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THE PLANIMETER

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Abstract

We show how the planimeter is used for the estimation area. For example, it could be used for measuring the area of trees' leaves, the size of a city or the area of a planar section of a human's organ cut.

Keywords: Planimeter, area, Green's theorem.

Resumen

Suponga que usted tiene un mapa del lago Arenal y desea medir su superficie. Si usted tiene la información que brinda la escala del mapa, podría usar un magnífico aparato llamado planímetro para determinar el área del lago.

El planímetro es un aparato de ingeniería usado para medir el área de una región plana acotada por una curva cerrada. Por ejemplo, se podría usar un planímetro para medir el área de hojas de árboles, el tamaño de una ciudad o el área de una sección planar del corte de un órgano humano.

En este artículo mostraremos como funciona un planímetro. Utilizaremos el Teorema de Green para explicar la teoría matemática detrás del concepto del planímetro.

Palabras clave: Planímetro, área, teorema de Green.

Mathematics Subject Classification: 51M25, 26B20.

1 Introduction

Suppose you had a map of Lake Arenal in Costa Rica and wanted to measure its area. If you knew the scale of the map, then the amazing planimeter could be used to measure the area.

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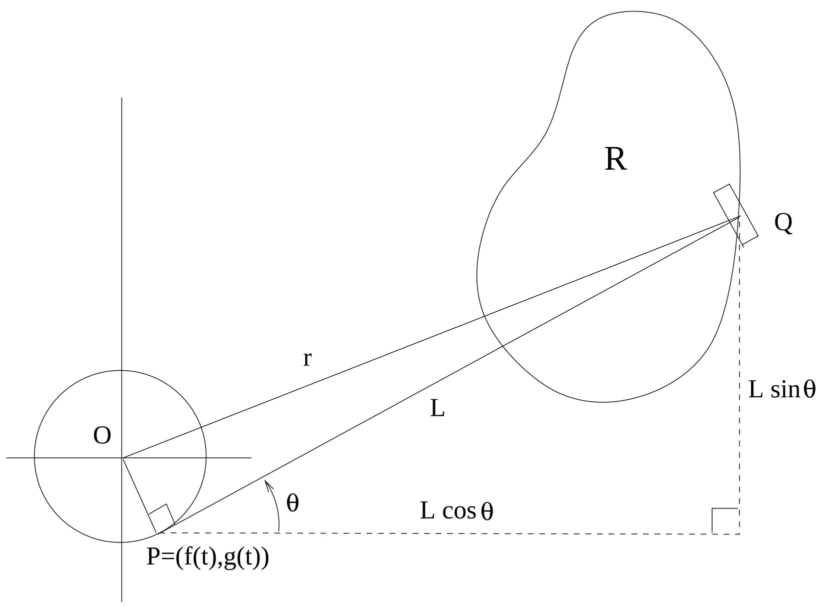
The planimeter is an engineering device for measuring the area of a planar region bounded by a closed curve. For example, you could use a planimeter to measure the area of leaves, the size of cities, or the cross section of a human organ.

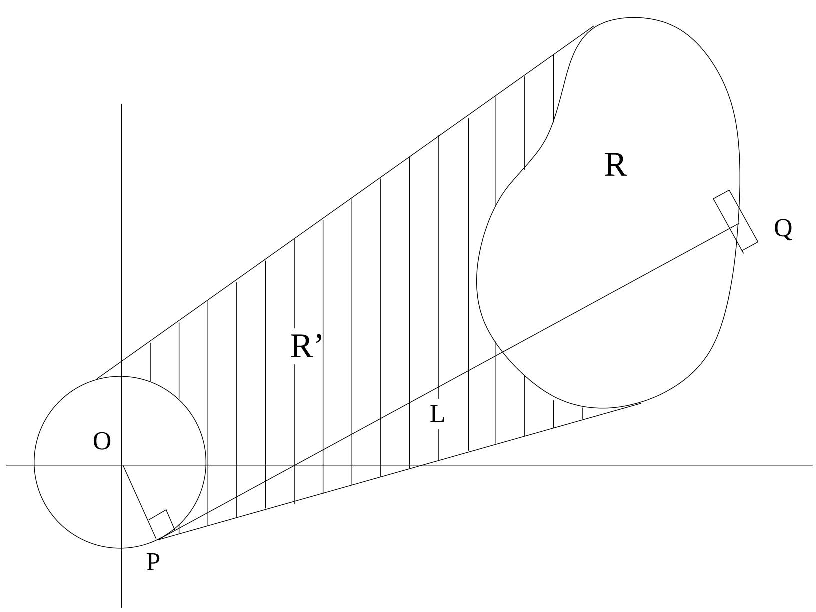
The planimeter is fixed at the origin O (but free to pivot) and has a hinge at P, as indicated in the figure below. The end of the tracer arm PQ moves counterclockwise around the region R. A small wheel at Q is perpendicular to PQ and a scale is marked on it to measure how much it rolls while Q traces out the boundary of the region R.



We will show in this paper that the area of R is given by the length L of the tracer arm PQ times the distance D that the wheel rolls. We will first show this using elementary arguments, and then using Green's Theorem from multivariable calculus.

2 Geometrical Analysis

Consider the planimeter in the figure below, where PQ has length L. Let D be the distance the wheel rotates as Q moves around the region R with area A.

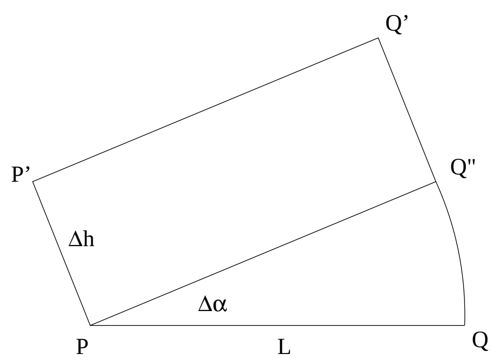


Assume that OP and PQ never rotate more than $360°$. That is, they never rotate more than a full circle. In this section we will use elementary arguments to show that the area A of the region is given by

$$A = LD.$$

As Q traces out the boundary of the region, the entire area that the arm sweeps out equals A, because the region R' is traced out twice, once in each direction.

Imagine that the arm PQ moves a short distance to position $P'Q'$, as indicated in the figure below



This movement can be decomposed into a small rotation from PQ to PQ'' through an angle $\Delta\alpha$, followed by a translation of distance Δh from PQ'' to $P'Q'$.

Let ΔD be the distance the wheel turns and ΔA the indicated area. Then

$$\Delta D = L\Delta\alpha + \Delta h.$$

Thus we have

$$\begin{aligned} \Delta A &= \frac{1}{2}L^2\Delta\alpha + L\Delta h && (1)\\ &= \frac{1}{2}L^2\Delta\alpha + (\Delta D - L\Delta\alpha)L && (2)\\ &= \Delta DL - \frac{1}{2}L^2\Delta\alpha && (3) \end{aligned}$$

Now add up all these small movements. Since PQ begins and ends at the same position, the total change in the angle α is 0. Thus, the total area is

$$A = LD.$$

3 Green's theorem argument

We now use Green's Theorem to show how the planimeter works. Place the fixed point O at the origin and let $(f(t), g(t)$ be the coordinates of the point P, as indicated in the first figure. As before, the length of the tracing arm PQ is L. Assume that Q traces out the entire boundary of the region R as $a \le t \le b$.

Let $\theta(t)$ be the angle that the arm PQ makes with the horizontal line at P. If $\mathbf{r}(t)$ is the position vector OQ, then

$$\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j} = [f(t) + L\cos\theta(t)]\ \mathbf{i} + [g(t) + L\sin\theta(t)]\ \mathbf{j},$$

where $\mathbf{r}(a) = \mathbf{r}(b)$. Green's Theorem implies that the area A of the region R is given by

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

where C is the entire boundary of the region R. We now evaluate this line integral and show that it equals LD. Differentiating the equations for $x(t)$ and $y(t)$ gives

$$dx = [f'(t) - L\sin\theta(t)\,\theta'(t)]\,dt$$

$$dy = [g'(t) + L\cos\theta(t)\,\theta'(t)]\,dt$$

$$\begin{aligned}
A &= \frac{1}{2}\int_a^b \Big([f + L\cos\theta]\left[g' + L\cos\theta\ \theta'\right] - [g + L\sin\theta]\left[f' - L\sin\theta\ \theta'\right]\Big)\,dt && (4)\\
&= \frac{1}{2}\int_a^b \left[fg' - gf'\right]\ dt && (5)\\
&\quad + \frac{1}{2}L\int_a^b \frac{d\theta}{dt}\,dt && (6)\\
&\quad + \frac{1}{2}L\int_a^b \left[f\cos\theta\ \theta' + g\sin\theta\ \theta'\right]\ dt && (7)\\
&\quad + \frac{1}{2}L\int_a^b \left[g'\ \cos\theta - f'\ \sin\theta\right]\ dt. && (8)
\end{aligned}$$

We now evaluate each of these four integrals. The first integral is 0 because the point P moves around a circle of radius OP, but never encloses any area.

The second integral also equals 0:

$$\frac{1}{2}L\int_a^b \frac{d\theta}{dt}\,dt = \frac{1}{2}L\ \theta|_a^b = \frac{1}{2}L[\theta(b) - \theta(a)] = 0.$$

The third integral equals the fourth integral. To see this, first note that

$$\begin{aligned}
\int_a^b [f\sin\theta - g\cos\theta]'\,dt &= [f(b)\sin\theta(b) - g(b)\cos\theta(b)] - [f(a)\sin\theta(a) - g(a)\cos\theta(a)]\\
&= [f(b)\sin\theta(b) - f(a)\sin\theta(a)] - [g(b)\cos\theta(b) - g(a)\cos\theta(a)]\\
&= 0 - 0 = 0.
\end{aligned}$$

Hence,

$$\int_a^b [f'\sin\theta + f\cos\theta\ \theta' - g'\cos\theta + g\sin\theta\ \theta']\,dt = 0$$

which implies that the third and fourth integrals are equal.

$$\int_a^b \left[f \cos\theta \ \theta' + g \sin\theta \ \theta'\right] \ dt = \int_a^b \left[g' \ \cos\theta - f' \ \sin\theta\right] \ dt.$$

So we now have

$$A = L \int_a^b \left[g' \ \cos\theta - f' \ \sin\theta\right] \ dt.$$

Let us now turn to the question of how much the wheel rotates as the point Q moves around the region R. The position of the wheel is given by

$$r(t) = x(t)\, i + y(t)\, j = [f(t) + L\cos\theta(t)]\, i + [g(t) + L\sin\theta(t)]\, j.$$

The wheel rotates in the direction of the unit normal vector n,

$$n = \cos(\theta + \pi/2)\, i + \sin(\theta + \pi/2)\, j = -\sin\theta\, i + \cos\theta\, j.$$

As Q moves a small distance dr,

$$dr = [(f' - L\sin\theta\,\theta')\, i + (g' + L\cos\theta\,\theta')\, j]\, dt.$$

Hence,

$$\begin{aligned} n \cdot dr &= [-f'\sin\theta + L\sin^2\theta\,\theta' + g'\cos\theta + L\cos^2\theta\,\theta']\, dt \\ &= [g'\cos\theta - f'\sin\theta + L\theta']\, dt. \end{aligned}$$

The distance rolled D is given by

$$\begin{aligned} D = \int_C n \cdot T\, ds &= \int_a^b n \cdot dr \\ &= \int_a^b (g'\cos\theta - f'\sin\theta)\, dt + L\int_a^b \theta'\, dt \\ &= \int_a^b (g'\cos\theta - f'\sin\theta)\, dt. \end{aligned}$$

So, A is equal to the product of L times the amount the wheel rolls,

$$A = LD.$$

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