb{C}$,

From where

$$t\alpha_0 = k$$

$$t\alpha_1 = 0$$

$$t\alpha_2 + (1-t) = 0$$

These equalities give $t = 0$ and $1 = 0$, which is an absurd.

Case 2:

If $\alpha_2 \neq 0$ we take the continuous path $\psi_2(t) = tX + (1-t)A$ for some $0 \leq t \leq 1$, and in a way similar to case 1 $\psi_2 \notin \langle I_3 \rangle$ for each $0 \leq t \leq 1$.

For cases $\beta_1 \neq 0$ and $\beta_2 \neq 0$ we take the continous paths:

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$\psi 3(t) = tY + (1-t)A^2$ and $\psi 4(t) = tY + (1-t)A$ for some $0 \leq t \leq 1$, respectively. In a similar way to case 1, it easy to observe that $\psi 3, \psi 4 \notin \langle I3 \rangle$ for each $0 \leq t \leq 1$.

Now, let's build paths to connect X with Y that won't pass through $\langle I3 \rangle$:

1. If $\alpha 1, \beta 1 \neq 0$ we take $\phi_1(t)$

$$\begin{cases} (1-2t)X + (2t)A^2, & if\ 0 \leq t \leq \frac{1}{2} \\ (2-2t)A^2 + (2t-1)Y, & if\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

2. If $\alpha 1, \beta 1 \neq 0$ we take $\phi 2(t)$

$$\begin{cases} (1-3t)X + (3t)A^2, & if\ 0 \leq t \leq \frac{1}{3} \\ (2-3t)A^2 + (3t-1)A, & if\ \frac{1}{3} \leq t \leq \frac{2}{3} \\ (3-3t)A + (3t-2)Y, & if\ \frac{2}{3} \leq t \leq 1 \end{cases}$$

Note that any path that connect A^2 with A is not present in $\langle I3 \rangle$ since $dim(C(A)) = 3$.

3. If $\alpha 2, \beta 1 \neq 0$ we take $\phi 3(t)$

$$\begin{cases} (1-3t)X + (3t)A, & if\ 0 \leq t \leq \frac{1}{3} \\ (2-3t)A + (3t-1)A^2, & if\ \frac{1}{3} \leq t \leq \frac{2}{3} \\ (3-3t)A^2 + (3t-2)Y, & if\ \frac{2}{3} \leq t \leq 1 \end{cases}$$

4. If $\alpha 2, \beta 2 \neq 0$ we take $\phi 4(t)$

$$\begin{cases} (1-2t)X + (2t)A, & if\ 0 \leq t \leq \frac{1}{2} \\ (2-2t)A + (2t-1)Y, & if\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

IV. tween two elements of $UC(A)$ an element of $\langle I3 \rangle$, since $\langle I3 \rangle \subseteq \cap C(A)$ and $C(A)$ is a vectorial $\mathbb{R}$-space.

V. Let

$$C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E = I_3 + C + C^2$$

And $F = I_3 + D + D^2$. Then

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}E + \frac{1}{2}B \notin UC(A).$$

Therefore $UC(A)$ is not a convex set.

Proposition 2.24.

$T3 - \langle I_3 \rangle$ Is a connect set.

Proof.

Let $\alpha 1, \alpha 2, \alpha 3$ be different complex scalars and

$$B = \begin{bmatrix} \alpha 1 & 0 & 0 \\ 0 & \alpha 2 & 0 \\ 0 & 0 & \alpha 3 \end{bmatrix}$$

A non-derogatory symmetric matrix that is not in $C(A)$ for each non null skew-symmetric matrix A from proposition 2.19. Let's take the path

$$\alpha(t) = tB + (1-t)\begin{bmatrix} \alpha 1 & 0 & 0 \\ 0 & \alpha 2 & 0 \\ 0 & 0 & \alpha 3 \end{bmatrix}$$

for some $t \in [0,1/2]$. It is easy to observe that $\alpha(t) \notin \langle I_3 \rangle$ for each $t \in [0,1/2]$. Now, if $t = 1/2$ we obtain a symmetric matrix $\bar{B}$ with two different eigenvalues, one of them with geometric multiplicity of 2, from proposition 2.19 there will exist a non null skew-symmetric matrix $\hat{A}$, such that $\hat{B} \in C(\hat{A})$ and from numeral (iii.) we can connect through a continuous path in $C(\hat{A}) - \langle I_3 \rangle \hat{B}$ with any element inside this set.

Bibliography

[1] D. A. Suprunenko, and R. I. Tyshkevich, "Commutative matrices", Academic Press, 1968.

[2] O. Taussky, "A generalization of matrix commutativity", Linear Algebra and its Applications, 2 pp. 349-353, 1969.

[3] L. Elsner, and C. R. Ikramov, "Normal matrices: An Update", Linear Algebra and its Applications, pp. 79-98, 1997.

[4] R. Grone, C. R. Johnson and H. Wolkovicz, Normal matrices, Linear Algebra and its Applications, pp. 213-225, 1987.

[5] G. Goodson, "Matrices That Commute with Their Con- jugatevand Transpose", Advances in Linear Algebra & Matrix Theory, 3 pp. 22-25, 2013.

[6] R. Horn, and C. Johnson, "Topics in Matrix Analysis" Cambridge U.P., 1991.

[7] K. Golden, G. Grimmett, R. James, G. Milton and P. Sen, "Mathematics of Multiscale Materials", Springer, pp. 18-19, 1998.

[8] Z. Horváth, and L. Palla, "Nonperturbative Quantum- field-theoretic Methods and Their Applications:" proceed- ings of the 24th Johns Hopkins workshop, Bolyai College, Budapest, Hungary, World Scientific, pp. 14-17, august 2000.

[9] R. Horn and C.Johnson, "Matrix Analysis", Cambridge U.P., 1985.

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