The investigation of approximate solutions of Deformed Klein-Fock-Gordon and Schrödinger Equations under Modified Equal Scalar and Vector Manning-Rosen and Yukawa Potentials by using the Improved Approximation of the Centrifugal term and Bopp's shift Method in NCQM Symmetries

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ISSN 1870-9095

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(Received 28 February 2021, accepted 30 May 2021)

Abstract

In the current work, we employed the elegant tool of Bopp's shift and standard perturbation theory methods to obtain a new relativistic and nonrelativistic approximate bound state solution of deformed Klein-Fock-Gordon and deformed Schrödinger equations using the modified equal vector scalar Manning-Rosen potential and Yukawa potential model. Furthermore, we have employed the improved approximation to the centrifugal term for some selected diatomic molecules such as (N2, I2, HCl, CH and LiH) in the symmetries of extended quantum mechanics to obtain the approximate solutions. It is seen that relativistic shift energy $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ and the perturbative nonrelativistic corrections $\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$ are sensitive to the quantum numbers (n, j, l, s, m), the potential range b, two dimensionless parameters (A and α), the strength of the potential V_0 in addition to the parameters of noncommutativity (Θ, σ, χ). We have highlighted three physical phenomena that automatically generate a result of the topological properties of noncommutativity, the first generated physical phenomena are the perturbative spin-orbit coupling, the second the magnetic induction while the third corresponds to the rotational proper phenomena. In both relativistic and nonrelativistic problems, we show that the corrections on the spectrum energy are smaller than the main energy in the ordinary cases of quantum field theory and quantum mechanics. A straightforward limit of our results to ordinary quantum mechanics shows that the present result under modified equal vector scalar Manning-Rosen potential and Yukawa potential is consistent with what is obtained in the literature. In the new symmetries of NCQM, is not possible to get the exact analytical solutions for $l = 0$ and $l \neq 0$, the approximate solutions are available. We have observed that the deformed Klein-Fock-Gordon equation under the modified equal vector scalar Manning-Rosen potential and Yukawa potential model has a physical behavior similar to the Duffin–Kemmer equation for meson with spin-1, it can describe a dynamic state of a particle with spin-1 in the symmetries of RNCQM. The NRNCQM and RNCQM results obtained within Bopp's shift and standard perturbation theory methods overlap entirely with the results obtained by ordinary NCQM, and it displays that the theoretical investigation in this study is excellent.

Keywords*:* Klein-Fock-Gordon equation, Schrödinger equation, Manning-Rosen**,** Yukawa potentials**;** the diatomic molecules, Noncommutative geometry, Bopp's shift method and star products.

Resumen

En el trabajo actual, empleamos la elegante herramienta de los métodos de teoría de perturbaciones estándar y de desplazamiento de Bopp para obtener una nueva solución de estado ligado aproximado relativista y no relativista de las ecuaciones de Klein-Fock-Gordon deformadas y de Schrödinger deformadas utilizando el potencial escalar de Manning-Rosen de vector igual modificado. y modelo potencial de Yukawa. Además, hemos empleado la aproximación mejorada al término centrífugo para algunas moléculas diatómicas seleccionadas como (N2, I2, HCl, CH y LiH) en las simetrías de la mecánica cuántica extendida para obtener las soluciones aproximadas. Se ve que la energía de desplazamiento relativista $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$y las correcciones perturbativas no relativistas $\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$son sensibles a los números cuánticos (n, j, l, s, m), el rango de potencial, dos parámetros adimensionales (A y α), la fuerza del potencial además de los parámetros de no conmutatividad (Θ, σ, χ). Hemos destacado tres fenómenos físicos que generan automáticamente un resultado de las propiedades topológicas de la no conmutatividad, el primer fenómeno físico generado son el acoplamiento perturbativo espín-órbita, el segundo la inducción magnética mientras que el tercero corresponde a los fenómenos rotacionales propios. Tanto en problemas relativistas como no relativistas, mostramos que las correcciones en la energía del espectro son menores que la energía principal en los casos ordinarios de la teoría cuántica de campos y la mecánica cuántica. Un límite directo de nuestros resultados a la mecánica cuántica ordinaria muestra que el resultado actual bajo el potencial escalar vectorial igual modificado de Manning-Rosen y el potencial de Yukawa es consistente con lo que se obtiene en la literatura. En las nuevas simetrías de NCQM, no es posible obtener las soluciones analíticas exactas para $l = 0$ y $l \neq 0$, las soluciones aproximadas están disponibles. Hemos observado que la ecuación de Klein-Fock-Gordon deformada bajo el modelo escalar de vector igual modificado de potencial de Manning-Rosen y potencial de Yukawa tiene un comportamiento físico similar a la ecuación de Duffin-Kemmer para mesón con espín-1, puede describir un estado dinámico de una partícula con spin-1 en las simetrías de RNCQM. Los resultados de NRNCQM y RNCQM obtenidos dentro de los métodos de teoría de perturbación estándar y de desplazamiento de Bopp se superponen completamente con los resultados obtenidos por NCQM ordinario, y muestra que la investigación teórica en este estudio es excelente.

Palabras clave: Ecuación de Klein-Fock-Gordon, Ecuación de Schrödinger, Manning-Rosen, Potenciales Yukawa, Moléculas diatómicas, Geometría no conmutativa, Método de cambio de Bopp y Productos estrella.

I. INTRODUCTION

It is well known that for searchers the physical phenomena at low energies requires the investigation of the nonrelativistic wave equation in the dynamics of quantum mechanics in the context of the Schrödinger equation [1], which can be applied to study fermionic or boson particles, regardless of the spin value. In the context of quantum field theory, at high energies, the role of the spin becomes a major reason for determining the type of relativistic equation used in the physical investigation, for a neutral or charged particle with spin-0, the ordinary Klien-Fock equation [2, 3, 4] is required, the Duffin–Kemmer equation [5] for meson with spin-1 while the Dirac equation [6] is suitable for particles such as electron and anti-particle (positron) with spin-1/2. The solutions to these equations are not always exact, but rather approximate in many potentials such as Coulomb potential [7, 8], Pöschl–Teller potential [9], Eckart potential [10], Rosen–Morse potential [11], The Manning-Rosen potential [12], Yukawa potential [13] and others among potentials. In our paper, we are interested in these two potentials. The Manning-Rosen potential one of the most important and oldest known to the searchers, it can be applied to many areas, for example, atomic field, condensed matter, particle, and nuclear physics in both relativistic and non-relativistic regimes [14, 15, 16, 17]. Also, this potential can be used to describe the vibrations of diatomic molecules such as N2, I2, HCl, CH and LiH [18]. Many researchers have investigated this potential in the nonrelativistic case, in both the s and *l*-waves cases (see for example [18, 19, 20]). Furthermore, it was also studied in the relativistic regimes of Klein–Gordon, and Dirac equations [20, 21, 22, 23].

The Yukawa potentials is another type of exponential-potentials (also known as static screened Coulomb potentials) that have received a great deal of attention, in many fields of physics such as nuclear physics, atomic physics, solid-state physics, and astrophysics and it was studied in both relativistic and non-relativistic quantum mechanics it is used to describe the interactions of Hydrogen-like atoms and neutral atoms [24, 25, 26, 27, 28, 29, 30, 31, 32].

Currently, there has been great interest in combining two or more potentials to have a large range of applications, for example, the Manning-Rosen plus a Class of Yukawa potential [33]. This combined is useful to examine the interactions of deformed-pair of the nucleus and spin-orbit coupling for a particle in the potential field, also, to the description of vibrations in the side of the hadronic system [33] (subatomic particles made of two or more quarks, for example, the proton (uud) and the neutron (udd) and mesons $q\overline{q}$). It should be noted that there is another combination between the family of the same potential that extended to the class of Yukawa potential known by the linear combination of Manning-Rosen plus a class of Yukawa potentials. This combined studied under the nonrelativistic Schrödinger equation and relativistic Klein-Gordon and Dirac equations with the Nikiforov-Uvarov, SUSYQM methods and the approximation scheme proposed by Greene and Aldrich [34, 35, 36, 37].

As a result of several considerations and many physical problems apparat at the level of the non-renormalizable of the electroweak interaction, the non-regularization of quantum field theories, quantum gravity, string theory, where the idea of non-commutativity resulting from properties of deformation of space-space (W. Heisenberg in 1930 is the first to suggest the idea and then it was developed by H. Snyder in 1947) was one of the major solutions to these problems.

In the past two decades, in particular, it has attracted a great attraction by researchers [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. Naturally, the topographical properties of the noncommutativity space-space and phase-phase have a clear effect on the various physical properties of quantum systems and this has been a very interesting subject in many fields of physics as mentioned previously.

In the last few years, we have investigated many interesting studies concerned the Yukawa potential in a nonrelativistic and relativistic regime [50, 51, 52, 53, 54] due to the importance of its applications in many fields, as previously indicated. From what we have seen so far that most of the studies concerning Manning-Rosen and Yukawa potentials were within the framework of ordinary quantum mechanics.

The above works motivated us to investigate the approximate solutions of the 3-dimensional deformed Klein-Fock-Gordon equation and deformed Schrödinger equation for the modified equal vector scalar Manning-Rosen potential and Yukawa potential offered by A.I. Ahmadov *et al*. [33] in RQM. The potential focus of study and interest can be applied for some selected diatomic molecules such as (N2, I2, HCl, CH and LiH) in RNCQM and NRNCQM symmetries. We hope to discover more investigation in the sub-atomic scales and from achieving more scientific knowledge of elementary particles in the field of nano-scales. The relativistic and nonrelativistic energy levels under the modified equal vector scalar Manning-Rosen potential and Yukawa potential have not been obtained yet in the RNCQM and NRNCQM symmetries, we hope to find new applications and profound physical interpretations using a new version model of this potential modeled in the new symmetries of NCQM as follows:

$$V_{my}(r)=\frac{\hbar^2}{2Mb^2}\left[\frac{\alpha(\alpha-1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{r}{b}}\right)^2}-\frac{Ae^{-\frac{r}{b}}}{1-e^{-\frac{r}{b}}}\right]-\frac{V_0\,exp(-\delta r)}{r},$$

$$\rightarrow V_{my}(r_{nc})\equiv V_{my}(r)-\frac{\partial V_{my}(r)}{\partial r}\frac{\vec{\mathrm{L}}\vec{\Theta}}{2r}+O\left(\Theta^2\right), \quad (1)$$

And

$$S_{my}(r)=\frac{\hbar^2}{2Mb^2}\left[\frac{\alpha(\alpha-1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{r}{b}}\right)^2}-\frac{Be^{-\frac{r}{b}}}{1-e^{-\frac{r}{b}}}\right]-\frac{S_0\,exp(-\delta r)}{r},$$

$$\rightarrow S_{my}(r_{nc}) \equiv S_{my}(r) - \frac{\partial S_{my}(r)}{\partial r}\frac{\vec{L}\vec{\Theta}}{2r} + O(\Theta^2), \quad (2)$$

where the parameter b relates to the potential range while (A and α) are two dimensionless parameters, V_0 is the strength of the potential and $1/\delta$ is its range, r_{nc} and r is the distance between the two particles in NCQM and QM symmetries. The coupling $\vec{L}\vec{\Theta}$ equals $L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}$ with L_x,L_y and L_z are present the usual components of angular momentum operator $\vec{L}$ while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. The new algebraic structure of covariant noncommutative canonical commutations relations NCNCCRs in the three representations of Schrödinger, Heisenberg, and interactions pictures, in the new symmetry of NCQM, as follows [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70]:

$$\left[\hat{x}_\mu^{(S,H,I)} \overset{*}{,} \hat{P}_\nu^{(S,H,I)}\right] \equiv \hat{x}_\mu^{(S,H,I)} * \hat{P}_\nu^{(S,H,I)} - \hat{P}_\nu^{(S,H,I)} * \hat{x}_\mu^{(S,H,I)} = i\hbar_{eff}\delta_{\mu\nu}, \quad (3.1)$$

and

$$\left[\hat{x}_\mu^{(S,H,I)} \overset{*}{,} \hat{x}_\nu^{(S,H,I)}\right] \equiv \hat{x}_\mu^{(S,H,I)} * \hat{x}_\nu^{(S,H,I)} - \hat{x}_\nu^{(S,H,I)} * \hat{x}_\mu^{(S,H,I)} = i\theta_{\mu\nu}. \quad (3.2)$$

With $\hat{x}_\mu^{(S,H,I)} \equiv \left(\hat{x}_\mu^S \vee \hat{x}_\mu^H(t) \vee \hat{x}_\mu^I(t)\right)$ and $\hat{P}_\mu^{(S,H,I)} \equiv \left(\hat{P}_\mu^S \vee \hat{P}_\mu^H(t) \vee \hat{P}_\mu^I(t)\right)$. It is important to note that Eq. (3.2) is a covariant equation (the same behavior of x_μ) under Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We generalize the NCNCCRs to include Heisenberg and interaction pictures. It should be noted that, in our calculation, we have used the natural units $c = \hbar = 1$. Here $\hbar_{eff} \approx \hbar$ is the effective Planck constant, $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ (θ is the non-commutative parameter), which is an infinitesimals parameter if compared to the energy values and elements of antisymmetric 3×3 real matrix and $\delta_{\mu\nu}$ is the identity matrix. The symbol $(*)$ denotes the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new deformed form $\hat{f}(\hat{x})\hat{g}(\hat{x})$ which expressed with the Weyl Moyal star product $f(x) * g(x)$ in the symmetries of NCQM as follows [66,67,68,69,70,71]:

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv (f * g)(x) = exp\left(i\theta\varepsilon^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}\right)f(x_\mu)g(x_\nu) \cong fg(x) - \frac{i\theta^{\mu\nu}}{2}\partial_\mu^x f\,\partial_\nu^x g\Big|_{x_\mu = x_\nu} + O(\theta^2). \quad (4)$$

The indices $\mu,\nu \equiv \overline{1,3}$ and $O(\theta^2)$ stand for the second and higher-order terms of the NC parameter. The second term $-\frac{i\theta^{\mu\nu}}{2}\partial_\mu^x f\,\partial_\nu^x g\Big|_{x_\mu = x_\nu}$ in the above equation gives the effects of space-space noncommutativity properties. Furthermore, it is possible to unify the operators $\hat{\vartheta}_\mu^H(t) = (\hat{x}_\mu \vee \hat{p}_\mu)(t)$ and $\hat{\vartheta}_\mu^I(t) = (\hat{x}_\mu^I \vee \hat{p}_\mu^I)(t)$ in Heisenberg and interaction pictures using the following projection relations, respectively:

$$\vartheta_\mu^H(t) = exp\left(i\hat{H}_r^{my}T\right)\vartheta_\mu^S\, exp\left(-i\hat{H}_r^{my}T\right),$$

$$\Rightarrow \hat{\vartheta}_\mu^H(t) = exp\left(i\hat{H}_{nc-r}^{my}T\right) * \hat{\vartheta}_\mu^S * exp\left(-i\hat{H}_{nc-r}^{my}T\right), \quad (5.1)$$

and

$$\vartheta_\mu^I(t) = exp\left(i\hat{H}_{or}^{my}T\right)\vartheta_\mu^S\, exp\left(-i\hat{H}_{or}^{my}T\right),$$

$$\Rightarrow \hat{\vartheta}_\mu^I(t) = exp\left(i\hat{H}_{nc-or}^{my}T\right) * \hat{\vartheta}_\mu^S * exp\left(-i\hat{H}_{nc-or}^{my}T\right). \quad (5.2)$$

Where $\hat{\vartheta}_\mu^H \equiv \hat{x}_\mu$ or $\hat{p}_\mu$ is the operator in Schrödinger picture, $T = t - t_0$ while $\vartheta_\mu^S \equiv (x_\mu \vee p_\mu)$, $\vartheta_\mu^H(t) \equiv (x_\mu \vee p_\mu)(t)$ and $\vartheta_\mu^I(t) \equiv (x_\mu^I \vee p_\mu^I)(t)$ are the corresponding unified operators in the ordinary QM symmetries. Moreover, the dynamics of new systems $\frac{d\hat{\vartheta}_\mu^H(t)}{dt}$ can be described from the following motion equations in the deformed Heisenberg picture as follows:

$$\frac{d\vartheta_\mu^H(t)}{dt} = \left[\vartheta_\mu^H(t), \hat{H}_r^{my}\right] + \frac{\partial\vartheta_\mu^H(t)}{\partial t}$$
$$\Rightarrow \frac{d\hat{\vartheta}_\mu^H(t)}{dt} = \left[\hat{\vartheta}_\mu^H(t) \overset{*}{,} \hat{H}_{nc-r}^{my}\right] + \frac{\partial\hat{\vartheta}_\mu^H(t)}{\partial t}. \quad (6)$$

Here ($\hat{H}_{or}^{my}$ and $\hat{H}_r^{my}$) are the free and total Hamiltonian operators for equal vector scalar Manning-Rosen potential and Yukawa potential while ($\hat{H}_{nc-or}^{my}$ and $\hat{H}_{nc-r}^{my}$) the corresponding Hamiltonians in the symmetries of NCQM. The present investigation aims at constructing a relativistic noncommutative effective scheme for the modified equal vector scalar Manning-Rosen potential and Yukawa potential model. On the other hand, the choice of these combined of Manning-Rosen and Yukawa potentials stems from the fact that it exhibits an almost exact behavior similar to the Morse [72] and Deng- Fan-Eckart [73] potentials and so considers it an excellent choice for the study of atomic interaction for diatomic molecules such as N2, I2, HCl, CH and LiH. Our current work is structured in six sections. The first one includes the scope and purpose of our investigation while the remaining parts of the paper are structured as follows. A review of the Klein-Fock-Gordon equation with equal vector scalar Manning-Rosen potential and Yukawa potential is presented in Sect. 2. Sect. 3 is devoted to studying the deformed Klein-Fock-Gordon equation by applying the ordinary Bopp's shift method and improved approximation of the centrifugal term to obtain the effective potential of Modified equal vector scalar Manning-Rosen potential and Yukawa potential. Besides, via perturbation theory we find the expectation values of some radial terms to calculate the energy shift produced with the effect of the perturbed effective potential of modified equal vector scalar Manning-Rosen potential and Yukawa potential. Sect. 4 is devoted to present the global energy shift and the global energy spectra produced with Modified equal vector scalar Manning-Rosen potential and Yukawa potential in the RNCQM symmetries. In Sect. 5, we apply our study for determining the energy spectra of some selected diatomic molecules such as (N2, I2, HCl, CH and LiH) under the modified equal vector scalar

Manning-Rosen potential and Yukawa potential in the RNCQM. In Sect. 6, the Summary and conclusion are presented.

II. REVIEWED OF KLIEN-FOCK-GORDON EQUATION UNDER EQUAL VECTOR AND SCALAR MANNING-ROSEN AND YUKAWA POTENTIAL IN RQM SYMMETRIES

The vector and scalar Manning-Rosen and Yukawa potentials in the symmetries of ordinary relativistic quantum mechanics are given by [33]:

$$V_{my}(r)=\frac{\hbar^2}{2Mb^2}\left[\frac{\alpha(\alpha-1)e^{-2r/b}}{\left(1-e^{-r/b}\right)^2}-\frac{Ae^{-r/b}}{1-e^{-r/b}}\right]-\frac{V_0\,exp(-\delta r)}{r}, \quad (7.1)$$

and

$$S_{my}(r)=\frac{\hbar^2}{2Mb^2}\left[\frac{\lambda(\lambda-1)e^{-2r/b}}{\left(1-e^{-r/b}\right)^2}-\frac{Be^{-r/b}}{1-e^{-r/b}}\right]-\frac{S_0\,exp(-\delta r)}{r} \quad (7.2)$$

The first two terms are the vector and scalar Manning-Rosen potential while the third term is the standard Yukawa potential. The 3-dimensional Klein-Fock-Gordon equation with a scalar potential $S_{my}(r)$ and a vector potential $V_{my}(r)$ for the diatomic molecule with reduced mass M and wave function $\Psi(r,\theta,\phi)$ is given as

$$\left\{\vec{\nabla}^2+\left(E_{nl}-V_{my}(r)\right)^2\right\}\Psi(r,\theta,\phi)=\left(M+\ S_{my}(r)\right)^2\Psi(r,\theta,\phi). \quad (8)$$

The vector potential $V_{my}(r)$ due to the four-vector linear momentum operator $A^{\mu}\left(V_{my}(r),\vec{A}=0\right)$ and the space-time scalar potential$S_{my}(r)$ whereas the interaction of scalar and vector bosons are considering by usual substitutions ($M\rightarrow M+S_{my}$ and $p^{\mu}\rightarrow p^{\mu}-A^{\mu}$), E_{nl} is the relativistic energy eigenvalues, $\vec{\nabla}$ is the ordinary 3-dimensional Nabla operator while ($n=0,1,2\ldots$ and l) are represents the principal and orbital quantum numbers, respectively. Since equal vector, scalar Manning-Rosen and Yukawa potentials have spherical symmetry, allowing the solutions of the time-independent Klein-Fock-Gordon equation of the known form$\Psi(r,\theta,\phi)=\frac{\chi_{nl}(r)}{r}Y_l^m(\Omega)$ to separate the radial $\chi_{nl}(r)$ and $Y_l^m(\Omega)$is the angular component of the wave function, thus Eq. (8) becomes:

$$\left(\begin{array}{l}\frac{d^2}{dr^2}-(M^2-E_{nl}^2)-2\left(E_{nl}V_{my}(r)+MS_{my}(r)\right)\\+V_{my}^2(r)-S_{my}^2(r)-\frac{l(l+1)}{r^2}\end{array}\right)\chi_{nl}(r)=0. \quad (9)$$

The shorthand notation $V_{eff}^{my}(r)\equiv 2\left(E_{nl}V_{my}(r)+MS_{my}(r)\right)-V_{my}^2(r)+S_{my}^2(r)+\frac{l(l+1)}{r^2}$ and$E_{eff}^{my}\equiv M^2-E_{nl}^2$, we obtain the following second-order Schrödinger-like equation:

$$\left(\frac{d^2}{dr^2}-\left(E_{eff}^{my}+V_{eff}^{my}(r)\right)\right)\chi_{nl}(r)=0. \quad (10)$$

When the vector potential is equal to the scalar potential $V_{my}(r)=S_{my}(r)$ the effective potential leads to the following simple form:

$$V_{eff}^{my}(r)\equiv 2(E_{nl}+M)V_{my}(r)+\frac{l(l+1)}{r^2}. \quad (11)$$

A. I. Ahmadov *et al.* [33] derived analytical expressions for the wave function and the corresponding energy values for the vector and scalar Manning-Rosen and Yukawa potentials using both Nikiforov-Uvarov, SUSYQM methods and the approximation scheme proposed by Greene and Aldrich as,

$$\Psi(r,\theta,\phi)=\frac{C_{nl}}{r}s^{\varepsilon_{nl}}(1-s)^{k_l}\frac{\Gamma(n+2\varepsilon_{nl}+1)}{n!\,\Gamma(2\varepsilon_{nl}+1)}\,{}_2F_1(-n,n+2\varepsilon_{nl}+2k_l;1+2\varepsilon_{nl},s)Y_l^m(\theta,\phi), \quad (12)$$

and

$$M^2-E_{nl}^2=\left[\delta\frac{\beta_{nl}^2-l(l+1)-½-n(n+1)-(2n+1)\sqrt{l(l+1)+¼+\alpha_{nl}^2}}{n+½+\sqrt{l(l+1)+¼+\alpha_{nl}^2}}\right]^2. \quad (13)$$

Where$s=exp(-2\delta r)$, $\varepsilon_{nl}\equiv\frac{\sqrt{M^2-E_{nl}^2}}{2\delta}$, $k_{nl}\equiv 1/2+\sqrt{¼+l(l+1)+\alpha_{nl}^2}$, $\alpha_{nl}^2=\left(\frac{E_{nl}+M}{2\delta^2}\right)V_{01}$, $V_{01}=\frac{2\delta^2\alpha(\alpha-1)}{M}$, $\beta_{nl}^2=\left(\frac{E_{nl}+M}{4\delta^2}\right)V_{023}$, $V_{023}=\frac{2\delta^2A}{M}+2\delta V_0$while ${}_2F_1(-n,n+2\varepsilon_{nl}+2k_l;1+2\varepsilon_{nl},s)$ are the hypergeometric polynomials. From the definition of Jacobi polynomials [74], we use the following relation:

$${}_2F_1(-n,n+2\varepsilon_{nl}+2k_{nl}-1+1;1+2\varepsilon_{nl},s)=\frac{n!\,\Gamma(2\varepsilon_{nl}+1)}{\Gamma(n+2\varepsilon_{nl}+1)}P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s). \quad (14)$$

To rewriting the wave function Eq. (12) as follows:

$$\Psi(r,\theta,\phi)=\frac{C_{nl}}{r}s^{\varepsilon_{nl}}(1-s)^{k_{nl}}P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)Y_l^m(\Omega), \quad (15)$$

while C_{nl} is the normalization constant is given by [33]:

$$C_{nl}=\sqrt{\frac{2\delta n!(n+k_{nl}+\varepsilon_{nl})\Gamma(2\varepsilon_{nl}+1)\Gamma(n+2\varepsilon_{nl}+k_{nl}-1)}{(n+k_{nl})\Gamma(2\varepsilon_{nl})\Gamma(n+2k_{nl})}}. \quad (16)$$

III. SOLUTIONS OF DEFORMED KLIEN-FOCK-GORDON EQUATION UNDER MODIFIED EQUAL VECTOR AND SCALAR MANNING-ROSEN AND YUKAWA POTENTIAL IN RNCQM SYMMETRIES

A. Review of Bopp's shift method

At the beginning of this section, we shall give and define a formula of the deformed equal vector scalar Manning-Rosen potential and Yukawa potential in the symmetries of relativistic noncommutative three-dimensional real space RNCQM symmetries. To achieve this goal, it is useful to write the deformed Klein-Fock-Gordon equation by applying the notion of the Weyl Moyal star product which have seen previously in Eq. (3), on the differential equation that satisfied by the radial wave function $\chi_{nl}(r)$ in Eq. (9), Thus, the radial wave function $\chi_{nl}(r)$ in RNCQM symmetries becomes as follows **[**75, 76, 77, 78, 79, 80, 81, 82, 83, 84**]**:

$$\left(\frac{d^2}{dr^2}-(M^2-E_{nl}^2)-2(E_{nl}+M)V_{my}(r)-\frac{l(l+1)}{r^2}\right)*\chi_{nl}(r)=0 \qquad (17)$$

It is established extensively in the literature and in a basic text [84, 85, 86] that star products can be simplified from Bopp's shift method. The physicist Fritz Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $x\rightarrow x-\frac{i}{2}\frac{\partial}{\partial p}$and $p\rightarrow p+\frac{i}{2}\frac{\partial}{\partial x}$instead of the ordinary correspondence $x\rightarrow x$and $p\rightarrow \frac{i}{2}\frac{\partial}{\partial x}$ [85, 86]. In physics literature, this is known by Bopp's shifts. This quantization procedure is called Bopp quantization. It is known to the specialists that Bopp's shift method [84, 85, 86], has been applied effectively and has succeeded in simplifying the three basic equations: deformed Klein-Fock-Gordon equation [50, 54, 57, 78, 79, 81], deformed Dirac equation [60, 82], deformed Shrodinger equation [51, 53, 55, 56, 70] and Duffin–Kemmer–Petiau equation [80]. With the notion of star product to the Klein-Fock-Gordon equation, the Dirac equation and the Schrödinger equation with the notion of ordinary product. Thus, Bopp's shift method is based on reducing second order linear differential equations of the deformed Klein-Fock-Gordon equation, the deformed Dirac equation, and the deformed Schrödinger equation with star product to second-order linear differential equations of Klein-Fock-Gordon equation, Dirac equation and Schrödinger equation without star product with simultaneous translation in the space-space. The CNCCRs with star product in Eqs. (2) and (3) become new CNCCRs without the notion of star product as follows (see, e.g., [75, 76,77,78, 79, 80, 81, 82, 83, 84, 85, 86]):

$$\left[\hat{x}_\mu^{(S,H,I)},\hat{x}_\nu^{(S,H,I)}\right]\equiv\hat{x}_\mu^{(S,H,I)}\hat{x}_\nu^{(S,H,I)}-\hat{x}_\nu^{(S,H,I)}\hat{x}_\mu^{(S,H,I)}=i\theta_{\mu\nu}. \qquad (18)$$

The generalized positions and momentum coordinates: $\left(\hat{x}_\mu^{(S,H,I)},\hat{p}_\mu^{(S,H,I)}\right)$, in the symmetries of RNCQM are defined in terms of the corresponding coordinates $\left(x_\mu^{(S,H,I)},p_\mu^{(S,H,I)}\right)$ in the symmetries of RQM via, respectively [76, 77, 78, 79, 80, 81, 82, 83]:

$$\left(x_\mu^{(S,H,I)},p_\mu^{(S,H,I)}\right)\Rightarrow\begin{pmatrix}\hat{x}_\mu^{(S,H,I)}=x_\mu^{(S,H,I)}-\frac{\theta_{\mu\nu}}{2}p_\nu^{(S,H,I)},\\ \hat{p}_\mu^{(S,H,I)}=p_\mu^{(S,H,I)}\end{pmatrix}. \qquad (19)$$

This allows us to find the operator r_{nc}^2 equal $r^2-\vec{L}\vec{\Theta}$ (see $\vec{L}\vec{\Theta}$ in the introduction) in NCQM symmetries [76, 77, 78].

B. New effective potential in RNCQM symmetries

According to the Bopp shift method, Eq. (17) becomes similar to the following like the Schrödinger equation (without the notions of star product):

$$\left(\frac{d^2}{dr^2}-(M^2-E_{nl}^2)-2(E_{nl}+M)V_{my}(r_{nc})-\frac{l(l+1)}{r_{nc}^2}\right)\chi_{nl}(r)=0. \qquad (20)$$

The new operators $V_{my}(r_{nc})$ and $\frac{1}{r_{nc}^2}$ are expressed as in RNCQMsymmetries as follows:

$$V_{my}(r_{nc})=V_{my}(r)-\frac{\vec{L}\vec{\Theta}}{2r}\frac{\partial V_{my}(r)}{\partial r}+O\left(\Theta^2\right), \qquad (21.1)$$

$$\frac{1}{r_{nc}^2}\approx\frac{1}{r^2}+\frac{\vec{L}\vec{\Theta}}{r^4}+O\left(\Theta^2\right). \qquad (21.2)$$

So we can rewrite:

$$(E_{nl}+M)V_{my}(r_{nc})=(E_{nl}+M)V_{my}(r)-(E_{nl}+M)\frac{\vec{L}\vec{\Theta}}{2r}\frac{\partial V_{my}(r)}{\partial r}+O\left(\Theta^2\right). \qquad (22)$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol$V_{pert}^{my}(r)$, thus the radial Eq. (20) becomes:

$$\left(\frac{d^2}{dr^2}-\left(E_{eff}^{my}+V_{nc-eff}^{my}(r)\right)\right)\chi_{nl}(r)=0, \qquad (23)$$

with:

$$V_{nc-eff}^{my}(r)=V_{eff}^{my}(r)+V_{pert}^{my}(r). \qquad (24)$$

Moreover, $V_{pert}^{my}(r)$ is given by the following relation:

$$V_{pert}^{my}(r)=\left(\frac{l(l+1)}{r^4}-(E_{nl}+M)\frac{1}{r}\frac{\partial V_{my}(r)}{\partial r}\right)\vec{L}\vec{\Theta}. \qquad (25)$$

It should be noted that when $l = 0$ Eq. (10) with Manning-Rosen and Yukawa potentials can be exactly solved, but for the case $l \neq 0$, A. I. Ahmadov *et al*. they had approximatively solved the equation using Eq. (10) using the Greene-Aldrich approximation scheme in RQM symmetries. In the new form of radial like-Schrödinger equation written in (23), we have terms including $\frac{1}{r}$ and $\frac{1}{r^4}$ which make this equation impossible to solve analytically for $l = 0$and $l \neq 0$, it can only be solved approximately. From this point of view, we can consider the improved approximation of the centrifugal term proposed by M. Badawi *et al*. [86], this method proved its power and efficiency when compared with Greene and Aldrich approximation [87]. The approximations type suggested by (Greene and Aldrich) and Dong *et al*. for a short-range potential is an excellent approximation to the centrifugal term and allows us to get a second order solvable differential equation. Unlike the following approximation used in the previous work [33, 34, 35, 36, 77, 82]:

$$\frac{1}{r^2} \approx \frac{4\delta^2 \exp(-2\delta r)}{(1-\exp(-2\delta r))^2} = \frac{4\delta^2 s}{(1-s)^2}. \qquad (26.1)$$

It is important to mention here that the above approximations are valid when δr<<1. This allows us to obtain:

$$\frac{1}{r} \approx \frac{2\delta \exp(-\delta r)}{1-\exp(-2\delta r)} = \frac{2\delta s^{\frac{1}{2}}}{1-s}. \qquad (26.2)$$

Now we rewrite the Manning-Rosen potential and Yukawa potentials under the assumption of $\frac{1}{b} = 2\delta$ as follows:

$$V_{my}(r) = \frac{\hbar^2}{2Mb^2}\left(\frac{\alpha(\alpha-1)\exp(-4\delta r)}{(1-\exp(-2\delta r))^2} - \frac{A\exp(-2\delta r)}{1-\exp(-2\delta r)}\right) - \frac{V_0 \exp(-\delta r)}{r}. \qquad (27)$$

After straightforward calculations we obtain $\frac{\partial V_{my}(r)}{\partial r}$ as follows:

$$\frac{\partial V_{my}(r)}{\partial r} = -\frac{8\delta^3}{M}\alpha(\alpha-1)(\frac{\exp(-4\delta r)}{(1-\exp(-2\delta r))^2} + \frac{\exp(-6\delta r)}{(1-\exp(-2\delta r))^3}) + \frac{4\delta^3 A}{M}\frac{\exp(-2\delta r)}{1-\exp(-2\delta r)} + \frac{4\delta^3 A}{M}\frac{\exp(-4\delta r)}{(1-\exp(-2\delta r))^2} + V_0\frac{\delta\exp(-\delta r)}{r} + V_0\frac{\exp(-\delta r)}{r^2}. \qquad (28)$$

We apply the approximations of Greene and Aldrich on the expression $\frac{\partial V_{my}(r)}{\partial r}$leads the following formula :

$$\frac{\partial V_{my}(r)}{\partial r} = \frac{\beta_1 s^2}{(1-s)^2} + \frac{\beta_2 s^3}{(1-s)^3} + \frac{\beta_3 s}{1-s} + \frac{\beta_4 s^{3/2}}{(1-s)^2}, \qquad (29)$$

with $\beta_1 \equiv \frac{4\delta^3 A}{M} + \beta_2$, $\beta_2 \equiv -\frac{8\delta^3\alpha(\alpha-1)}{M}$, $\beta_3 \equiv \frac{4\delta^3 A}{M} + 2V_0\delta^2$ and $\beta_4 \equiv 4V_0\delta^2$. Simplifying further Eq. (29) becomes:

$$\frac{(E_{nl}+M)}{r}\frac{\partial V_{my}(r)}{\partial r} = 2\delta(E_{nl}+M)(\frac{\beta_1 s^{3/2}}{(1-s)^3} + \frac{\beta_2 s^{7/2}}{(1-s)^4} + \frac{\beta_3 s^{3/2}}{(1-s)^2} + \frac{\beta_4 s^2}{(1-s)^3}). \qquad (30)$$

By making the substitution Eq. (30) into Eq. (25), we find the perturbed effective potential generated from noncommutativity properties of space-space $V_{pert}^{my}(r)$ in the symmetries of RNCQM as follows:

$$V_{pert}^{my}(r) = \begin{Bmatrix} 16\delta^4 l(l+1)\frac{s^2}{(1-s)^4} - 2\delta(E_{nl}+M) \\ (\frac{\beta_1 s^{3/2}}{(1-s)^3} + \frac{\beta_2 s^{7/2}}{(1-s)^4} + \frac{\beta_3 s^{3/2}}{(1-s)^2} + \frac{\beta_4 s^2}{(1-s)^3}) \end{Bmatrix} \vec{L}\vec{\Theta}. \qquad (31)$$

We have replaced the term $\frac{l(l+1)}{r^4}$ with the approximations of Greene and Aldrich. The Manning-Rosen potential and Yukawa potentials are extended by including new terms proportional to the radial terms$\frac{s^2}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^3}$,$\frac{s^{7/2}}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^2}$ and $\frac{s^2}{(1-s)^3}$ become the deformed Manning-Rosen potential and Yukawa potentials in RNCQM symmetries. The generated new effective potential $V_{pert}^{my}(r)$ is also proportional to the infinitesimal vector$\vec{\Theta}$. This allows us to consider $V_{pert}^{my}(r)$as a perturbation potential compared with the main potential (parent potential operator$V_{eff}^{my}(r)$ in the symmetries of RNCQM, that is, the inequality $V_{pert}^{my}(r)$<<$V_{eff}^{my}(r)$has become achieved. That is all the physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states.

C. The expectation values in RNCQM symmetries

In this sub-section, we want to apply the perturbative theory, in the case of RNCQM, we find the expectation values of the radial terms$\frac{s^2}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^3}$,$\frac{s^{7/2}}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^2}$and $\frac{s^2}{(1-s)^3}$ taking into account the wave function which we have seen previously in Eq. (12). Thus, after straightforward calculations, we obtain the following results:

$$\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)} = C_{nl}^2\int_0^{+\infty} s^{2\varepsilon_{nl}}(1-s)^{2k_{nl}}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)\right]^2\frac{s^2 dr}{(1-s)^4}, \qquad (32.1)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)} = C_{nl}^2\int_0^{+\infty} s^{2\varepsilon_{nl}}(1-s)^{2k_{nl}}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)\right]^2\frac{s^{3/2} dr}{(1-s)^3}, \qquad (32.2)$$

$$\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} = C_{nl}^2\int_0^{+\infty} s^{2\varepsilon_{nl}}(1-s)^{2k_{nl}}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)\right]^2\frac{s^{7/2} dr}{(1-s)^4}, \qquad (32.3)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)} = C_{nl}^2\int_0^{+\infty} s^{2\varepsilon_{nl}}(1-s)^{2k_{nl}}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)\right]^2\frac{s^{3/2}dr}{(1-s)^2}, \quad (32.4)$$

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(n,l,m)} = C_{nl}^2\int_0^{+\infty} s^{2\varepsilon_{nl}}(1-s)^{2k_{nl}}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(1-2s)\right]^2\frac{s^2dr}{(1-s)^3}. \quad (32.5)$$

We have used useful abbreviations $\langle n,l,m|\widehat{D}|n,l,m\rangle \equiv \langle\widehat{D}\rangle_{(n,l,m)}$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\sin(\theta)\,d\theta d\phi = \delta_{ll'}\delta_{mm'}$. We have $s = exp(-2\delta r)$, this allows us to obtain $dr = -\frac{1}{2\delta}\frac{ds}{s}$. After introducing a new variable $z = 1-2s$, we have $s = \frac{1-z}{2}$, $dr = \frac{1}{2\delta}\frac{dz}{1-z}$ and $1-s = \frac{z+1}{2}$. From the asymptotic behavior of $s = exp(-2\delta r)$ and $z = 1-2s$, when $r \to 0$ $(z \to -1)$ and $r \to +\infty$ $(z \to 1)$, this allows to reformulating Eqs. $(32, i = \overline{1,6})$ as follows:

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(n,l,m)} = \frac{C_{nl}^2}{2^{2\varepsilon_{nl}+2k_{nl}}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+1}(1+z)^{2k_{nl}-3}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(z)\right]^2 dz, \quad (33.1)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)} = \frac{2C_{nl}^2}{2^{2\varepsilon_{nl}+2k_{nl}+1/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+\frac{1}{2}}(1+z)^{2k_{nl}-3}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(z)\right]^2 dz, \quad (33.2)$$

$$\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} = \frac{C_{nl}^2}{2^{2\varepsilon_{nl}+2k_{nl}+3/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+5/2}(1+z)^{2k_{nl}-4}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(z)\right]^2 dz \quad (33.3)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)} = \frac{C_{nl}^2}{2^{2\varepsilon_{nl}+2k_{nl}+1/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+\frac{1}{2}}(1+z)^{2k_{nl}-2}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(z)\right]^2 dz, \quad (33.4)$$

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(n,l,m)} = \frac{C_{nl}^2}{2^{2\varepsilon_{nl}+2k_{nl}}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+1}(1+z)^{2k_{nl}-3}\left[P_n^{(2\varepsilon_{nl},2k_{nl}-1)}(z)\right]^2 dz. \quad (33.5)$$

For the ground state $n = 0$, we have $P_{n=0}^{(2\varepsilon_{0l},2k_{0l}-1)}(z) = 1$, thus the above expectation values in Eqs. $(33, i = \overline{1,6})$ are reduced to the following simple form:

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2^{2\varepsilon_{0l}+2k_{0l}}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{0l}+1}(1+z)^{2k_{0l}-3}\,dz, \quad (34.1)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{2C_{0l}^2}{2^{2\varepsilon_{0l}+2k_{0l}+1/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{0l}+\frac{1}{2}}(1+z)^{2k_{0l}-3}\,dz, \quad (34.2)$$

$$\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2^{2\varepsilon_{0l}+2k_{0l}+3/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{0l}+5/2}(1+z)^{2k_{0l}-4}\,dz, \quad (34.3)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2^{2\varepsilon_{0l}+2k_{0l}+1/2}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{0l}+1/2}(1+z)^{2k_{0l}-2}\,dz, \quad (34.4)$$

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2^{2\varepsilon_{0l}+2k_{0l}}\delta}\int_{-1}^{+1}(1-z)^{2\varepsilon_{0l}+1}(1+z)^{2k_{0l}-3}\,dz, \quad (34.5)$$

where $\varepsilon_{0l} \equiv \frac{\sqrt{M^2-E_{0l}^2}}{2\delta}$, $k_{0l} \equiv 1/2 + \sqrt{¼ + l(l+1) + \alpha_{0l}^2}$, $\alpha_{0l}^2 = \left(\frac{E_{0l}+M}{2\delta^2}\right)V_{01}$, and E_{0l} determined from the following formula:

$$M^2 - E_{0l}^2 = \delta^2\left[\frac{\beta_{0l}^2 - l(l+1) - ½ - \sqrt{l(l+1)+¼+\alpha_{0l}^2}}{½ + \sqrt{l(l+1)+¼+\alpha_{0l}^2}}\right]^2, \quad (35)$$

with $\beta_{0l}^2 = \left(\frac{E_{0l}+M}{2\delta^2}\right)V_{023}$. Comparing Eqs. $(34, i = \overline{1,3})$ with the integral of the form [87]:

$$\int_{-1}^{+1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\,\delta_{mn} \Rightarrow \int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)} \text{ for } (n = 0,1,....). \quad (36)$$

A direct calculation gives the expectation values in Eqs. $(34, i = \overline{1,5})$ as follows:

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2\delta}\frac{\Gamma(2\varepsilon_{0l}+2)\Gamma(2k_{0l}-2)}{(\gamma_{0l}-1)\Gamma(\gamma_{0l}-1)}, \quad (37.1)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{\sqrt{2}\delta}\frac{\Gamma(2\varepsilon_{0l}+3/2)\Gamma(2k_{0l}-2)}{(\gamma_{0l}-3/2)\Gamma(\gamma_{0l}-3/2)}, \quad (37.2)$$

$$\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{4\delta}\frac{\Gamma(2\varepsilon_{0l}+7/2)\Gamma(2k_{0l}-3)}{(\gamma_{0l}-1/2)\Gamma(\gamma_{0l}-1/2)}, \quad (37.3)$$

$$\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2\delta}\frac{\Gamma(2\varepsilon_{0l}+3/2)\Gamma(2k_{0l}-1)}{(\gamma_{0l}-1/2)\Gamma(\gamma_{0l}-1/2)}, \quad (37.4)$$

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{C_{0l}^2}{2\delta}\frac{\Gamma(2\varepsilon_{0l}+2)\Gamma(2k_{0l}-2)}{(\gamma_{0l}-1)\Gamma(\gamma_{0l}-1)}. \quad (37.5)$$

Where $\gamma_{0l}=2\varepsilon_{0l}+2k_{0l}$. For the first excited state $n=1$, the Jacobi polynomial reduced to $P_{n=1}^{(2\varepsilon_{1l},2k_{1l}-1)}(z)=\mathrm{N}_{1l}+\Lambda_{1l}(1-z)$, here $\mathrm{N}_{1l}=2k_{1l}$, $\Lambda_{1l}=-(2\varepsilon_{1l}+2k_{1l}+1)$, with $\varepsilon_{nl}\equiv\frac{\sqrt{M^2-E_{nl}^2}}{2\delta}$, $k_{1l}\equiv 1/2 \quad +\sqrt{¼+l(l+1)+\alpha_{1l}^2}$, $\alpha_{1l}^2=\left(\frac{E_{1l}+M}{2\delta^2}\right)V_{01}$ while E_{1l} presenting the energy of first excited states:

$$M^2-E_{1l}^2=\delta^2\left[\frac{\beta_{1l}^2-l(l+1)-5/2-3\sqrt{l(l+1)+¼+\alpha_{1l}^2}}{3/2+\sqrt{l(l+1)+¼+\alpha_{1l}^2}}\right]^2. \quad (38)$$

And $\beta_{nl}^2=\left(\frac{E_{nl}+M}{2\delta^2}\right)V_{023}$. Thus, the expectation values in Eqs. $(33, i=\overline{1,5})$ are reduced to the following simple form:

$$\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(1,l,m)}=\lambda_1^{(1)}+\lambda_1^{(2)}+\lambda_1^{(3)},$$

$$\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(1,l,m)}=\lambda_2^{(1)}+\lambda_2^{(2)}+\lambda_2^{(3)}$$

$$\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(1,l,m)}=\lambda_3^{(1)}+\lambda_3^{(2)}+\lambda_3^{(3)}, \quad (39.1)$$

And

$$\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)}=\lambda_4^{(1)}+\lambda_4^{2}+\lambda_4^{(3)},$$

$$\left\langle\frac{s^2}{(1-y)^3}\right\rangle_{(1,l,m)}=\lambda_5^{(1)}+\lambda_5^{(2)}+\lambda_5^{(3)}. \quad (39.2)$$

The 15-factors $\lambda_i^{(j)}(i=\overline{1,5}, j=1,2,3)$ are given by:

$$\begin{pmatrix}\lambda_1^{(1)}\\ \lambda_1^{(2)}\\ \lambda_1^{(3)}\end{pmatrix}=\frac{C_{1l}^2}{2^{2\varepsilon_{1l}+2k_{1l}}\delta}\begin{pmatrix}N_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+1}(1+z)^{2k_{1l}-3}\,dz\\ 2N_{1l}\Lambda_{1l}\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+2}(1+z)^{2k_{1l}-3}\,dz\\ \Lambda_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+3}(1+z)^{2k_{1l}-3}\,dz\end{pmatrix}, \quad (40.1)$$

$$\begin{pmatrix}\lambda_2^{(1)}\\ \lambda_2^{(2)}\\ \lambda_2^{(3)}\end{pmatrix}=\frac{C_{1l}^2}{2^{2\varepsilon_{1l}+2k_{1l}+1/2}\delta}\begin{pmatrix}2N_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+1/2}(1+z)^{2k_{1l}-3}\,dz\\ 4N_{1l}\Lambda_{1l}\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+3/2}(1+z)^{2k_{1l}-3}\,dz\\ 2\Lambda_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+5/2}(1+z)^{2k_{nl}-3}\,dz\end{pmatrix}, \quad (40.2)$$

$$\begin{pmatrix}\lambda_3^{(1)}\\ \lambda_3^{(2)}\\ \lambda_3^{(3)}\end{pmatrix}=\frac{C_{1l}^2}{2^{2\varepsilon_{1l}+2k_{1l}+3/2}\delta}\begin{pmatrix}N_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{nl}+5/2}(1+z)^{2k_{nl}-4}\,dz\\ 2N_{1l}\Lambda_{1l}\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+7/2}(1+z)^{2k_{1l}-4\ 2}dz\\ \Lambda_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+9/2}(1+z)^{2k_{1l}-4}\,dz\end{pmatrix}, \quad (40.3)$$

$$\begin{pmatrix}\lambda_4^{(1)}\\ \lambda_4^{(2)}\\ \lambda_4^{(3)}\end{pmatrix}=\frac{C_{1l}^2}{2^{2\varepsilon_{1l}+2k_{1l}+1/2}\delta}\begin{pmatrix}N_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+1/2}(1+z)^{2k_{1l}-2}\,dz\\ 2N_{1l}\Lambda_{1l}\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+3/2}(1+z)^{2k_{1l}-2}\,dz\\ \Lambda_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+5/2}(1+z)^{2k_{1l}-2\ 2}dz\end{pmatrix}, \quad (40.4)$$

and

$$\begin{pmatrix}\lambda_5^{(1)}\\ \lambda_5^{(2)}\\ \lambda_5^{(3)}\end{pmatrix}=\frac{C_{1l}^2}{2^{2\varepsilon_{1l}+2k_{1l}}\delta}\begin{pmatrix}N_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+1}(1+z)^{2k_{1l}-3}\,dz\\ 2N_{1l}\Lambda_{1l}\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+2}(1+z)^{2k_{1l}-3\ 2}dz\\ \Lambda_{1l}^2\int_{-1}^{+1}(1-z)^{2\varepsilon_{1l}+3}(1+z)^{2k_{1l}-3}\,dz\end{pmatrix}. \quad (40.5)$$

To evaluate the above differential equations, we will apply the special integration relationship that we saw in Eq. (32) one get:

$$\begin{pmatrix}\lambda_1^{(1)}\\ \lambda_1^{(2)}\\ \lambda_1^{(3)}\end{pmatrix}=\frac{C_{1l}^2\Gamma(2k_{1l}-2)}{\delta}\begin{pmatrix}\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+2)}{2(\Omega_{1l}-1)\Gamma(\Omega_{1l}-1)}\\ \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+3)}{\Omega_{1l}\Gamma(\Omega_{1l})}\\ \frac{\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+4)}{2(\Omega_{1l}+1)\Gamma(\Omega_{1l}+1)}\end{pmatrix}, \quad (41.1)$$

$$\begin{pmatrix}\lambda_2^{(1)}\\ \lambda_2^{(2)}\\ \lambda_2^{(3)}\end{pmatrix}=\frac{C_{1l}^2\Gamma(2k_{1l}-2)}{\delta}\begin{pmatrix}\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+3/2)}{2(\Omega_{1l}-3/2)\Gamma(\Omega_{1l}-3/2)}\\ \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+5/2)}{(\Omega_{1l}-1/2)\Gamma(\Omega_{1l}-1/2)}\\ \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)}\end{pmatrix}, \quad (41.2)$$

$$\begin{pmatrix}\lambda_3^{(1)}\\ \lambda_3^{(2)}\\ \lambda_3^{(3)}\end{pmatrix}=\frac{C_{1l}^2\Gamma(2k_{1l}-3)}{\delta}\begin{pmatrix}\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{4(\Omega_{1l}-1/2)\Gamma(\Omega_{1l}-1/2)}\\ \frac{N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+9/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)}\\ \frac{\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+11/2)}{(\Omega_{1l}+3/2)\Gamma(\Omega_{1l}+3/2)}\end{pmatrix}, \quad (41.3)$$

$$\begin{pmatrix} \lambda_4^{(1)} \\ \lambda_4^{(2)} \\ \lambda_4^{(3)} \end{pmatrix} = \frac{C_{1l}^2 \Gamma(2k_{1l}-1)}{\delta} \begin{pmatrix} \frac{N_{1l}^2 \Gamma(2\varepsilon_{1l}+3/2)}{2(\Omega_{1l_{1l}}-1/2)\Gamma(\Omega_{1l}-1/2)} \\ \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+5/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)} \\ \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{(\Omega_{1l}+3/2)\Gamma(\Omega_{1l}+3/2)} \end{pmatrix}, \quad (41.4)$$

and

$$\begin{pmatrix} \lambda_5^{(1)} \\ \lambda_5^{(2)} \\ \lambda_5^{(3)} \end{pmatrix} = \frac{C_{1l}^2 \Gamma(2k_{1l}-2)}{\delta} \begin{pmatrix} \frac{N_{1l}^2 \Gamma(2\varepsilon_{1l}+2)}{2(\Omega_{1l}-1)\Gamma(\Omega_{1l}-1)} \\ \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+3)}{\Omega_{1l}\Gamma(\Omega_{1l})} \\ \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+4)}{(2\Omega_{1l}+1)\Gamma(\Omega_{1l}+1)} \end{pmatrix}. \quad (41.5)$$

With$\Omega_{1l} \equiv 2\varepsilon_{1l} + 2k_{1l}$. On substitution of Eqs. (41) Into Eqs. (39), we obtain the following expectation values in the first excited state $(1, l, m)$as follows:

$$\left\langle \frac{s^2}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{C_{1l}^2}{\delta}\Gamma(2k_{1l}-2)\Big(\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+2)}{2(\Omega_{1l}-1)\Gamma(\Omega_{1l}-1)} + \\ + \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+3)}{\Omega_{1l}\Gamma(\Omega_{1l})} + \frac{\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+4)}{2(\Omega_{1l}+1)\Gamma(\Omega_{1l}+1)}\Big), \quad (42.1)$$

$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{C_{1l}^2}{\delta}\Gamma(2k_{1l} - 2)\Big(\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+3/2)}{2(\Omega_{1l}-3/2)\Gamma(\Omega_{1l}-3/2)} + \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+5/2)}{(\Omega_{1l}-1/2)\Gamma(\Omega_{1l}-1/2)} + \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)}\Big), \quad (42.2)$$

$$\left\langle \frac{s^{7/2}}{(1-s)^4} \right\rangle_{(1,l,m)} = \frac{C_{1l}^2}{\delta}\Gamma(2k_{1l} - 3)\Big(\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{4(\Omega_{1l}-1/2)\Gamma(\Omega_{1l}-1/2)} + \frac{N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+9/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)} + \frac{\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+11/2)}{(\Omega_{1l}+3/2)\Gamma(\Omega_{1l}+3/2)}\Big), \quad (42.3)$$

$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} = \frac{C_{1l}^2}{\delta}\Gamma(2k_{1l} - 1)\Big(\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+3/2)}{2(\Omega_{1l_{1l}}-1/2)\Gamma(\Omega_{1l}-1/2)} + \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+5/2)}{(\Omega_{1l}+1/2)\Gamma(\Omega_{1l}+1/2)} + \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+7/2)}{(\Omega_{1l}+3/2)\Gamma(\Omega_{1l}+3/2)}\Big), \quad (41.4)$$

and

$$\left\langle \frac{s^2}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{C_{1l}^2}{\delta}\Gamma(2k_{1l}-2)\Big(\frac{N_{1l}^2\Gamma(2\varepsilon_{1l}+2)}{(\Omega_{1l}-1)\Gamma(\Omega_{1l}-1)} + \frac{2N_{1l}\Lambda_{1l}\Gamma(2\varepsilon_{1l}+3)}{\Omega_{1l}\Gamma(\Omega_{1l})} + \frac{2\Lambda_{1l}^2\Gamma(2\varepsilon_{1l}+4)}{(2\Omega_{1l}+1)\Gamma(\Omega_{1l}+1)}\Big). \quad (42.5)$$

D. The energy shift for the deformed equal vector scalar Manning-Rosen potential and Yukawa potential in RNCQM symmetries

The global energy shift for the deformed equal vector scalar Manning-Rosen potential and Yukawa potential in RNCQM symmetries is composed of three principal parts. The first one is produced from the effect of the generated spin-orbit effective potential. This effective potential is obtained by replacing the coupling of the angular momentum operator $\vec{L}\vec{\Theta}$ with the new equivalent coupling $\Theta\vec{L}\vec{S}$ (with$\Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{1/2}$,). This degree of freedom came considering that the infinitesimal vector $\vec{\Theta}$is arbitrary. We have chosen it to a parallel of the spin of the diatomic molecules under deformed equal vector scalar Manning-Rosen and of Yukawa potentials. Furthermore, we replace $\Theta\vec{L}\vec{S}$ with the corresponding physical form$(\Theta/2)G^2$, with $G^2 = \vec{J}^2 - \vec{L}^2 - \vec{S}^2$. Moreover, in quantum mechanics, the operators $(\hat{H}_{nc-r}^{my}, J^2, L^2, S^2$and $J_z)$ forms a complete set of conserved physics quantities, the eigenvalues of the operator G^2 are equal the values $\tau(j,l,s) \equiv (j(j+1) - l(l+1) - s(s+1))/2$, with $j \in [|l-s|, |l+s|]$. Consequently, the energy shift $\Delta E_{my}^{so}(n=0,\Theta,j,l,s) \equiv \Delta E_{my}^{so}(0,\Theta,j,l,s)$ and $\Delta E_{my}^{so}(n=1,\Theta,j,l,s) \equiv \Delta E_{my}^{so}(1,\Theta,j,l,s)$ due to the perturbed effective potential produced $V_{pert}^{my}(r)$ for the ground state and the first excited state, respectively, in RNCQM symmetries as follows:

$$\Delta E_{my}^{so}(0,\Theta,j,l,s) = (j(j+1) - l(l+1) - s(s+1))(\Theta/2)\langle X\rangle_{(0,l,m)}^R, \quad (43.1)$$

$$\Delta E_{my}^{so}(1,\Theta,j,l,s) = (j(j+1) - l(l+1) - s(s+1))(\Theta/2)\langle X\rangle_{(1,l,m)}^R. \quad (43.2)$$

The global expectation value $\langle X\rangle_{(0,l,m)}^R$is determined from the following expression:

$$\langle X\rangle_{(0,l,m)}^R = 16\delta^4 l(l+1)\left\langle \frac{s^2}{(1-s)^4}\right\rangle_{(0,l,m)} - 2\delta(E_{nl}+M)\Big(\beta_1\left\langle \frac{s^{3/2}}{(1-s)^3}\right\rangle_{(0,l,m)} + \beta_2\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} + \beta_3\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)} + \beta_4\left\langle \frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)}\Big) \quad (44)$$

While $\langle X\rangle_{(1,l,m)}^R = \langle X\rangle_{(0\to 1,l,m)}^R$. This allows us to generalize the above results $\Delta E_{my}^{so}(n,\Theta,j,l,s)$ to the case of n^{th} excited states in RNCQM symmetries as follows:

$$\Delta E_{my}^{so}(n,\Theta,j,l,s) = (j(j+1) - l(l+1) - s(s+1))(\Theta/2)\langle X\rangle_{(n,l,m)}^R. \quad (45)$$

We can express the general expectation value $\langle X\rangle_{(n,l,m)}^{R}$ as follows:

$$\langle X\rangle_{(n,l,m)}^{R}=16\delta^{4}l(l+1)\left\langle\frac{s^{2}}{(1-s)^{4}}\right\rangle_{(n,l,m)}-2\delta(E_{nl}+M)(\beta_{1}\left\langle\frac{s^{3/2}}{(1-s)^{3}}\right\rangle_{(n,l,m)}+\beta_{2}\left\langle\frac{s^{7/2}}{(1-s)^{4}}\right\rangle_{(n,l,m)}+\beta_{3}\left\langle\frac{s^{3/2}}{(1-s)^{2}}\right\rangle_{(nl,m)}+\beta_{4}\left\langle\frac{s^{2}}{(1-s)^{3}}\right\rangle_{(n,l,m)}). \quad (46)$$

The second part of the relativistic energy shift is obtained from the magnetic effect of perturbative effective potential under the deformed equal vector scalar Manning-Rosen and Yukawa potentials. This effective potential is achieved when we replace both ($\vec{L}\vec{\Theta}$ and Θ_{12}) by (σ B $\hat{L}_z$ and σB), respectively, here (B and σ) are symbolize the intensity of the magnetic field induced by the effect of deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of σB, we have also need to apply $\langle n,l,m|\hat{L}_z|n',l',m'\rangle=m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $(m,m')\in[-(l,l'),+(l,l')]$). All of this data allows for the discovery of the new energy shift $\Delta E_{my}^{m}(n=0,\sigma,l,m)\equiv E_{my}^{m}(0,\sigma,l,m)$ and $\Delta E_{my}^{m}(n=1,\sigma,l,m)\equiv \Delta E_{my}^{m}(1,\sigma,l,m)$ due to the perturbed Zeeman effect which created by the influence of the deformed equal vector scalar Manning-Rosen potential and Yukawa potential for the ground state and the first excited state in RNCQMsymmetries,s as follows:

$$\Delta E_{my}^{m}(0,\sigma,l,m)=B\langle X\rangle_{(0,l,m)}^{R}\sigma m, \quad (47.1)$$

$$\Delta E_{my}^{m}(1,\sigma,l,m)=B\langle X\rangle_{(1,l,m)}^{R}\sigma m. \quad (47.2)$$

Thus, we can generalize the above particular cases to the general case $\Delta E_{my}^{m}(n,\sigma,l,m)$ which correspond to the n^{th} excited states in RNCQMsymmetries as follows:

$$\Delta E_{my}^{m}(n,\sigma,l,m)=B\langle X\rangle_{(n,l,m)}^{R}\sigma m. \quad (48)$$

Now, for our purposes, we are interested in finding a new third automatically important symmetry for deformed equal vector scalar Manning-Rosen potential and Yukawa potential at zero temperature in RNCQM symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential $V_{pert}^{my}(r)$ which we have seen in Eq. (31). We discover these important physical phenomena when our studied system consists of N non-interacting is considered as Fermi gas, it is formed from all the particles in their gaseous state (N2, I2, HCl, CH, and LiH) undergoing rotation with angular velocity $\vec{\Omega}$ if we make the following two simultaneous transformations to ensure that previous calculations are not repeated:

$$\vec{\Theta}\rightarrow\chi\vec{\Omega}\ . \quad (49)$$

Here χ is just infinitesimal real proportional constants. We can express the effective potential $V_{pert-rot}^{my}(r)$ which induced the rotational movements of the diatomic molecules as follows:

$$V_{pert-rot}^{my}(r)=\chi\left\{\begin{matrix}\frac{16\delta^{4}l(l+1)s^{2}}{(1-s)^{4}}-2\delta(E_{nl}+M)(\frac{s^{3/2}\beta_{1}}{(1-s)^{3}}\\+\frac{s^{7/2}\beta_{2}}{(1-s)^{4}}+\frac{s^{3/2}\beta_{3}}{(1-s)^{2}}+\frac{s^{2}\beta_{4}}{(1-s)^{3}})\end{matrix}\right\}\vec{\Omega}\vec{L} \quad (50)$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\vec{\Omega}=\Omega e_z$. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\chi h(s)\vec{\Omega}\vec{L}\rightarrow\chi h(s)\Omega L_z, \quad (51)$$

with

$$h(s)=\frac{16\delta^{4}l(l+1)s^{2}}{(1-s)^{4}}-2\delta(E_{nl}+M)(\frac{s^{3/2}\beta_{1}}{(1-s)^{3}}+\frac{s^{7/2}\beta_{2}}{(1-s)^{4}}+\frac{s^{3/2}\beta_{3}}{(1-s)^{2}}+\frac{s^{2}\beta_{4}}{(1-s)^{3}}). \quad (52)$$

All of this data allows for the discovery of the new energy shift $\Delta E_{my}^{f}(n=0,\chi,l,m)\equiv\Delta E_{my}^{f}(0,\chi,l,m)$ and $\Delta E_{my}^{f}(n=1,\chi,l,m)\equiv\Delta E_{my}^{f}(1,\chi,l,m)$ due to the perturbed Fermi gas effect which generated automatically by the influence of the deformed equal vector scalar Manning-Rosen potential and Yukawa potential for the ground state and the first excited state in RNCQMsymmetries as follows:

$$\Delta E_{my}^{f}(0,\chi,l,m)=\langle X\rangle_{(0,l,m)}^{R}\chi\Omega m \quad (53.1)$$

$$\Delta E_{my}^{f}(1,\chi,l,m)=\langle X\rangle_{(1,l,m)}^{R}\chi\Omega m. \quad (53.2)$$

Thus, we can generalize the above particular cases to the general case $\Delta E_{my}^{f}(n,\chi,l,m)$ which correspond to the n^{th} excited states in RNCQMsymmetries as follows:

$$\Delta E_{my}^{f}(n,\chi,l,m)=\langle X\rangle_{(n,l,m)}^{R}\chi\Omega m. \quad (54)$$

It is worth mentioning that K. Bencheikh *et al*. [88, 89] were studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $\chi h(s)\vec{\Omega}\vec{L}$ automatically appears due to the large symmetries resulting from the deformation of space-phase.

IV. RESULTS OF RELATIVISTIC STUDY

In this section of the paper, we summarize our obtained results $(\Delta E_{my}^{so}(0,\Theta,j,l,s),\Delta E_{hk}^{so}(1,\Theta,j,l,s))$, $(\Delta E_{my}^{m}(0,\sigma,l,m),\ \Delta E_{my}^{m}(1,\sigma,l,m))$ and $(\Delta E_{my}^{f}(0,\chi,l,m)$ and $\Delta E_{my}^{f}(1,\chi,l,m))$ for the ground state and first excited state due to the spin-orbital complying, modified Zeeman effect, and perturb Fermi gas potential which induced by $V_{eff}^{my}(r)$ on based to the superposition principle. This allows us to deduce the additive energy shift $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)$ and $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,1,j,l,s,m)$ under the influence of Modified equal vector scalar Manning-Rosen potential and Yukawa potential in RNCQMsymmetries as follows:

$$\Delta E_{my}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)=\langle X\rangle_{(0,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}, \quad (55.1)$$

$$\Delta E_{my}^{tot}(\Theta,\sigma,\chi,1,j,l,s,m)=\langle X\rangle_{(1,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}. \quad (55.2)$$

It is easily to generalized the above special cases to the n^{th} excited states $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under the influence of Modified equal vector scalar Manning-Rosen potential and Yukawa potential in RNCQM symmetries as follows:

$$\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)=\langle X\rangle_{(n,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}. \quad (56)$$

The above results present the global energy shift, which is generated with the effect of noncommutativity properties of space-space; it depended explicitly on the noncommutativity parameters(Θ,σ,χ), the parameters of equal vector scalar Manning-Rosen and Yukawa potentials(b,A,η,V_0) in addition to the atomic quantum numbers (n,j,l,s,m). We observed that the obtained global effective energy $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under Modified equal vector scalar Manning-Rosen potential and Yukawa potential has a carry unit of energy because it is combined from the carrier of energy $(M^2-E_{nl}^2)$. As a direct consequence, the energy$E_{r-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ produced with Modified equal vector scalar Manning-Rosen potential and Yukawa potential, in the symmetries of (RNC: 3D-RS), corresponding the generalized n^{th} excited states, the sum of the roots quart $\left[\Delta E_{my}^{tot}(\Theta,\sigma,n,j,l,s,m)\right]^{1/2}$ of the shift energy and E_{nl} due to the effect of equal vector scalar Manning-Rosen potential and Yukawa potential in RQM, which determined from Eq. (12), as follows:

$$E_{r-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)=E_{nl}-M$$
$$+\left[\langle X\rangle_{(n,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}\right]^{1/2}. \quad (57)$$

For the ground state and first excited state, the above equation can be reduced to the following form:

$$E_{r-nc}^{my}(\Theta,\sigma,b,A,\eta,V_0,n=0,j,l,s,m)=E_{0l}-M$$
$$+\left[\langle X\rangle_{(0,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}\right]^{1/2}, \quad (58.1)$$

and

$$E_{r-nc}^{my}(\Theta,\sigma,b,A,\alpha,V_0,n=1,j,l,s,m)=E_{1l}-M$$
$$+\left[\langle X\rangle_{(1,l,m)}^{R}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}\right]^{1/2}. \quad (58.2)$$

Eq. (57) can de describe the relativistic energy of some diatomic molecules such as HCl, CH and LiH under the deformed equal vector scalar Manning-Rosen potential and Yukawa potential in RNCQMsymmetries.

After examining the bound state solutions of any l-state deformed Klein-Fock-Gordon equation with a modified modified Manning-Rosen potential and modified Yukawa potential, now we discuss the particular cases below.

First : Setting$\alpha=1$ or $\alpha=0$, $A=0$and $V_0\neq 0$, as a direct result, the parameters of perturbative effective potential $\beta_1=\beta_2=0$and $\beta_4=2\beta_3=4V_0\delta^2$, allows us to obtain the reduced perturbative effective potential as flows:

$$V_{pert}^{my}(r)\rightarrow V_{pert}^{yp}(r)=4\delta^3\left\{\frac{4\delta l(l+1)s^2}{(1-s)^4}-V_0(E_{nl}+M)\left(\frac{s^{3/2}}{(1-s)^2}+\frac{2s^2}{(1-s)^3}\right)\right\}\vec{L}\vec{\Theta}. \quad (59)$$

This perturbative potential induced the total energy shift for the n^{th} excited states $\Delta E_{RYP}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under the influence of modified equal vector scalar Yukawa potential in RNCQM symmetries as follows:

$$\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)\rightarrow\Delta E_{RYP}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$$
$$=\langle X\rangle_{(n,l,m)}^{RYP}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}, \quad (60)$$

with

$$\langle X\rangle_{(n,l,m)}^{RYP}=16\delta^4 l(l+1)\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)}-4(E_{nl}+M)V_0\delta^3\left(\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(nl,m)}+2\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(n,l,m)}\right). \quad (61)$$

This result identical to our model in reference [50] when the third parameter $\chi\rightarrow 0$, it is described the deformed Klein-Gordon equation under modified Yukawa Potentials.

Second: Setting$V_0\rightarrow 0$, the potential turn to the deformed Manning potential in RNCQM symmetries. The parameters of perturbative effective potential $\beta_1\equiv\frac{4\delta^3 A}{M}+\beta_2$, $\beta_2\equiv-\frac{8\delta^3\alpha(\alpha-1)}{M}$ and $\beta_3\equiv\frac{4\delta^3 A}{M}$ allows us to obtain the reduced perturbative effective potential as flows:

$$V_{pert}^{my}(r) \rightarrow V_{pert}^{mp}(r) = \begin{Bmatrix} \frac{16\delta^4 l(l+1)s^2}{(1-s)^4} - 2\delta(E_{nl}+M) \\ (\frac{\beta_1 s^{3/2}}{(1-s)^3} + \frac{\beta_2 s^{7/2}}{(1-s)^4} + \frac{\beta_3 s^{3/2}}{(1-s)^2}) \end{Bmatrix} \vec{L}\vec{\Theta}. \quad (62)$$

This perturbative potential induced the total energy shift for the n^{th} excited states $\Delta E_{MP}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under the influence of modified equal vector scalar Yukawa potential in RNCQM symmetries as follows:

$$\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m) \rightarrow \Delta E_{MP}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$$

$$= \langle X\rangle_{(n,l,m)}^{MP}\{\tau(j,l,s)\Theta + B\sigma m + \chi\Omega m\}, \quad (63)$$

with

$$\langle X\rangle_{(n,l,m)}^{RMP} = 16\delta^4 l(l+1)\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)} - 2\delta(E_{nl}+M)(\beta_1\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)} + \beta_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} + \beta_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(nl,m)}). \quad (64)$$

This result identical to our model in reference [under publication].

V. NONRELATIVISTIC STUDY OF MODIFIED MANNING-ROSEN AND YUKAWA POTENTIALS

In this section, we want to derive the nonrelativistic spectrum, which is produced with the effect of deformed equal vector scalar Manning-Rosen potential and Yukawa potential for the diatomic molecules such as N2, I2, HCl, CH and LiH. From Eqs. (1.1) and (7.1), we can write modified Manning-Rosen and Yukawa potentials in the nonrelativistic noncommutative three-dimensional real space NRNCQM symmetries as follows:

$$V_{nc}^{my}(r) = V_{my}(r) + V_{my}^{pert}(r) \text{ with } V_{my}^{pert} << V_{my}(r). \quad (65)$$

Where $V_{my}^{pert}(r)$ is the perturbative potential in nonrelativistic noncommutative three-dimensional real space NRNCQM symmetries:

$$V_{my}^{pert}(r) = -\frac{\partial V_{my}(r)}{\partial r}\frac{\vec{L}\vec{\Theta}}{2r} + O(\Theta^2). \quad (66)$$

We have applied the approximations type suggested by (Greene and Aldrich) and Dong *et al*. for a short-range potential that is an excellent approximation to the centrifugal term for Manning-Rosen and Yukawa potentials (see Eq. (26.1)) and we calculate $\frac{\partial V_{my}(r)}{\partial r}$ (see Eq. (29)). Now, substituting Eq. (29) into Eq. (66) and we replace $\frac{1}{r}$by its corresponding approximation in Eq. (26.2), we get the perturbative potential in NRNCQM symmetries as follows:

$$V_{my}^{pert}(r) = -\delta\left[\frac{s^{\frac{3}{2}}\beta_1}{(1-s)^3} + \frac{s^{\frac{7}{2}}\beta_2}{(1-s)^4} + \frac{s^{\frac{3}{2}}\beta_3}{(1-s)^2} + \frac{s^2\beta_4}{(1-s)^3}\right]\vec{L}\vec{\Theta} + O(\Theta^2). \quad (67)$$

Thus, we need the expectation values of $\frac{s^2}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^3}$, $\frac{s^{7/2}}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^2}$and $\frac{s^2}{(1-s)^3}$ to find the nonrelativistic energy corrections produced with the perturbative potential$V_{my}^{pert}(r)$. By using wave function in Eq. (15) and the expectations values in Eqs.$(37, i=\overline{1,5})$, and Eqs.$(42, i=\overline{1,5})$ for the ground state and first excited state, respectively, we get the corresponding global expectation values $\langle X\rangle_{(0,l,m)}^{NR}$ and $\langle X\rangle_{(1,l,m)}^{NR}$ as follows:

$$\langle X\rangle_{(0,l,m)}^{NR} = -\delta\left[\beta_1\left\langle\frac{s^{\frac{3}{2}}}{(1-s)^3}\right\rangle_{(0,l,m)} + \beta_2\left\langle\frac{s^{\frac{7}{2}}}{(1-s)^4}\right\rangle_{(0,l,m)} + \beta_3\left\langle\frac{s^{\frac{3}{2}}}{(1-s)^2}\right\rangle_{(0,l,m)} + \beta_4\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(0,l,m)}\right], \quad (68.1)$$

and

$$\langle X\rangle_{(1,l,m)}^{NR} = -\delta\begin{bmatrix}\beta_1\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(1,l,m)} + \beta_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(1,l,m)} \\ +\beta_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)} + \beta_4\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(1,l,m)}\end{bmatrix}. \quad (68.2)$$

By following the same physical methodology that we devoted in our relativistic previous study, the energy corrections $\Delta E_{my}^{NR}(0,j,l,s,m)$ and $\Delta E_{my}^{NR}(1,j,l,s,m)$ for the ground state and first excited state due to the spin-orbit complying, modified Zeeman effect and nonrelativistic perturbed Fermin gas potential which induced by $V_{my}^{pert}(r)$ under the influence of Modified equal vector scalar Manning-Rosen potential and Yukawa potential in NRNCQM symmetries as follows:

$$\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n=0,j,l,s,m)$$

$$= \langle X\rangle_{(0,l,m)}^{NR}\{\tau(j,l,s)\Theta + B\sigma m + \chi\Omega m\}, \quad (69.1)$$

and

$$\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n=1,j,l,s,m)$$

$$= \langle X\rangle_{(1,l,m)}^{NR}\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}. \quad (69.2)$$

It is easily to generalized the above special cases to the n^{th} excited states $\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$ under the influence of the Modified equal vector scalar Manning-Rosen potential and Yukawa potential model in NRNCQM symmetries as follows:

$$\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$$
$$=\langle X\rangle_{(n,l,m)}^{NR}\{\tau(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m\}. \quad (70)$$

With $\langle X\rangle_{(n,l,m)}^{NR}$ is given by:

$$\langle X\rangle_{(n,l,m)}^{NR}=-\delta\left[\beta_1\left\langle\frac{s^{\frac{3}{2}}}{(1-s)^3}\right\rangle_{(n,l,m)}+\beta_2\left\langle\frac{s^{\frac{7}{2}}}{(1-s)^4}\right\rangle_{(n,l,m)}+\beta_3\left\langle\frac{s^{\frac{3}{2}}}{(1-s)^2}\right\rangle_{(n,l,m)}+\beta_4\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(n,l,m)}\right]. \quad (71)$$

The nonrelativistic energy $E_{nr-nc}^{my}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$ for the diatomic molecules (HCl, CH and LiH) produced with Modified equal vector scalar Manning-Rosen potential and Yukawa potential, in the symmetries of NRNCQM, corresponding the generalized n^{th} excited states, the sum of the nonrelativistic energy E_{nl}^{nr} due to the effect of equal vector scalar Manning-Rosen potential and Yukawa potential in NRQM, and the corrections produced with the perturbed spin-orbit interaction and modified Zeeman effect, as follows:

$$E_{nr-nc}^{my}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)=E_{nl}^{nr}(n,l,V_0,V_1)$$
$$+\langle X\rangle_{(n,l,m)}^{NR}\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}. \quad (72)$$

The nonrelativistic energy $E_{nl}^{nr}(n,l,V_0,V_1)$ due to the effect of equal vector scalar Manning-Rosen potential and Yukawa potential in NRQM is determined directly from the study of B.I. Ita *et al.* [90] given by:

$$E_{nl}^{nr}=-\frac{\alpha^2}{2\mu}\left\{\frac{\frac{2\mu C}{\alpha}+\frac{2\mu V_0}{\alpha}-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\Sigma}{(2n+1)+2\Sigma}\right\}, \quad (73)$$

here $\Sigma=\sqrt{\frac{1}{4}-\frac{2\mu C}{\alpha^2}-\frac{2\mu D}{\alpha^2}}$, with $\alpha\to\delta^{-1}$, $D\to-\frac{\alpha(\alpha-1)}{2Mb^2}$ to match the notations used in the two references are identical.

Now, considering composite systems such as molecules made of $N=2$ particles of masses $m_n(n=1{,}2)$ in the frame of noncommutative algebra, it is worth taking into account features of descriptions of the systems in the non-relativistic case, it was obtained that composite systems with different masses are described with different noncommutative parameters [91, 92, 93, 94]:

$$\left[\hat{x}_\mu^{(S,H,I)}\stackrel{*}{,}\hat{x}_\nu^{(S,H,I)}\right]\equiv\hat{x}_\mu^{(S,H,I)}*\hat{x}_\nu^{(S,H,I)}-\hat{x}_\nu^{(S,H,I)}*\hat{x}_\mu^{(S,H,I)}=i\theta_{\mu\nu}^c, \quad (74)$$

the new noncommutativity parameter $\theta_{\mu\nu}^c$ is determined from the following relation:

$$\theta_{\mu\nu}^c=\sum_{n=1}^{2}\mu_n^2\theta_{\mu\nu}^{(n)}, \quad (75)$$

with $\mu_n=\frac{m_n}{\sum_n m_n}$, the indices $(n=1{,}2)$ label the particle, and $\theta_{\mu\nu}^{(n)}$ is the new parameter of noncommutativity, corresponding to the particle of mass m_n. Note that in the case of a system of two particles with the same mass $m_1=m_2$ such as the diatomic (N_2 and I_2) molecules under the Modified equal vector scalar Manning-Rosen potential and Yukawa potential model, the parameter $\theta_{\mu\nu}^{(n)}=\theta_{\mu\nu}$. Thus, the three parameters Θ, σ and χ which appears in Eq. (70) are changed to the new form Θ^c, σ^c and χ^c as follows:

$$\Theta^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\Theta_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\Theta_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\Theta_{13}^{(n)}\right)^2, \quad (76.1)$$

$$\sigma^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\sigma_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\sigma_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\sigma_{13}^{(n)}\right)^2, \quad (76.2)$$

$$\chi^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\chi_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\chi_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\chi_{13}^{(n)}\right)^2. \quad (76.3)$$

As it is mentioned above, in the case of a system of two particles with the same mass $m_1=m_2$, we have $\Theta_{\mu\nu}^{(n)}=\Theta_{\mu\nu}$, $\sigma_{\mu\nu}^{(n)}=\sigma_{\mu\nu}$ and $\chi_{\mu\nu}^{(n)}=\chi_{\mu\nu}$. Allows us to generalize the nonrelativistic global energy $E_{nr-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ under the modified Manning-Rosen and Yukawa potentials model considering that composite systems with different masses are described with different noncommutative parameters for the diatomic molecules (HCl, CH and LiH) as:

$$E_{nr-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)=E_{nl}^{nr}(n,l,V_0,V_1)$$
$$+\langle X\rangle_{(n,l,m)}^{NR}(k(j,l,s)\Theta^c+B\sigma^c m+\chi^c\Omega m). \quad (77)$$

The KGE, as the most well-known relativistic wave equation describing spin-zero particles, but its extension in RNCQM symmetries deformed Klein-Fock-Gordon equation under modified Manning-Rosen and Yukawa potentials model has a physical behavior similar to the Duffin–Kemmer equation for meson with spin-1, it can describe a dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. This is one of the most important new results of this research. Worthwhile it is better to mention that for the two simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$ and $(\Theta^c,\sigma^c,\chi^c)\to(0,0,0)$, we recover the results of the in Refs. [33, 90]. and it displays that the mathematical implementation of SUSY quantum mechanics is quite perfect.

VI. SUMMARY AND CONCLUSION

In this work, we have found the approximate bound state solutions of deformed Klein-Fock-Gordon equation and deformed Schrödinger equation using the tool of Bopp's shift and standard perturbation theory methods of modified equal vector scalar Manning-Rosen potential and Yukawa potential in both relativistic and nonrelativistic regimes which correspond to high and low energy physics. We have employed the improved approximation scheme to deal with the centrifugal term to obtain the new relativistic bound state solutions $E_{r-nc}^{ym}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ corresponding to the generalized n^{th} excited states that appear as a sum of the total shift energy $\Delta E_{my}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ and the relativistic energy E_{nl} of the equal vector scalar Manning-Rosen and Yukawa potentials. Furthermore, we have obtained the new nonrelativistic global energy $E_{nr-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ of some diatomic molecules such as (N2, I2, HCl, CH and LiH) in NRNCQM symmetries as a sum of the nonrelativistic energy E_{nl}^{nr} and the perturbative corrections $\Delta E_{my}^{NR}(\Theta,\sigma,b,A,\alpha,V_0,n,j,l,s,m)$. Furthermore, we state that the new relativistic energy eigenvalues the new relativistic bound state solutions $E_{r-nc}^{ym}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ and the new nonrelativistic global energy $E_{nr-nc}^{my}(\Theta,\sigma,\chi,b,A,\alpha,V_0,n,j,l,s,m)$ are quite sensitive with potential parameters for the quantum states (b,A,α,V_0) and the discreet atomic quantum numbers (j,l,s,m) in addition to noncommutativity three parameters (Θ,σ and χ). This behavior is similar to the perturbed both modified Zeeman effect and modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V_{pert}^{my}(r)$ in the symmetries of RNCQM and NRNCQM. Furthermore, we can conclude that the deformed Klein-Fock-Gordon equation under the modified equal vector scalar Manning-Rosen potential and Yukawa potential model becomes similar to Duffin–Kemmer equation for meson with spin-1, it can describe a dynamic state of a particle with spin one in the symmetries of RNCQM. Furthermore, we have applied our results to composite systems such as molecules made of $N=2$ particles of masses $m_n(n=1,2)$. It is worth mentioning that, for all cases, when to make the two simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$ and $(\Theta^c,\sigma^c,\chi^c)\to(0,0,0)$, the ordinary physical quantities are recovered. Furthermore, our research findings could also be applied in, atomic physics, vibrational and rotational spectroscopy, mass spectra, nuclear physics, and other applications. Finally, given the effectiveness of the methods that we followed in achieving our goal in this research, we advise researchers to apply the same methods to delve more deeply, whether in the relativistic and nonrelativistic regimes for others potentials.

ACKNOWLEDGEMENTS

This work was supported by the Research Fund of Laboratory of Physics and Material Chemistry, University of M'sila, and DGRSDT in Algeria with the project number B00L02UN280120180001. We would like to thank the kind referees for their helpful comments and suggestions which have greatly improved the paper.

REFERENCES

[1] Schrödinger, E., *An Undulatory Theory of the Mechanics of Atoms and Molecules,* Phys. Rev. **28**, 1049–1070 (1926). doi:10.1103/physrev.28.1049

[2] Klein, O., *Quantentheorie und fünfdimensionale Relativitätstheorie*, Z. Physik **37**, 895–906 (1926). https://doi.org/10.1007/BF01397481

[3] Fock, V., Zur Schrödingerschen Wellenmechanik, Z. Physik **38**, 242–250 (1926). https://doi.org/10.1007/BF01399113

[4] Gordon, W., *Der Comptoneffekt nach der Schrödingerschen Theorie*. Z. Physik **40**, 117–133 (1926). https://doi.org/10.1007/BF01390840

[5] Kemmer, N., *The Particle Aspect of Meson Theory*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **173**, 91–116 (1939). doi:10.1098/rspa.1939.0131

[6] Dirac, P. A. M., *The Quantum Theory of the Electron*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **117**, 610–624 (1928). doi:10.1098/rspa.1928.0023

[7] Alhaidari, A. D., Bahlouli, H. and Al-Hasan, A., *Dirac and Klein–Gordon equations with equal scalar and vector potentials*, Phys. Lett. A **34**9, 87–97 (2006). doi:10.1016/j.physleta.2005.09.008

[8] Nagiyev, S. M., Ahmadov, A. I. and Tarverdiyeva, V. A., *Approximate Solutions to the Klein-Fock-Gordon Equation for the Sum of Coulomb and Ring-Shaped-Like Potentials*. Advances in High Energy Physics, **2020**, 1–10 (2020). doi: 10.1155/2020/1356384

[9] Qiang, W. C. and Dong, S. H., *Analytical approximations to the l-wave solutions of the Klein–Gordon equation for a second Pöschl–Teller like potential*, Phys. Lett. A **372**, 4789–4792 (27-28) (2008). doi: 10.1016/j.physleta.2008.05.020

[10] Wei, G. F., Dong, S. H. and Bezerra, V. B., *The relativistic bound and scattering states of the Eckart potential with a proper new approximate scheme for the centrifugal term*, International Journal of Modern Physics A **24**, 161–172 (2009). doi:10.1142/s0217751x09042621

[11] Yi, L. Z., Diao, Y. F., Liu, J. Y. and Jia, C. S., *Bound states of the Klein–Gordon equation with vector and scalar Rosen–Morse-type potentials*, Phys. Lett. A **333**, 212–217 (2004). doi:10.1016/j.physleta.2004.10.054

[12] Manning, M. F. and Rosen, N., *A potential function for the vibrations of diatomic molecules,* Phys. Rev. **44**, 953 (1933). doi:10.1103/physrev.44.951

[13] Yukawa, H., *On the interaction of elementary particles I*, Proc. Phys. Math. Soc. Japan **17**, 48 (1935).

[14] Manning, M. F., *A potential function for the vibrations of diatomic molecules*, Phys. Rev. **44**, 951 (1933). doi: 10.1103/physrev.44.951

[15] Manning, M. F., *Energy Levels of a Symmetrical Double Minima Problem with Applications to the NH3 and ND3 Molecules*, The Journal of Chemical Physics **3**, 136–138 (1935). DOI: 10.1063/1.1749619

[16] Manning, M. F., *Exact Solutions of the Schrödinger Equation*, *Phys. Rev*. **48**, 161–164 (1935). DOI: 10.1103/physrev.48.161

[17] Falaye, B. J., Oyewumi, K. J., Ibrahim, T. T., Punyasena, M. A. and Onate, C. A., *Bound state solutions of the Manning–Rosen potential*, Canadian Journal of Physics **91**, 98–104 (2013). doi: 10.1139/cjp-2012-0330

[18] Dong, S. H. and García-Ravelo, J., *Exact solutions of the s-wave Schrödinger equation with Manning–Rosen potential*, Phys. Scr. **75**, 307 (2007). DOI: https://doi.org/10.1088/0031-8949/75/3/013

[19] Ikhdair, S. M., On the bound-state solutions of the Manning–Rosen potential including an improved approximation to the orbital centrifugal term, *Phys. Scr*. **83**, 015010 (2011). DOI: https://doi.org/10.1088/0031-8949/83/01/015010

[20] Hatami, N. and Setare, M. R., *Analytical solutions of the Klein–Gordon equation for Manning–Rosen potential with centrifugal term through Nikiforov–Uvarov method*, Indian Journal of Physics **91**, 1229–1232 (2017). doi:10.1007/s12648-017-1020-4

[21] Wei, G. F., Zhen, Z.Z. and Dong, S.H., *The relativistic bound and scattering states of the Manning-Rosen potential with an improved new approximate scheme to the centrifugal term*, Cent. Eur. J. Phys. **7,** 175–183 (2009). https://doi.org/10.2478/s11534-008-0143-9

[22] Wei, G. F. and Dong, S. H., *Approximately analytical solutions of the Manning–Rosen potential with the spin–orbit coupling term and spin symmetry*, Phys. Lett. A **373**, 49 (2008). https://doi.org/10.1016/j.physleta.2008.10.064

[23] Bahar, M. K. and Yasuk, F., *Approximate Solutions to the Dirac Equation with Effective Mass for the Manning–Rosen Potential in N Dimensions*, *Few-Body Syst* **53**, 515–524 (2012). https://doi.org/10.1007/s00601-012-0461-8

[24] Vrscay, E. R., *Hydrogen atom with a Yukawa potential: Perturbation theory and continued-fractions–Padé approximants at large order*, *Phys. Rev. A* **33**, 1433–1436 (1986). doi: 10.1103/physreva.33.1433

[25] Ugalde, J. M., Sarasola, C. and Lopez, X., *Atomic and molecular bound ground states of the Yukawa potential*, Phys. Rev. A **56**, 1642–1645 (1997). doi:10.1103/physreva.56.1642

[26] Gazeau, J. P. and Maquet, A., *Bound states in a Yukawa potential: A Sturmian group-theoretical approach*, Phys. Rev. A **20**, 727-739 (1979). DOI: https://doi.org/10.1103/PhysRevA.20.727

[27] Totsuji, H., *Theory of Critical Screening Radius of Energy Levels of Hydrogen-Like Atoms in Plasmas*, Journal of the Physical Society of Japan **31**, 584–590 (1971). doi:10.1143/jpsj.31.584

[28] Smith, C. B., *Bound States in a Debye-Hückel Potential*, Phys. Rev. **134**, A1235 (1964) DOI: https://doi.org/10.1103/PhysRev.134.A1235

[29] Harris, G. M., *Attractive Two-Body Interactions in Partially Ionized Plasmas*, Phys. Rev. **125**, 1131 (1962). DOI: https://doi.org/10.1103/PhysRev.125.1131

[30] McEnnan, J., Kissel, L. and Pratt, R. H., *Analytic perturbation theory for screened Coulomb potentials: Nonrelativistic case*, Phys. Rev. A **13**, 532–559 (1976). doi:10.1103/physreva.13.532

[31] Mehta, C. H. and Patil, S. H., *Nonperturbative approach to screened Coulomb potentials*, Phys. Rev. A **17**, 34–42 (1978). doi:10.1103/physreva.17.34

[32] Dutt, R. and Varshni, Y. P., *An analytic approximation for the energy levels of neutral atoms*, Z. Phys. Atoms and Nuclei **313**, 143-145 (1983). DOI: 10.1007/bf01417219

[33] Ahmadov, A. I., Demirci, M. and Aslanova, S. M., *Bound state solutions of the Klein-Fock-Gordon equation with the sum of Manning-Rosen potential and Yukawa potential within SUSYQM*, Journal of Physics: Conference Series **1416**, 01200 (2019). doi:10.1088/1742-6596/1416/1/012001

[34] Ita, B.I., Louis, H., Akakuru, O. U., Nzeata-Ibe, N. A., Ikeuba, A. I., Magu, T. O., Amos, P. I., Edet, C. O., *Approximate Solution to the Schrödinger Equation with Manning-Rosen plus a Class of Yukawa Potential via WKBJ Approximation Method*, Bulg. J. Phys. **45**, 323–333 (2018).

[35] Ahmadov, A.I., Demirci, M., Aslanova, S.M. and Mustamin, M.F., *Arbitrary ℓ-state solutions of the Klein-Gordon equation with the Manning-Rosen plus a Class of Yukawa potentials*, Phys. Lett. A **384**, 126372 (2020). doi:10.1016/j.physleta.2020.126372

[36] Magu, T. O., Ita, B.I., Nyong, B.E., Hitler, L., Nelson, N., Joseph, I. and Amos, P. I., *Analytic spin and pseudospin solutions to the Dirac equation for the Manning-Rosen plus a Class of Yukawa potential and Coulomb-like tensor interaction*, Journal of Nigerian Association of Mathematical Physics **42**, 1–8 (2017).

[37] Louis, H., Ita, B. I., Amos, P. I., Akakuru, O. U., Orosum, M. M., Nzeata-Ibe, N. A., and Philip, A., *Bound State Solutions of Klein-Gordon Equation with Manning-Rosen Plus a Class of Yukawa Potential Using Pekeris-Like Approximation of the Coulomb Term and Parametric Nikiforov-Uvarov*, Physical Science International Journal IJCPS **7**, 33-37 (2018). DOI: 10.9734/PSIJ/2017/34330

[38] Capozziello, S., Lambiase, G. and Scarpetta, G., *Generalized uncertainty principle from quantum geometry*, Int. J. Theor. Phys. **39**, 15-22 (2000). DOI: 10.1023/A:1003634814685

[39] Doplicher, S., Fredenhagen, K. and Roberts, J. E., *Spacetime quantization induced by classical gravity*, Phys. Lett. B. **331**, 39–44 (1994). doi:10.1016/0370-2693(94)90940-7

[40] Witten, E., *Refection on the fate spacetime*, Phys. Today **49**, 24 (1996). https://doi.org/10.1063/1.881493

[41] Kempf, A., Mangano, G. and Mann, R. B., *Hilbert space representation of the minimal length uncertainty relation*, Phys. Rev. D. **52**, 1108–1118 (1995). doi:10.1103/physrevd.52.1108.

[42] Scardigli, F., *Some heuristic semi-classical derivations of the Planck length, the Hawking effect and the unruh effect*, Il Nuovo Cimento B Series **11**, 1029–1034 (1995). doi:10.1007/bf02726152

[43] Adler, R. J. and Santigo, D. I., *On gravity and the uncertainty principal*, Mod. Phys. Lett. A **14**, 1371–138 (1999). Doi: 10.1142/s0217732399001462.

[44] Kanazawa, T., Lambiase, G., Vilasi, G. and Yoshioka, A., *Noncommutative Schwarzschild geometry and generalized uncertainty principle*, The European Physical Journal C. **79**(2) (2019).
doi:10.1140/epjc/s10052-019-6610-1

[45] Scardigli, F., *Generalized uncertainty principle in quantum gravity from micro-black hole Gedanken experiment*, Physics Letters B. **452**, 39–44 (1999). DOI: 10.1016/s0370-2693(99)00167-7

[46] Ho, P. M. and Kao, H. C., *Noncommutative quantum mechanics from noncommutative quantum field Theory*, Phys. Rev. Letters **88**, 151602-1 (2002).
doi:10.1103/physrevlett.88.151602

[47] Gnatenko, P., *Parameters of noncommutativity in Lie-algebraic noncommutative space*, Phys. Rev. D. **99**, 026009-1 (2019). doi:10.1103/physrevd.99.026009

[48] Bertolami, O., Rosa, J. G., Dearagao, C. M. L., Castorina, P. and Zappala, D., *Scaling of variables and the relation between noncommutative parameters in noncommutative quantum mechanics*, Mod. Phys. Lett. A **21**, 795–802 (2006). doi:10.1142/s0217732306019840

[49] Connes, A., Douglas, M. R. and Schwarz, A., *Noncommutative geometry and Matrix theory*, Journal of High Energy Physics, 1998(02), 003–003. doi:10.1088/1126-6708/1998/02/003

[50] Abdelmadjid Maireche, *A model of modified Klein-Gordon equation with modified scalar–vector Yukawa potential*, Afr. Rev Phys. **15**, 1-11 (2020). http://lamp.ictp.it/index.php/aphysrev/article/view/1777/618.

[51] Abdelmadjid Maireche, *Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics*, International Journal of Geometric Methods in Modern Physics **17**, 2050067 (2020).
DOI: 10.1142/S021988782050067X

[52] Abdelmadjid Maireche, New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms, Afr. Rev Phys. **12**, 130-143 (2017). http://lamp.ictp.it/index.php/aphysrev/article/view/1533/564.

[53] Abdelmadjid Maireche, *Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles*, International Frontier Science Letters **11**, 29-44 (2017). DOI: https://doi.org/10.18052/www.scipress.com/IFSL.11.29

[54] Abdelmadjid Maireche, *Modified unequal mixture scalar vector Hulthén–Yukawa potentials model as a quark–antiquark interaction and neutral atoms via relativistic treatment using the improved approximation of the centrifugal term and Bopp's shift method*, Few-Body Syst. **61**, 30 (2020).
https://doi.org/10.1007/s00601-020-01559-z.

[55] Abdelmadjid Maireche, *A New Approach to the approximate analytic solution of the three-dimensional Schrödinger equation for Hydrogenic and neutral atoms in the generalized Hellmann potential model*, Ukr. J. Phys. **65**, 987 (2020).
https://doi.org/10.15407/ujpe65.11.987

[56] Abdelmadjid Maireche, *Effects of two-dimensional noncommutative theories on bound states Schrödinger diatomic molecules under New modified Kratzer-type interactions*, International Letters of Chemistry, Physics and Astronomy **76**, 1-11 (2017).
https://doi.org/10.18052/www.scipress.com/ILCPA.76.1

[57] Abdelmadjid Maireche, *Any l-States solutions of the modified Schrodinger equation with generalized Hellmann–Kratzer potential model in the symmetries of NRNCQM*, To Physics Journal **4**, 16-32 (2019).
https://purkh.com/index.php/tophy/article/view/521

[58] Gamboa, J., Loewe, M, and Rojas, J. C., *Noncommutative quantum mechanics*, Phys. Rev. D. **64**, 067901 (2001).
DOI: https://doi.org/10.1103/PhysRevD.64.067901.

[59] Djemaï, E. F. and Smail, H., *On quantum mechanics on noncommutative quantum phase space*, Commun. Theor. Phys. (Beijing, China) **41**, 837–844 (2004).
doi:10.1088/0253-6102/41/6/837

[60] Yi, Y., Kang, L., Jian-Hua, W. and Chi-Yi, C., S*pin-1/2 relativistic particle in a magnetic field in NC phase space*, Chinese Physics C. **34**, 543–547 (2010).
doi:10.1088/1674-1137/34/5/005

[61] Bertolami, O. and Leal, P., *Aspects of phase-space noncommutative quantum mechanics*, Phys. Lett. B. **750**, 6–11 (2015). doi:10.1016/j.physletb.2015.08.024

[62] Bertolami, O., Rosa, J. G., de Aragão, C. M. L., Castorina, P. and Zappalà, D., *Noncommutative gravitational quantum well*, Phys. Rev. D. **72**, 025010-1 (2005).
doi: 10.1103/physrevd.72.025010

[63] Zhang, J., *Fractional angular momentum in non-commutative spaces*, Phys. Lett. B. **584**, 204–209 (2004). DOI:10.1016/j.physletb.2004.01.049

[64] Chaichian, M., Sheikh-Jabbari and Tureanu, A., *Hydrogen atom spectrum and the Lamb Shift in noncommutative QED*, Phys. Rev. Letters **86**, 2716–2719 (2001). doi:10.1103/physrevlett.86.2716.

[65] De Andrade, M. A. and Neves, C., *Noncommutative mapping from the symplectic formalism*, Journal of Mathematical Physics **59**, 012105 (2018).
doi: 10.1063/1.4986964

[66] Abreu, E. M. C., Neves, C. and Oliveira, W., *Noncommutativity from the symplectic point of view*, Int. J. Mod. Phys. A. **21**, 5359 (2006).
DOI: 10.1142/s0217751x06034094

[67] Abreu, E. M. C., Neto, J. A., Mendes, A. C. R., Neves, C., Oliveira, W. and Marcial, M. V., *Lagrangian formulation for noncommutative nonlinear systems*, Int. J. Mod. Phys. A. **27**, 1250053 (2012). DOI: 10.1142/s0217751x12500534

[68] Wang, J. and Li, K., *The HMW effect in noncommutative quantum mechanics*, Journal of Physics A: Mathematical and Theoretical **40**, 2197–2202 (2007).
doi:10.1088/1751-8113/40/9/021

[69] Li, K. and Wang, J., *The topological AC effect on non-commutative phase space*, The European Physical Journal C. **50**, 1007–1011 (2007).

doi:10.1140/epjc/s10052-007-0256-0

[70] Abdelmadjid Maireche, *A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, Sri Lankan Journal of Physics **21**, 11-35 (2020). DOI: http://DOI.org/ 10.4038/sljp.v20i0

[71] Abdelmadjid Maireche, *Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics*, J. Nano- Electron. Phys. **10**, 06015-1 (2018). DOI: https://doi.org/10.21272/jnep.10(6).06015

[72] Morse, P. M., *Diatomic Molecules According to the Wave Mechanics. II. Vibrational Levels*, Phys. Rev. **34**, 57–64 (1929). doi:10.1103/physrev.34.57

[73] Amadi, P. O. Ikot, A. N., Okorie, U. S. Rampho, G. J., Abdullah, H. Y., Lütfüoğlu, B. C., *Information entropies for H2 and ScF diatomic molecules with Deng- Fan-Eckart potential*, Revista Mexicana de Fısica **66**, 742–748 (2020). DOI: https://doi.org/10.31349/RevMexFis.66.742

[74] Abramowitz, M., Stegum, I. A., Handbook of mathematical function with formulas, graphs, and mathematical tables (National Bureau of Standards Applied Mathematics Series-55) (1964).

[75] Maireche, A., *The Klein–Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space*, Modern Physics Letters A. **35**, 052050015 (2020). doi:10.1142/s0217732320500157

[76] Abdelmadjid Maireche, *The Klein–Gordon equation with modified Coulomb potential plus inverse-square–root potential in three-dimensional noncommutative space*, To Physics Journal **3**, 186-196 (2019). https://purkh.com/index.php/tophy/article/view/489

[77] Abdelmadjid Maireche, *Bound state solutions of Klein-Gordon and Shrodinger equations with linear combination of Hulthén and Kratzer potentials*, Afr. Rev Phys. **15**, 19-31 (2020). http://lamp.ictp.it/index.php/aphysrev/article/view/1779/620.

[78] Motavalli, H. and Akbarieh, A. R., *Klein-Gordon equation for the Coulomb potential in noncommutative space*, Mod. Phys. Lett. A **25**, 2523–2528 (2010). doi:10.1142/s0217732310033529

[79] Darroodi, M., Mehraban, H. and Hassanabadi, H., *The Klein–Gordon equation with the Kratzer potential in the noncommutative space,* Mod. Phys. Lett. A. **33**, 1850203 (2018). doi:10.1142/s0217732318502036

[80] Saidi, A. and Sedra, M. B., *Spin-one (1+3)-dimensional DKP equation with modified Kratzer potential in the non-commutative space*, Mod. Phys. Lett. A **35**, 2050014 (2019). doi:10.1142/s0217732320500145

[81] Abdelmadjid Maireche, *Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics*, Sri Lankan Journal of Physics **22**, 1–19 (2021). DOI: http://doi.org/10.4038/sljp.v22i1.8079

[82] Abdelmadjid Maireche, *Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Nonrelativistic treatment in the NCQM symmetries*, Lat. Am. J. Phys. Educ. **14**, 3310-1 (2020).

[83] Gouba, L., *A comparative review of four formulations of noncommutative quantum mechanics*, International Journal of Modern Physics A **31**, 1630025 (2016). DOI:10.1142/s0217751x16300258

[84] Mezincescu, L., "*Star operation in quantum mechanics*" e-print arXiv: hep-th/0007046v2.

[85] Bopp, F., *La mécanique quantique est-elle une mécanique statistique classique particulière?*, Ann. Inst. Henri Poincaré 15, 81 (1956).

[86] Badawi, M., Bessis, N. and Bessis, G., *On the introduction of the rotation-vibration coupling in diatomic molecules and the factorization method*, Journal of Physics B: Atomic and Molecular Physics **5**, L157–L159 (1972). doi:10.1088/0022-3700/5/8/004

[87] Gradshteyn, S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, 7th. ed.: eds. A. Jeffrey D. Zwillinger (Elsevier, UK, 2007).

[88] Medjedel, S. and Bencheikh, K., *Exact analytical results for density profile in Fourier space and elastic scattering function of a rotating harmonically confined ultra-cold Fermi gas*. Phys. Lett. A **383**, 1915-1921 (2019). https://doi.org/10.1016/j.physleta.2019.03.021

[89] Bencheikh, K., Medjedel, S. and Vignale, G., *Current reversals in rapidly rotating ultracold Fermi gases*, Phys. Rev. A **89**, 063620 (2014). doi:10.1103/physreva.89.063620

[90] Ita, B. I., Louis, H., Nzeata-ibe, N. A., Magu, T. O., and Orosun, M. M., *Bound State Solutions of the Schrodinger's Equation with ManningRosen plus Yukawa Potential using Pekeris-like Approximation of the Coulomb Term and Parametric Nikoforov-Uvarov*, Zimbabwe Journal of Science and Technology **12**, 99-103 (2017).

[91] Gnatenko, K. P. and Tkachuk, V. M., *Upper bound on the momentum scale in noncommutative phase space of canonical type*, EPL (Europhysics Letters) **127**, 20008 (2019). doi:10.1209/0295-5075/127/2000

[92] Gnatenko, K.P. and Tkachuk, *Composite system in rotationally invariant noncommutative phase space*, International Journal of Modern Physics A **33**, 1850037 (2018). doi:10.1142/s0217751x18500379

[93] Gnatenko, K. P., *Composite system in noncommutative space and the equivalence principle*, Phys. Lett. A **377**, 3061-3066 (2013). doi:10.1016/j.physleta.2013.09.036

[94] Abdelmadjid Maireche, *Bound-state solutions of the modified Klein–Gordon and Schrödinger equations for arbitrary l-state with the modified Morse potential in the symmetries of noncommutative quantum mechanics*, Journal of Physical Studies **25**, 1002 (2021). DOI: 10.30970/jps.25.1002