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ON POLYNOMIAL RATES OF DECAY AND GROWTH OF SOLUTIONS TO NONLINEAR DIFFERENCE EQUATIONS

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Abstract

We discuss stability and growth of the solutions of an autonomous scalar difference equations. We determine the exact rates of decay and growth of the solutions when autonomous difference equation has a polynomial nonlinearity.

Keywords: non-linear difference equation, stability, rates of decay and growth.

Resumen

Se discute sobre la estabilidad y crecimiento de las soluciones de ecuaciones autónomas escalares en diferencias. Determinamos las tasas exactas de decaimiento y crecimiento de las soluciones cuando un ecuación autónoma en diferencias tiene una no linealidad polinomial

Palabras clave: ecuaciones en diferencias no lineales, estabilidad, tasa de decaimiento y crecimiento.

Mathematics Subject Classification: 39A11.

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1 Introduction

Due to the numerous applications the problems of stability for difference equations have been extensively studied in recent years. Appropriate references can be found *e.g.* in [1, 2, 3, 4] .

In this paper we study convergence to the equilibrium or to infinity for non-linear difference equations. This equilibrium can be taken to be zero, without loss of generality. In this difference equation, linearization of the equation close to the equilibrium does not determine the asymptotic behavior, because the terms which depend on the state of the system are $o(x)$ as $x \to 0$. As a consequence of the fact that the equation admits a trivial linearization at the equilibrium, which can yield only a weak restoring force towards the equilibrium, it must be suspected that the convergence of the difference equation to its equilibrium cannot take place at an exponentially fast rate. For autonomous difference equation with polynomial nonlinearity at the origin we obtain exact decay rate.

We also study convergence of solutions of non-linear difference equations to infinity. When the terms which depend on the state is $o(x)$ as $x \to \infty$ and decreases polynomially, we obtain exact rate of growth of the solution.

The paper is organized as follows: in Section 2 we discuss stability and infinite growth of the solution of nonlinear difference equation. Section 3, which is subdivided into two subsections, is devoted to the exact polynomial rates of decay and growth of the solution. In Section 4 we present examples and show some numerical simulations which illustrate our results.

2 Stability and infinite growth

We consider equation

$$x_{n+1} = x_n(1 - f(x_n)), \quad n = 0, 1, \dots, \tag{1}$$

with arbitrary initial condition $x_0 \in \mathbb{R}$ and continuous function $f : \mathbb{R} \to \mathbb{R}$.

Theorem 1. *Let x_n be a solution to equation (1).*

- *(a) If there is any $u^* \in \mathbb{R}$ such that $0 \le f(u) < 2$ when $|u| < |u^*|$, and $f(u) = 0 \Rightarrow u = 0$, then $\lim_{n\to\infty} x_n = 0$ for any initial condition x_0 such that $|x_0| < |u^*|$. Furthermore, if $0 \le f(u) \le 1$ $\forall u \in (-u^*, u^*)$, then the solution does not change sign, while if $1 < f(u) < 2$ $\forall u \in (-|u^*|, |u^*|)$, then the solution oscillates.*

- *(b) If there is any $u^* \in \mathbb{R}$ such that i) $f(u) > 2$ $\forall u \in (-|u^*|, \infty) \cap (|u^*|, \infty)$, or ii) $f(u) < 0$ $\forall u \in (-|u^*|, \infty) \cap (|u^*|, \infty)$, then $\lim_{n\to\infty} x_n^2 = \infty$ for any initial condition x_0, $|x_0| \ge |u^*|$ and the solution oscillates in case i) and does not oscillate in case ii).*

Proof. We are going to prove just part of item (a), namely that $\lim_{n\to\infty} x_n = 0$. The rest of the proof is quite straightforward, and can be obtained by applying Theorems 1.12 (see [1], page 22) and Theorem 7.9 (see [1], page 302, or [3]).

We square both parts of equation (1) and get

$$x_{n+1}^2 = x_n^2(1 - 2f(x_n) + f^2(x_n)). \tag{2}$$

We set

$$F(v) = 2f(v) - f^2(v), \quad v \in \mathbb{R}, \tag{3}$$

and write equation (2) as

$$x_{n+1}^2 = x_n^2(1 - F(x_n)), \quad n = 0, 1, \dots. \tag{4}$$

For every $n \in \mathbf{N}$ we have

$$x_{n+1}^2 = x_0^2 - \sum_{i=0}^{n} x_i^2 F(x_i) \le x_0^2. \tag{5}$$

Suppose that there is a number $c_0 > 0$ and a sequence $\{n_k\}$ such that $x_{n_k}^2 > c_0$. We define

$$K(n) = \text{number of members of sequence } \{n_k\} \le n,$$

and note that $K(n) \to \infty$ when $n \to \infty$. Since by (5) $x_{n_k}^2 \le x_0^2$, the hypotheses of part (a) imply that there are numbers $b_0 > 0$ and $e_0 > 0$ such that $f(x_{n_k}) \ge b_0$ and $2 - f(x_{n_k}) \ge e_0$. Therefore

$$\sum_{i=0}^{n} x_i^2 F(x_i) \ge \sum_{i: n_i \le n} x_{n_i}^2 F(x_{n_i}) \ge e_0 b_0 c_0 \sum_{i: n_i \le n} 1,$$

which implies that $x_{n+1}^2 \le x_0^2 - e_0 c_0 b_0 K(n) \to -\infty$, when $n \to \infty$. The contradiction obtained proves that $\lim_{n\to\infty} x_n = 0$. □

3 Rates of decay and growth

To prove results on exact rates of decay and growth of the solutions we make use of the following lemma (see *e.g.* [5], page 390).

Lemma 1 (Toeplitz Lemma). *Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of nonnegative real numbers and b_n be a sequence of partial sums of a_n: $b_n = \sum_{i=1}^{n} a_i$. Let $(\kappa_n)_{n\in\mathbf{N}}$ be a sequence convergent to κ_∞ as $n \to +\infty$. If $b_n \to \infty$, then*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=0}^{n} a_i \kappa_i = \kappa_\infty.$$

In the following subsections we take the convergence of $x_n \to 0$ (or $x_n \to \infty$) as $n \to \infty$ as a hypothesis. We do this so that we may concentrate on the rate of decay or growth of solutions to zero (*or* to infinity) *when convergence takes place*, and to separate hypotheses which simply ensure convergence, from those which determine the convergence.

3.1 Polynomial rate of decay

In this subsection we suppose that f has polynomial behavior in an open neighborhood of zero, in the sense that there is some $\mu > 1$ and $c > 0$ such that

$$\lim_{u\to 0} \frac{f(u)}{|u|^{\mu-1}} = c. \tag{6}$$

Theorem 2. *Let condition (6) hold. Let x be a solution to equation (1) such that $\lim_{n\to\infty} x_n = 0$. Then, there either exists an $\bar{n} \in \mathbf{N}$ such that $x_n = 0$ for all $n \geq \bar{n}$, or*

$$\lim_{n\to\infty} n^{\frac{1}{\mu-1}}|x_n| = [c(\mu-1)]^{-\frac{1}{\mu-1}}. \tag{7}$$

Proof. If $x_{\bar{n}} = 0$ for some $\bar{n} \in \mathbf{N}$, then $x_n = 0$ for all $n \geq \bar{n}$. Suppose on the other hand that $x_n \neq 0$ for all $n \in \mathbf{N}$. We define

$$G(u) = u^{\frac{1-\mu}{2}}, \quad u > 0,$$

so that

$$G'(u) = \frac{1-\mu}{2}u^{-\frac{1+\mu}{2}}, \quad G''(u) = \frac{\mu^2-1}{4}u^{-\frac{3+\mu}{2}}, \quad u > 0.$$

Let $y_n = G(x_n)$: then, by taking a second–order Taylor expansion, we obtain

$$\begin{aligned} y_{n+1} &= G(x_{n+1}^2) = G(x_n^2 - x_n^2F(x_n)) \\ &= G(x_n^2) + G'(x_n^2)(-x_n^2F(x_n)) + \frac{1}{2}G''(\eta_n)x_n^4F^2(x_n), \end{aligned} \tag{8}$$

where F is defined by (3) and there is an η_n such that

$$|\eta_n - x_n^2| \leq x_n^2|F(x_n)|.$$

Substituting values for derivatives G' and G'' in (8) we arrive at

$$y_{n+1} = y_n + \frac{\mu-1}{2}\left[x_n^2\right]^{-\frac{1+\mu}{2}+1}F(x_n) + \frac{\mu^2-1}{8}\eta_n^{-\frac{3+\mu}{2}}x_n^4F^2(x_n). \tag{9}$$

Since, as $n \to \infty$,

$$\frac{F(x_n)}{|x_n|^{\mu-1}} = \frac{2f(x_n) - f^2(x_n)}{|x_n|^{\mu-1}} \to 2c, \quad \text{and} \quad |x_n|^{\mu-1} \to 0,$$

we obtain

$$\begin{aligned} &\lim_{n\to\infty}\left[x_n^2\right]^{-\frac{1+\mu}{2}+1}F(x_n) = \lim_{n\to\infty}\frac{F(x_n)}{|x_n|^{\mu-1}} = 2c, \\ &\limsup_{n\to\infty}\left|\frac{\eta_n}{x_n^2} - 1\right| \leq \limsup_{n\to\infty}\frac{|F(x_n)|}{|x_n|^{\mu-1}} \times |x_n|^{\mu-1} = 0, \\ &\lim_{n\to\infty}\eta_n^{-\frac{3+\mu}{2}}x_n^4F^2(x_n) = \lim_{n\to\infty}\left(\frac{F(x_n)}{|x_n|^{\mu-1}}\right)^2\left(\frac{x_n}{\eta_n}\right)^{\frac{3+\mu}{2}}x_n^{\frac{3\mu+1}{2}} = 0. \end{aligned}$$

Therefore (9) implies that

$$y_{n+1} - y_n \to (\mu - 1)c, \quad \text{as} \quad n \to \infty. \tag{10}$$

By summation and dividing by n, we get

$$\frac{y_n}{n} = \frac{y_0}{n} + \frac{1}{n}\sum_{i=0}^{n-1}[y_{i+1} - y_i]$$

and so by using (10), and applying Lemma 1, we conclude that

$$y_n/n \to (\mu - 1)c, \quad \text{as} \quad n \to \infty.$$

Substituting $G(x_n^2) = y_n$, we obtain $|x_n|^{1-\mu}/n \to (\mu - 1)c$ as $n \to \infty$, which rearranges to give (7). ∎

3.2 Polynomial rate of growth

In this section we suppose that f has a polynomial behavior at infinity: there is some $\beta > -1$ and $b > 0$ such that

$$\lim_{|u|\to\infty} f(u)|u|^{\beta+1} = -b. \tag{11}$$

Theorem 3. *Let condition (11) hold. Let x be a solution to equation (1) and $\lim_{n\to\infty} |x_n| = \infty$. Then*

$$\lim_{n\to\infty} n^{-\frac{1}{\beta+1}}|x_n| = (b(\beta+1))^{\frac{1}{\beta+1}}.$$

Proof. Since $|x_n| \to \infty$ as $n \to \infty$, there exists $n^* \in \mathbf{N}$ such that $|x_n| > 1$ for all $n \geq n^*$, and therefore $x_n \neq 0$ for all $n \geq n^*$. Now, square both parts of equation (1) and define F by (3). We let $G(u) = u^{\frac{\beta+1}{2}}, \quad u > 0$. For $y_n = G(x_n^2) = |x_n|^{\beta+1}, \quad n \geq n^*$, we obtain recursion (8), which takes the form:

$$y_{n+1} = y_n - \frac{\beta+1}{2}|x_n|^{\beta+1}F(x_n) + \frac{(\beta^2-1)}{8}\eta_n^{\frac{\beta-3}{2}}x_n^4F^2(x_n), \quad |\eta_n - x_n^2| \leq x_n^2|F(x_n)|. \tag{12}$$

From (11) we conclude that $\lim_{|u|\to\infty} f(u) = 0$. Therefore, as $n \to \infty$,

$$\lim_{n\to\infty} F(x_n)|x_n|^{\beta+1} = \lim_{n\to\infty}\big(2f(x_n) - f^2(x_n)\big)|x_n|^{\beta+1} = -2b,$$
$$\limsup_{n\to\infty}\left|\frac{\eta_n}{x_n^2} - 1\right| \leq \limsup_{n\to\infty}|F(x_n)| = 0,$$
$$\lim_{n\to\infty}\eta_n^{\frac{\beta-3}{2}}x_n^4F^2(x_n) = \lim_{n\to\infty}\left(\frac{\eta_n}{x_n^2}\right)^{\frac{\beta-3}{2}} \times \Big(|x_n|^{\beta+1}F(x_n)\Big)^2 \times |x_n|^{-(\beta+1)} = 0,$$

which together with (12) implies that $y_{n+1} - y_n \to (\beta+1)b$, as $n \to \infty$. Now we complete the proof in the same way as in Theorem 2. ∎

4 Examples and simulation

In this section we present some simple examples which illustrate Theorems 1-3. In each of Figures 1(a)-5(a), 5(b) below, we plot x_n versus n. In each of Figures 1(b), 3(b), we try to determine whether the theoretical asymptotic rate of decay given by Theorem 2 is exhibited by simulations. Similarly, in Figures 2(b), 4(b), we try to determine whether the theoretical asymptotic rate of growth given by Theorem 3 is exhibited by simulations. In Figures 5(a)-5(b) we present the simulations of solutions, when the equation is perturbed by random noise.

Example 1. For the equation

$$x_{n+1} = x_n(1 - x_n^2), \quad n = 1, 2, \dots,$$

we apply Theorem 1,(a). We simulate the solution with the initial value $x_0 = 1.41$ (see Figure 1(a)). Figure 1(b) suggests that $\sqrt{n}x_n \to -\frac{1}{\sqrt{2}}$, when $n \to \infty$, which is consistent with Theorem 2. The dashed horizontal line is $\frac{1}{\sqrt{2}} \approx 0.707$ units below the horizontal axis.

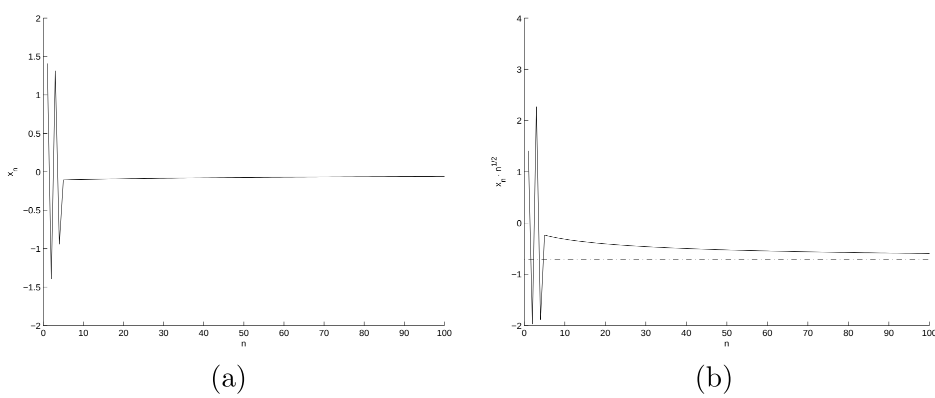


Figure 1: Example 1.

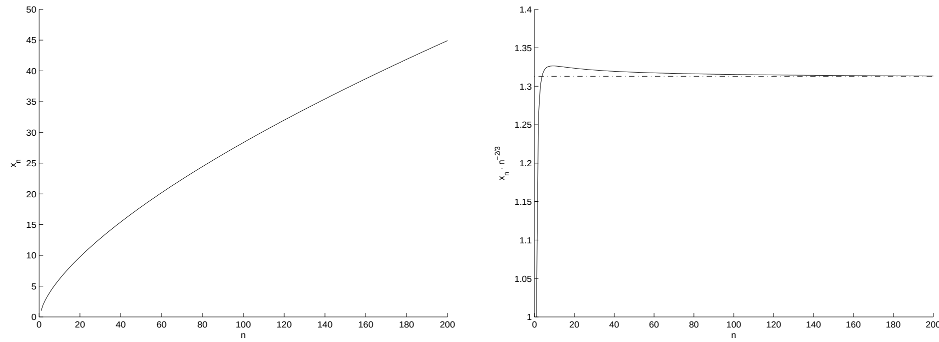
Example 2. For the equation

$$x_{n+1} = x_n + x_n^{-1/2}, \quad n = 1, 2, \dots,$$

we apply Theorem 1,(b). We simulate solution with the initial value $x_0 = 1$ (see Figure 2(a)). Figure 2(b) suggests that $n^{-2/3}x_n \to \left(\frac{3}{2}\right)^{\frac{2}{3}} \approx 1.31$, when $n \to \infty$ as predicted by Theorem 2. The dashed horizontal line is $\left(\frac{3}{2}\right)^{\frac{2}{3}} \approx 1.31$ units above the horizontal axis.

Example 3. For the equation

$$x_{n+1} = x_n(1 - f(x_n)), \tag{13}$$



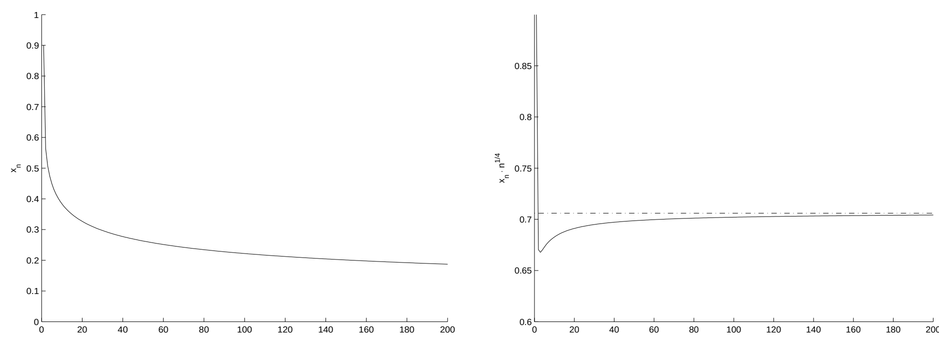
(a) (b)

Figure 2: Example 2.

with

$$f(u) = \begin{cases} u^4 - u^{12}, & u \in (0, 2^{\frac{1}{8}}), \\ -\frac{2}{u^4}, & 2^{\frac{1}{8}} \le u, \end{cases} \tag{14}$$

$f(-u) = f(u)$, we apply Theorems 1-3. When $x_0 = 0.9$ the solution tends to zero and $n^{\frac{1}{4}} x_n \to 4^{-1/4} \approx 0.71$ (see Figures 3(a)-3(b)), while when $x_0 = 1.1$ the solution grows to infinity and $n^{-\frac{1}{4}} x_n \to 8^{1/4} \approx 1.68$ (see Figures 4(a)-4(b)). The dashed horizontal lines on Figures 3(b) and 4(b) are respectively $4^{-1/4} \approx 0.71$ units and $8^{1/4} \approx 1.68$ above the horizontal axis.



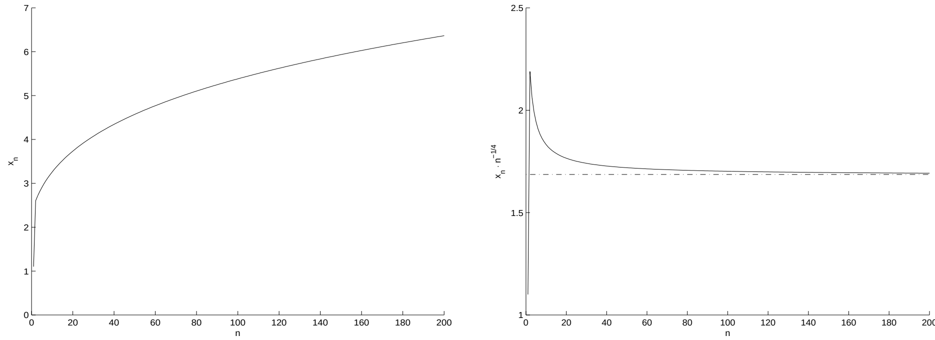
(a) (b)

Figure 3: Example 3, $x_0 = 0.9$.

Example 4. Consider the equation

$$x_{n+1} = x_n(1 - f(x_n) + h\xi_{n+1}), \quad n = 1, 2, \dots, \tag{15}$$

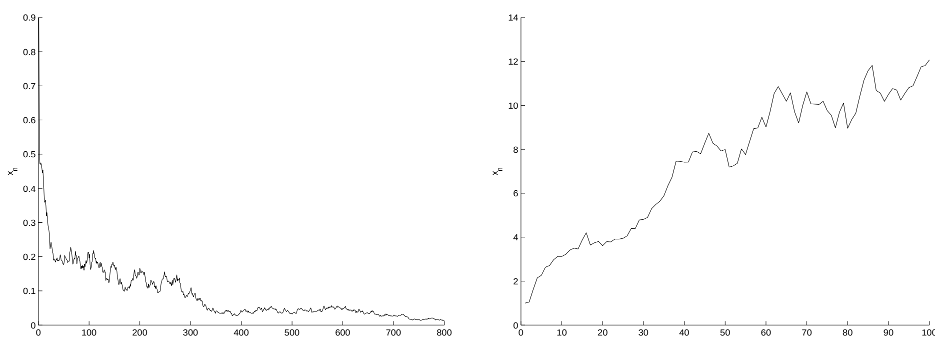
where f is defined by (14), $h = 0.6$ and (ξ_n) are independent and identically distributed random variables with zero mean and unit variance. We simulate solutions with $x_0 = 0.9$ and $x_0 = 0.9999$. Despite the fact that the noise intensity h is the same, the behavior of



(a) (b)

Figure 4: Example 3, $x_0 = 1.1$.

solutions is different: x_n tends to 0 in the first case and x_n has tendency to grow in the second case (see Figures 5(a)-(b)). We note that the original deterministic equation (13) (with f defined by (14)) possesses only a local stability property: by Theorem 1, solution x_n of (13) tends to zero if $|x_0| < 1$ and $|x_n|$ tends to infinity if $|x_0| > 1$. Figures 5(a)-5(b) demonstrate that when stochastic disturbances are present the situation is different: the solution of (15) changes its behavior when the initial value is too close to unity. To the best of our knowledge this type of result for local stability of stochastic difference equation is not yet known. Based upon the consistency exhibited between theoretical predictions and simulations presented here, we anticipate that for this open question, such simulations not only help to determine the admissible level of noise, but also will provide useful guidance in forming conjectures concerning the asymptotic behavior.



(a) (b)

Figure 5: Example 4; (a) $x_0 = 0.9, h = 0.6$; (b) $x_0 = 0.9999, h = 0.6$.

All simulations above were done with programs Mathcad and Matlab. The code was implemented in C programming language. To generate random noise we used “rnorm” generator of pseudorandom numbers from Matcad, which generated numbers with zero mean and unit variance.

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