PREFERENCE STRUCTURES, CHOICE FUNCTIONS AND THEIR LOGICAL FORM

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Abstract

Binary relations and choice functions are two different tools for describing DM preferences on a finite set of alternatives. Normal choice functions are the link between these two tools. These choice functions are easily identified using the logical form. In this paper, we analyse normal choice functions from the viewpoint of their logical form.

Keywords: Binary relation, choice function, logical form.

Resumen

Las relaciones binarias y las funciones de elección constituyen dos herramientas diferentes para describir las preferencias del Decisor en un conjunto finito de alternativas. Las funciones de elección normales representan el puente entre estas dos herramientas. Estas funciones de elección se identifican fácilmente utilizando su forma lógica. En este artículo, analizamos las funciones de elección normales a través de su forma lógica.

Palabras clave: Relación binaria, función de elección, forma lógica.

1   Introduction

Let us suppose a preference modelling problem in which the Decision Maker (DM) has to make a choice from a finite set of alternatives, Ω. Binary relations and choice functions define two different tools for describing DM preferences in such problems.

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A decision-making process based on pairwise comparisons introduces a relative evaluation of every pair of alternatives, modelled by a binary relation, R

$$xRy \text{ if and only if "} x \text{ is at least as good as } y\text{"}.$$

Three binary relations can be defined from this reflexive binary relation: strict preference, indifference and incomparability. These binary relations are known as the Preference Structure associated with R.

In a general choice situation, the DM has to choose from a subset, $X \subseteq \Omega$. This leads to the choice set $C(X)$. The function connecting $(X, C(X))$ is known as a choice function. An obvious way of structuring a choice function is to list the choices made by the DM from each subset of alternatives. However, the logical form provides a shorter form for representing this choice function (Makarov *et al.*, 1987). The logical form and other necessary concepts, will be presented in Section 2.

The Normal Choice Function concept represents a "bridge" between binary relations and choice functions. A normal choice function is a function C that can be represented by a binary relation as follows:

$$C(X) = \{x \in X \ / \ \forall y \in X \ \ xRy\} \quad \forall X \subseteq \Omega$$

Normal choice functions can be easily identified using the logical form.

The purpose of this paper is to explore new aspects of logical forms in a decision-making context. We first give the translation of the logical form to the language of choice functions in the case of a normal choice function. In addition, we establish a result by means of which the logical form of a normal choice function can be built from the binary relation that generating this function, and we analyse the relation between the preference structure associated with a binary relation R and the choice function generated by R. This will be addressed in Section 3.

For many applications, it is important to consider pairwise choice, that is, the values of a choice function on sets of two elements (Herzberger, 1973). In Section 4, we study the choice function when restricted to subsets of one and two elements. Because of the importance of the normal choice function, we introduce the normalizable concept to mean restricted choice functions that can be extended to a normal choice function, and then we establish when this is possible. Finally, some conclusions are given in Section 5.

2   Basic definitions

Let Ω be a finite set of alternatives in a decision-making problem, $\Omega = \{x_1, \ldots, x_n\}$. Every binary relation R on Ω can be represented by a matrix

$$A(R) = (a_{ij}(R))_{i,\,j=1,\ldots,n}$$

where

$$a_{ij}(R) = \begin{cases} 1 & \text{if } (x_i, x_j) \in R \\ 0 & \text{if } (x_i, x_j) \notin R \end{cases} \quad i, j = 1, \ldots, n$$

or a directed graph $G(R) = (\Omega, R)$, where Ω is the set of vertices and R is the set of arcs.

Let R be a reflexive binary relation on Ω, representing the relation "at least as good as". Taking R, let us define the relations P, I and N as follows (Roubens and Vincke, 1985):

- xPy iff xRy and $y\overline{R}x$.

- xIy iff xRy and yRx.

- xNy iff $x\overline{R}y$ and $y\overline{R}x$.

P is called *strict preference relation* and is the subset of pairs (x, y) such that the statement "x is preferred to y" is true.

I is called *indifference relation* and is the subset of pairs (x, y) such that "x is indifferent to y".

N is the *incomparability or noncomparability relation*, and it is formed by the pairs (x, y) such that "x is not comparable to y" (the individual is unable to compare the elements x and y).

Definition 1: *The triplet of relations $\{P, I, N\}$ will be called preference structure.*

Now, we give the formal definition of a choice function.

Definition 2: *A choice function C on Ω is a function $C : \quad \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ such that $\forall X \subset \Omega \ C(X) \subset X$.*

$C(X)$ is the set of alternatives chosen by the DM from X.

An alternative x is termed singular in respect of C if $C(\{x\}) = \emptyset$ and nonsingular if $C(\{x\}) = \{x\}$ (Aizerman, 1985). Furthermore, a choice function C is called reflexive if all the alternatives are nonsingular, i.e., $C(\{x\}) = \{x\} \ \forall x \in \Omega$.

The most straightforward form of representing C is to list the choices made by the DM from each subset of alternatives. However, Boolean algebra can be used to get a shorter form of expressing a choice function: the *logical form*.

Each subset X of Ω is associated with the vector $\beta(X) = (\beta_1(X), \ldots, \beta_n(X))$ of Boolean variables, where

$$\beta_i(X) = \left\{ \begin{array}{ll} 1 & \text{if } x_i \in X \\ 0 & \text{if } x_i \notin X \end{array} \right. \quad \forall i = 1, \ldots, n$$

Clearly, $\beta(\Omega) = (1, \ldots, 1)$ and $\beta(\emptyset) = (0, \ldots, 0)$.

Let C be a choice function defined on Ω.

Definition 3: *The logical form of the choice function C $(LFC(C))$ is the family of Boolean functions[1] $(f_1, \ldots, f_n)$ of $n - 1$ variables, such that*

$$\beta_i(X) \wedge f_i(\beta(X)) = 1 \iff x_i \in C(X)$$

where

$$\begin{array}{lll} f_i(\beta) & = & f_i(\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n) \text{ for } i \neq 1, n \\ f_1(\beta) & = & f_1(\beta_2, \ldots, \beta_n) \\ f_n(\beta) & = & f_n(\beta_1, \ldots, \beta_{n-1}) \end{array}$$

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[1] A Boolean function f is a mapping $f : B_2^m = \underbrace{B_2 \times \ldots \times B_2}_{m} \to B_2$, where $B_2 = \{0, 1\}$.

If $x_i \notin X$, it makes no sense to check whether or not x_i is chosen from the subset X. Hence, this definition is simplified if we assume that implicitly $\beta_i = 1$ in $f_i(\beta)$. Thus, we have that

$$x_i \in C(X) \iff f_i(\beta(X)) = 1.$$

(if $\beta_i = 0$, we know that $x_i \notin C(X)$, because $x_i \notin X$).

A choice function can be associated to every binary relation, however, the inverse is not true; i.e., there are choice functions that cannot be represented by a binary relation.

Normal choice functions represent the link between binary relations and choice functions.

Definition 4: *A choice function is normal if there is a binary relation R such that $C = C_R$, where*

$$C_R(X) = \{x \in X \ / \ \forall y \in X \ \ xRy\} \quad \forall X \subseteq \Omega$$

In this case, we say that the choice function C is generated by the binary relation R.

3   Normal Choice Function

In this section, a normal choice function and its associated preference are connected by the logical form.

The following result connects binary choices with a general one.

Theorem 1: *(Rodríguez-Galiano and González-Pachón, 2001) Let C be a choice function on Ω. C is normal iff*

$$C(X) = \bigcup_{x \in X} \left[\bigcap_{y \in X} C(\{x, y\}) \right] \quad \forall X \subseteq \Omega.$$

This result describes what the choice functions that can be represented by a binary relation are like: they are the functions where the selection criteria are summarized in pairwise comparisons (including the unitary comparisons).

The logical form of a choice function provides an easy method for determining when there is an associated binary relation. The following result is useful for identifying normal choice functions.

Theorem 2: *(Makarov et al., 1987) A choice function C on $\Omega = \{x_1, x_2, \ldots, x_n\}$ is normal iff there exists a partition of the set $J = \{1, \ldots, n\}$ into the subsets I, I_0, I_1 (any one of them might be empty) such that*

1. $\forall i \in I \quad \exists J_i \subset J, \ J_i \neq \varnothing \ \ s.t. \ \ f_i = \bigwedge_{j \in J_i} \overline{\beta}_j.$

2. $\forall i \in I_0 \quad f_i \equiv 0.$

3. $\forall i \in I_1 \quad f_i \equiv 1.$

Thus, a choice function C on Ω is a normal choice function iff the Boolean expression of the f_i's that formed its logical form are 0, 1 or a conjunction of refuted terms, i.e., $f_i \in \{0, 1, \bigwedge_{j \in J_i} \overline{\beta}_j\} \; \forall i$.

The interpretation of the previous theorem is as follows: a choice function is generated by a binary relation iff, given an element $x_i \in \Omega$, one of the following affirmations is true:

1. x_i is chosen whenever none of the elements $x_{i_1}, \ldots, x_{i_k}$ (if $J_i = \{i_1, \ldots, i_k\}$) are in the proposed subset X, i.e., it is chosen if $\{x_{i_1}, \ldots, x_{i_k}\} \bigcap X = \emptyset$.

$$x_i \in C(X) \iff (x_{i_1} \notin X, \; x_{i_2} \notin X, \ldots, \; x_{i_k} \notin X)$$

2. x_i is never chosen, i.e., $x_i \notin C(X) \quad \forall X \subseteq \Omega$.

3. x_i is always chosen, i.e., $x_i \in C(X) \quad \forall X \subseteq \Omega$ with $x_i \in X$.

As mentioned above, a choice function can be represented in a shorter and more manageable manner using the logical rather than explicit form. Given a binary relation R, it would be useful to get the logical form of the choice function it generates, without having to determine this explicitly. Let us see how the choice function can be calculated from the matrix $A(R)$ or, equivalently, from the graph $G(R)$.

Proposition 1: *Let R be a binary relation represented by the matrix $A(R)$ and let $LFC(C_R) = (f_1, f_2, \ldots, f_n)$ be the logical form of the choice function generated by R. Then*

- $f_i = 0 \iff a_{ii} = 0$.

- $f_i = 1 \iff a_{ij} = 1 \;\; \forall j$.

- $f_i = \bigwedge_{j \in J_i} \overline{\beta}_j$ *with* $J_i = \{h \; / \; a_{ih} = 0\} \iff a_{ii} = 1$ *and* $\exists j \neq i$ *s.t.* $a_{ij} = 0$.

Proof:

$f_i = 0 \iff a_{ii} = 0$
  $a_{ii} = 0 \Leftrightarrow x_i \overline{R} x_i \Leftrightarrow x_i \notin C_R(X) \;\; \forall X \subseteq \Omega \Leftrightarrow f_i = 0$.

$f_i = 1 \iff a_{ij} = 1 \;\; \forall j$
  $a_{ij} = 1 \;\; \forall j \Leftrightarrow x_i R x_j \;\; \forall j \Leftrightarrow x_i \in C_R(X) \;\; \forall X \subseteq \Omega \Leftrightarrow f_i = 1$.

$f_i = \bigwedge_{j \in J_i} \overline{\beta}_j$ with $J_i = \{h \; / \; a_{ih} = 0\} \iff a_{ii} = 1$ and $\exists j \neq i$ s.t. $a_{ij} = 0$

  ($\Longrightarrow$) If $f_i = \bigwedge_{j \in J_i} \overline{\beta}_j$, we have that $J_i \neq \emptyset$ and then $\exists k \neq i$ such that $x_i \overline{R} x_k$. Thus, $a_{ik} = 0$.
  $a_{ii} = 1$ since if $a_{ii} = 0$, $f_i = 0$, which is a contradiction.

  ($\Longleftarrow$) $a_{ii} = 1 \Leftrightarrow x_i R x_i$.
  Let $J_i = \{h \; / \; a_{ih} = 0\}$. Since $\exists k \neq i$ s.t. $a_{ik} = 0$, J_i is not empty. Moreover,

$$a_{ij} = 0 \;\; \forall j \in J_i \Leftrightarrow x_i \overline{R} x_j \;\; \forall j \in J_i.$$

Given $X \subset \Omega$, we have that $x_i \in C_R(X) \Leftrightarrow x_j \notin X$ $\forall j \in J_i \Leftrightarrow \beta_j(X) = 0$ $\forall j \in J_i$. Then, taking into account that $x_i \in C_R(X) \Leftrightarrow f_i(\beta(X)) = 1$, we get $f_i = \bigwedge_{j \in J_i} \overline{\beta}_j$. ∎

Obviously, it is enough to observe the ith row of matrix $A(R)$ to get f_i.

Example 1: Let $\Omega = \{x_1, x_2, x_3, x_4\}$ and the binary relation R be defined by the following matrix:

$$A(R) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying the previous proposition, we get that the logical form of the choice function generated by R is:

$$\begin{aligned}
f_1(\beta_2, \beta_3, \beta_4) &= \overline{\beta}_3 \overline{\beta}_4 \\
f_2(\beta_1, \beta_3, \beta_4) &= 1 \\
f_3(\beta_1, \beta_2, \beta_4) &= \overline{\beta}_2 \\
f_4(\beta_1, \beta_2, \beta_3) &= \overline{\beta}_1 \overline{\beta}_2 \overline{\beta}_3
\end{aligned}$$

That is,

$$LFC(C_R) = (\overline{\beta}_3 \overline{\beta}_4,\ 1,\ \overline{\beta}_2,\ \overline{\beta}_1 \overline{\beta}_2 \overline{\beta}_3).$$

□

From both the explicit and logical form of the normal choice function C, we can get the binary relations that form the preference structure.

First, let us see the relation between preference structure and a normal choice function.

Given a reflexive binary relation R, let us see what $C_R(X)$ indicates for each subset X of Ω.

- If $C_R(X) = \emptyset$ with $|X| = 2$, the elements of X are not comparable (then we get the relation N of the preference structure $\{P, I, N\}$ associated with R).

- If $C_R(X) = \emptyset$ with $|X| > 2$, the elements of X are either not comparable or indifferent (there are non-comparable elements, and there could be indifferent elements), or some elements are preferred to others but not to all elements.

- If $C_R(X) = X$ the elements of X are indifferent to each other (thus, we get the relation I).

- If $C_R(X) = \{x_i\}$ with $x_i \in X$, then x_i is the element preferred to all elements of the subset X, i.e., from subset X, we choose the alternative x_i (then we get the relation P).

- If $C_R(X) = X_i$ with $X_i \subset X$ and $1 < |X_i| < |X|$, the elements of X_i are indifferent to each other, but they are preferred to the elements of $X \setminus X_i$.

Thus, we can get the preference structure $\{P, I, N\}$ from the choice function C_R, as follows:

Consider $x, y \in \Omega$, $x \neq y$

- xPy iff $C_R(\{x,y\}) = \{x\}$.

- xIy iff $C_R(\{x,y\}) = \{x,y\}$.

- xNy iff $C_R(\{x,y\}) = \emptyset$.

Moreover, $xIx \;\; \forall x \in \Omega$.

Example 2: Let us consider the binary relation R of the previous example. R generates the choice function shown in Table 1. Then, the preference structure associated with R is

Table 1: C_R

X	$C_R(X)$
x_1	x_1
x_2	x_2
x_3	x_3
x_4	x_4
x_1, x_2	x_1, x_2
x_1, x_3	x_3
x_1, x_4	$\emptyset$
x_2, x_3	x_2
x_2, x_4	x_2
x_3, x_4	x_3
x_1, x_2, x_3	x_2
x_1, x_2, x_4	x_2
x_1, x_3, x_4	x_3
x_2, x_3, x_4	x_2
x_1, x_2, x_3, x_4	x_2

given below.

$$
\begin{aligned}
P &= \{(x_2, x_3), (x_2, x_4), (x_3, x_1), (x_3, x_4)\} \\
I &= \{(x_1, x_2), (x_2, x_1), (x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\} \\
N &= \{(x_1, x_4), (x_4, x_1)\}
\end{aligned}
$$

$\square$

If, instead of the explicit form, we have the logical form of C_R, we get the following result.

Proposition 2: *Consider $\Omega = \{x_1, \ldots, x_n\}$ and let R be a reflexive binary relation on Ω. If $LFC(C_R) = (f_1, \ldots, f_n)$ is the logical form of the choice function generated by R and $\{P, I, N\}$ is the preference structure associated with R, then*
$\forall i, j \in J = \{1, \ldots, n\}, \; i \neq j$

- $x_i P x_j$ *iff* $\exists L_j \subset J$ *with* $i \in L_j$ *s.t.* $f_j = \bigwedge_{l \in L_j} \overline{\beta}_l$ *and* $(f_i = 1$ *or* $\exists L_i \subset J$ *with* $j \notin L_i$

  and $L_i \neq \emptyset$ *s.t.* $f_i = \bigwedge_{l \in L_i} \overline{\beta}_l)$ *(in an abridged form, we write:*

  $f_i \in \{1, \bigwedge_{l \in L_i} \overline{\beta}_l$ *with* $j \notin L_i\}$ *and* $f_j = \bigwedge_{l \in L_j} \overline{\beta}_l$ *with* $i \in L_j).$

- $x_i I x_j$ *iff* $f_i \in \{1, \bigwedge_{l \in L_i} \overline{\beta}_l$ *with* $j \notin L_i\}$ *and* $f_j \in \{1, \bigwedge_{l \in L_j} \overline{\beta}_l$ *with* $i \notin L_j\}$.

- $x_i N x_j$ *iff* $f_i = \bigwedge_{l \in L_i} \overline{\beta}_l$ *with* $j \in L_i$ *and* $f_j = \bigwedge_{l \in L_j} \overline{\beta}_l$ *with* $i \in L_j$.

Proof: Since $R = P \cup I$ is reflexive, $f_i \neq 0$, $\forall i = 1, \ldots, n$ in the logical form of C_R, and so

$$f_i \in \{1, \bigwedge_{j \in J_i} \overline{\beta}_j\}, \ \ \forall i = 1, \ldots, n.$$

$x_i P x_j$ iff $C_R(\{x_i, x_j\}) = \{x_i\}$ iff x_i is chosen from subset $\{x_i, x_j\}$ and x_j is not chosen when x_i is in the proposed subset (see Theorem 1).

Let us see how these last two affirmations are expressed in terms of the logical form:

1. x_i is chosen from subset $\{x_i, x_j\}$
   There are two possible cases.

   (a) x_i is always chosen. This happens if and only if $f_i = 1$.
   (b) x_i is not always chosen, but it is, at least, chosen from subset $\{x_i, x_j\}$ iff
      $\exists L_i \subset J_i$, $L_i \neq \emptyset$ s.t. $j \notin L_i$ and $f_i = \bigwedge_{l \in L_i} \overline{\beta}_l$.

2. x_j is not chosen when x_i is in the proposed subset.
   In this case, $\exists L_j \subset J_i$, $L_j \neq \emptyset$ with $i \in L_j$ s.t. $f_j = \bigwedge_{l \in L_j} \overline{\beta}_l$.

For relations I and N we have,
$x_i I x_j$ iff $C_R(\{x_i, x_j\}) = \{x_i, x_j\}$ iff x_i is chosen from subset $\{x_i, x_j\}$ and x_j is chosen from subset $\{x_i, x_j\}$.

$x_i N x_j$ iff $C_R(\{x_i, x_j\}) = \emptyset$ iff x_i is not chosen when x_j is in the proposed subset and x_j is not chosen when x_i is in the subset in question.

Then, applying the above results for the preference relation P to relations I and N, the proposition is proved. ∎

Example 3: Consider the same choice function C_R as in the previous examples, whose logical form is (see Example 1):

$$LFC(C_R) = (\overline{\beta}_3 \overline{\beta}_4, \ 1, \ \overline{\beta}_2, \ \overline{\beta}_1 \overline{\beta}_2 \overline{\beta}_3).$$

Then,

$$L_1 = \{3, 4\}, \ L_3 = \{2\}, \ L_4 = \{1, 2, 3\}.$$

We get P by observing that $f_2 = 1$ and then searching for $x_j \in \Omega$ such that $x_2 P x_j$, i.e., the elements for which $f_j = \bigwedge_{l \in L_j} \overline{\beta_l}$ with $2 \in L_j$. We have that $2 \in L_3 \cap L_4$, so that $x_2 P x_3$ and $x_2 P x_4$.

For $f_3 = \overline{\beta_2}$, we have that $1, 4 \notin L_3$. Thus, we focus on f_1 and f_4. Since $3 \in L_1 \cap L_4$, we get that $x_3 P x_1$ and $x_3 P x_4$.

For I, we have that $f_2 = 1$ and $2 \notin L_1$, so that $x_2 I x_1$ and $x_1 I x_2$. Furthermore, we know that I is reflexive.

For N, we observe that $f_1 = \overline{\beta_3}\overline{\beta_4}$ and $1 \in L_4$, therefore $x_1 N x_4$ and $x_4 N x_1$.
Thus,

$$
\begin{aligned}
P &= \{(x_2, x_3),\ (x_2, x_4),\ (x_3, x_1),\ (x_3, x_4)\} \\
I &= \{(x_1, x_2),\ (x_2, x_1),\ (x_1, x_1),\ (x_2, x_2),\ (x_3, x_3),\ (x_4, x_4)\} \\
N &= \{(x_1, x_4),\ (x_4, x_1)\}
\end{aligned}
$$

which is the same as we got before (in Example 2).   □

4   Restriction of Choice Functions to Pairwise Comparison

Now, we study choice functions when the DM has to make a choice in subsets of one or two elements.

Let C_{cp} be a choice function known only in Ω subsets, whose cardinal is smaller or equal than 2.

Definition 5: *The restricted choice function C_{cp} is normalizable if there is a binary relation R such that $C_{cp}(X) = C_R(X)$ $\forall X \subseteq \Omega$ with $card(X) = 1, 2$.*

We use the term normalizable, because if $\exists R$ such that $C_{cp}(X) = C_R(X)$ $\forall X \subseteq \Omega$ with $|X| = 1, 2$, we can extend the choice function C_{cp} to the other subsets so that $C_{cp}(X) = C_R(X)$ $\forall X \subseteq \Omega$, getting a normal choice function.

Proposition 3: *C_{cp} is normalizable iff*

$$
C_{cp}(\{x\}) = \varnothing \Rightarrow x \notin C_{cp}(\{x, y\})\ \ \forall y \neq x.
$$

Proof: According to Theorem 1, we have that a choice function is normal iff

$$
C(X) = \bigcup_{x \in X} \bigcap_{y \in X} C(\{x, y\})\ \ \forall X \subseteq \Omega.
$$

In particular, it should be true for the subsets of cardinal smaller or equal than two. Thus, we have that C_{cp} is normalizable iff

$$
C_{cp}(X) = \bigcup_{x \in X} \bigcap_{y \in X} C_{cp}(\{x, y\})\ \ \forall X \subseteq \Omega \text{ with } card(X) = 1, 2.
$$

iff $C_{cp}(\{x,y\}) = \bigcup_{x \in X} \bigcap_{y \in X} C_{cp}(\{x,y\}) = (C_{cp}(\{x\}) \cap C_{cp}(\{x,y\})) \cup (C_{cp}(\{y\}) \cap C_{cp}(\{x,y\}))$.
Therefore, it is clear that

$$C_{cp} \text{ is normalizable iff } C_{cp}(\{x\}) = \emptyset \Rightarrow x \notin C_{cp}(\{x,y\}) \ \forall y \neq x.$$

∎

Thus, if the DM is coherent and he/she rejects an alternative x, the problem would be equivalent to the selection problem in $\Omega \setminus \{x\}$. That is, we can remove singular alternatives from the problem. Nevertheless, we find that this is not always possible, since the presence of such an alternative can influence the choice of other alternatives (Bauman, 1977). Only if the alternative does not affect to the selection process can it be excluded.

Moreover, we can extend the restricted choice function C_{cp} to the normal choice function C, as follows:

$$
\begin{aligned}
C(X) &= C_{cp}(X) \ \forall X \subseteq \Omega \text{ s.t. } |X| \leq 2 \\
C(X) &= \bigcup_{x \in X} \bigcap_{y \in X \setminus \{x\}} C(\{x,y\}) \ \forall X \subseteq \Omega \text{ s.t. } |X| > 2
\end{aligned}
$$

Obviously, the extension is unique.

Example 4: Consider the restricted choice functions represented in Table 2.

Table 2: Restricted functions

X	$C_{cp_1}(X)$	$C_{cp_2}(X)$	$C_{cp_3}(X)$
x_1	x_1	x_1	x_1
x_2	$\emptyset$	x_2	$\emptyset$
x_3	x_3	x_3	x_3
x_4	$\emptyset$	x_4	$\emptyset$
x_1, x_2	x_1, x_2	x_1, x_2	x_1
x_1, x_3	x_1	x_1	x_1
x_1, x_4	x_4	x_4	$\emptyset$
x_2, x_3	x_2	x_2	$\emptyset$
x_2, x_4	x_4	x_4	$\emptyset$
x_3, x_4	x_3	x_3	x_3

C_{cp_1} is not normalizable, since $C_{cp_1}(\{x_2\}) = C_{cp_1}(\{x_4\}) = \emptyset$ and

$$
\begin{aligned}
x_2 &\in C(\{x_2, y\}), & y = x_1, \ x_3 \\
x_4 &\in C(\{x_4, y\}), & y = x_1, \ x_2
\end{aligned}
$$

However, C_{cp_2} and C_{cp_3} are normalizable, and their extensions are the choice functions C_2 and C_3 respectively, shown in Table 3.

□

Table 3: C_2 and C_3

X	$C_2(X)$	$C_3(X)$
x_1	x_1	x_1
x_2	x_2	$\emptyset$
x_3	x_3	x_3
x_4	x_4	$\emptyset$
x_1, x_2	x_1, x_2	x_1
x_1, x_3	x_1	x_1
x_1, x_4	x_4	$\emptyset$
x_2, x_3	x_2	$\emptyset$
x_2, x_4	x_4	$\emptyset$
x_3, x_4	x_3	x_3
x_1, x_2, x_3	x_1, x_2	x_1
x_1, x_2, x_4	x_4	$\emptyset$
x_1, x_3, x_4	$\emptyset$	$\emptyset$
x_2, x_3, x_4	$\emptyset$	$\emptyset$
x_1, x_2, x_3, x_4	$\emptyset$	$\emptyset$

5   Conclusions

The logical form of a choice function is a good tool for representing a choice function in a shorter way. In this paper, we have shown how the logical form of a normal choice function can be obtained directly from the associated binary relation. Moreover, we have shown how the preference structure related to the binary relation related to a choice function can be obtained from the logical form.

If we consider only subsets of cardinal less than or equal to two, it is not always possible to represent the restricted choice function using a binary relation. Provided that the choice is made according to a normal choice function, the problem would be equivalent to the selection problem in the subset of nonsingular alternatives.

Acknowledgement

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