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# Iterative valuation process in the method of the two beta distributions

J. García Pérez[1], S. Cruz Rambaud[2]* and C. B. García García[3]

[1] *Departamento de Economía Aplicada. Facultad de Ciencias Económicas y Empresariales. Universidad de Almería. La Cañada de San Urbano, s/n. 04120 Almería. Spain*

[2] *Departamento de Dirección y Gestión de Empresas. Facultad de Ciencias Económicas y Empresariales. Universidad de Almería. La Cañada de San Urbano, s/n. 04120 Almería. Spain*

[3] *KPMG Auditores. Paseo de la Castellana, 95. 28046 Madrid. Spain*

## Abstract

In the literature on PERT methodology, four subfamilies of beta distributions have appeared: classical, of constant variance, mesokurtic and Caballer. To date, these four subfamilies have been used independently to resolve economic valuation problems. The only differences between using one or another lie in the means or variances obtained by each. For example, following a criterion of prudence the maximum variance is required, and for a riskier criterion the minimum variance is preferred. With respect to the mean, we are interested in the one closest to the centre of the interval, i.e. the model that provides a more centered expected value and hence more moderate estimations. This work focuses on the field of valuation, more specifically on the valuation method of the two distribution functions (recommended when there are limited data). The aim of this work was to develop an iterative process that uses the four families of beta distributions simultaneously with the objective of using all the information provided by each of them. The practical application of this process can conclude either with an interval of possible values or a precise valuation. Then the concepts of stability and convergence of the valuation process appear.

**Key words:** classical beta, constant variance, mesokurtic, Caballer, confidence interval.

## Resumen

### Proceso iterativo de valoración en el método de las dos distribuciones beta

En la literatura sobre la metodología PERT han aparecido cuatro subfamilias de distribuciones beta: clásica, de varianza constante, mesocúrtica y Caballer. Hasta ahora, estas cuatro subfamilias han sido utilizadas independientemente en la resolución de problemas de valoración económica, discriminándose su uso sólo en función de las medias y las varianzas obtenidas en cada uno de ellas. En efecto, podemos utilizar un criterio de prudencia en cuyo caso consideraremos la varianza máxima, o una posición más arriesgada en cuyo caso preferiremos la varianza mínima. Con respecto a la media, nos interesa la más cercana al centro del intervalo, es decir, el modelo que proporciona un valor esperado más centrado y por tanto más moderado en sus estimaciones. Este trabajo se enmarca en el ámbito de la valoración, concretamente en el método de valoración de las dos funciones de distribución (aconsejable cuando se dispone de pocos datos). El objetivo de este trabajo fue desarrollar un proceso iterativo que emplee simultáneamente las cuatro familias de betas con el propósito de utilizar toda la información proporcionada por cada una de ellas. En la aplicación del proceso, que puede concluir con una valoración por intervalo o una valoración puntual, aparecen de forma natural los conceptos de estabilidad y convergencia del proceso de valoración.

**Palabras clave:** beta clásica, varianza constante, mesocúrtica, Caballer, intervalo de confianza.

## Introduction

Not only in the field of valuation in general, but also in a more specific field such as that of agricultural valuation, the two distribution functions method is well-known. This was an original idea of Ballestero (1971, 1973) and was later developed by Caballer (1993) and Romero (1977), who contributed some practical applications, and is described in several textbooks: Ballestero (1991a, 1991b) and Caballer (1994). Moreover, it has been studied in several doctoral theses such

* Corresponding author: scruz@ual.es

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as Lozano's (1996) and Herrerías' (2002). In the last few years, a series of papers have been published which present new distributions (García *et al.*, 1999a, b; Herrerías *et al.*, 1999, 2001), or where the method is expanded to the multi-index or multivariate case (García *et al.*, 2000, 2002b).

Nowadays, some works currently under review attempt to introduce statistical tests for the representativeness of the indices and the adaptation of the distributions that model them (García *et al.*, 2003b). All these works and others included in the references of these papers attempt to introduce mathematical models in the discipline of valuation. This is illustrated by Segura *et al.* (1998) according to whom: «Research in the field of valuation should not only pay attention to a rigorous formulation of the concepts and methods of the real state valuation and their extension to other fields, but incorporate to the theoretical body of the discipline the use of mathematical models, introducing in the valuation a methodology that has been shown productive in other fields.»

We can, therefore, ask ourselves if the valuation process of a building or a property (in general, an asset) must conclude with obtaining a specific value or with the establishment of an interval where the value we try to obtain lies with a certain probability. Therefore, should we value in a punctual way or should we value by interval? For example, in the field of fiscal valuation whose final objective, according to Segura *et al.* (1998, *op. cit.*), is «the verification of the value declared by the liable person since this value will be the taxable base in certain taxes», if the valuation fixed by the Treasury is a punctual valuation, the probability that the value declared by the taxpayer coincides with the one reached by the Treasury is zero, if we consider that the domain of possible values is the set of real numbers. The fact is that the Treasury only issues a parallel liquidation when the deviation between the declared value and the checked one exceeds a certain percentage (20% in many cases), but we wonder if it would not be more convenient to obtain a confidence interval for the value of the good and then to check if the one declared by the taxpayer lies in this interval. The problem of verifying values of an agricultural nature has been discussed by García (1995) and Olmeda (1977, 1978).

The method of the two distribution functions is appropriate for works of massive valuation (Lozano, 1996), with limited information, motivated by the behaviour of certain objective indexes that guarantee the taxpayer's defence. But this method, as known to date, produces a punctual value of the good or asset to be valued. Only the case of the econometric application in the multi-index model (García *et al.*, 2002b, *op. cit.*) includes a procedure, which can generate a confidence interval, using the prediction.

Estimate by interval has never been used in the two distribution functions method in the field of valuation, even when other methods of valuation have been used with different econometric models. For example, Calatrava and Cañero (2000) and Segura *et al.* (1998, *op. cit.*) estimate the models with different methods, using data of different origin, although they do not consider the possibility of estimating the final value using an interval with a certain confidence.

The aim of this work is to develop, in the method of the two distribution functions, a poly-stage model of valuation that generates a confidence interval for the value to be determined, which will be called the *negotiation interval.* On the other hand, reiteration of the process will lead to a concrete value that could be useful for the purpose of some reports.

The poly-stage valuation process uses, as the only family of distribution functions, the family of beta distributions from which different subfamilies are selected: classical, mesokurtic, of constant variance and Caballer. The practical application is done by computer program. This paper has the following organisation: Section 2 describes the different subfamilies of betas to be used; Section 3 presents the algorithm of valuation eliminating the need for tables; Section 4 presents the poly-stage valuation method in the context of the two distribution functions method and, finally, Section 5 describes the solution of a practical case. Section 6 summarises and concludes the findings.

## Description of the families of beta distributions determined by the classical values *a, m* and *b*

This Section synthesises the use of the different subfamilies of beta distributions. A complete justification and development can be found in García *et al.* (2002a, 2003a). After the expert or the available data have contributed to values $a$ (maximum), $m$ (most likely) and $b$ (minimum), the standardised mode $M$ must be calculated:

$$M = \frac{m - a}{b - a} \quad ,$$

where we can estimate $\mu$ and $\sigma^2$ as follows:

1. *Classical model:*

$$\mu = \frac{4M+1}{6} \quad \text{and} \quad \sigma^2 = \frac{1}{36} \; .$$

2. *Caballer model*:

If $M > \frac{1}{2}$, then $h = 1 + \frac{\sqrt{2}}{2M-1}$ and if $M < \frac{1}{2}$, then $h = 1 - \frac{\sqrt{2}}{2M-1}$. So:

$$\mu_C = \frac{(2h-2)M+1}{2h} \quad \text{and} \quad \sigma_C^2 = \frac{h^2-2}{2h^2(2h+1)} \; .$$

3. *Constant variance model*: Starting from the cubic equation (see Herrerías *et al.*, 1999a, *op. cit.*):

$$k^3 + k^2\left[7 - 36(M - M^2)\right] - 20k - 24 = 0 \; ,$$

the unique positive value of $k$ is obtained and hence:

$$\mu_{CV} = \frac{1+kM}{k+2} \quad \text{and} \quad \sigma_{CV}^2 = \frac{k^2M(1-M)+(k+1)}{(k+3)(k+2)^2} \; .$$

4. *Mesokurtic model* (only for values of $M$ greater than 0.723606 or less than 0.27639): Starting from the cubic equation (see Herrerías *et al.*, 1999a, *op. cit.*):

$$k^3(5M^2 - 5M + 1) + k^2(16M^2 - 16M + 2) - 5k - 4 = 0$$

and, from the unique positive value of $k$, we obtain:

$$\mu_M = \frac{1+kM}{k+2} \quad \text{and} \quad \sigma_M^2 = \frac{k^2M(1-M)+(k+1)}{(k+3)(k+2)^2} \; .$$

## An algorithm of valuation through beta distributions

The well-known *two betas valuation method* or, in general, the *two distribution functions valuation method*, introduced by Ballestero (1971, *op. cit.*), corresponds to an improvement of the synthetic method and was formalised later on by its author; Ballestero (1973, *op. cit.*) who describes it as follows: «Statistically, the variable market value of a good will follow the distribution function $F$. On the other hand, the index, parameter or explicative variable will statistically follow another distribution function $G$. Assuming that the density functions $F' = f$ and $G' = g$ are bell-shaped or similar, then the method of the two betas states a relationship between both variables.»

To do this, it is necessary to accept the following hypothesis: if the index $L_i$ corresponding to an asset $F_i$ is greater than the index $L_j$ corresponding to another asset $F_j$, the market value $V_i$ of the first asset will be also greater than the market value $V_j$ of the second one. Starting from this assumption, once the distributions $F$ and $G$ corresponding to the market value and the index, respectively, are known, the market value $V_k$ corresponding to an index $L_k$ is determined through the transformation:

$$V_k = \phi(L_k) \Leftrightarrow F(V_k) = G(L_k) \; .$$

Palacios *et al.* (2000) have presented a rigorous formalisation of this method. Given an asset to be valued, two random variables related with the asset are considered: the variable $L$ which represents a quality index of the asset, and the variable $V$, the market value of the asset. It is assumed that $V$ is a function of its quality, i.e. $V = \phi(L)$, where $\phi$ is a strictly increasing function in a given interval $[L_1, L_2]$, the range of variable $L$. If $L$ has the distribution function $G(l)$, then $V$ is a random variable with the distribution function:

$$F(v) = P[V \le v] = P[\phi(L) \le v] = P\left[L \le \phi^{-1}(v)\right] = G(\phi^{-1}(v))$$

or, equivalently,

$$G(l) = P[L \le l] = P[\phi(L) \le \phi(l)] = P[V \le \phi(l)] = F(\phi(l)),$$

where $\phi$ is strictly increasing. It is obvious that if $F$ is strictly increasing in the interval $[\phi(L_1), \phi(L_2)]$, then $F$ is invertible in this interval, hence, starting from the last expression, the function $[L_1, L_2] \to [\phi(L_1), \phi(L_2)]$ is obtained such that $\phi(l) = F^{-1}(G(l))$, that is a bijection which transforms qualities into market values. Thus, if $L_0$ is the quality of a good, then its market value would be:

$$V_0 = \phi(L_0) = F^{-1}(G(L_0)) \; .$$

We are now going to estimate the value ($x$) of an asset according to the value ($x'$) of an index, starting from the values $a$, $m$ and $b$, for the asset, and $a'$, $m'$ and $b'$ for the index (with $x' \in (a', b')$), by using any of the beta distributions determined with the classical estimations, i.e.: classical, of constant variance, mesokurtic and Caballer. In all cases, we can proceed as follows: by standardising the original values through the transformation $t = \frac{x-a}{b-a}$ (or $t' = \frac{x'-a'}{b-a}$), the values $a$, $m$ and $b$ would become 0, $M$ and 1, and the values $a'$, $m'$, $b'$ and $x'$ become 0, $M'$, 1 and $Z'$, so the distribution function corresponding to the standardised index would be:

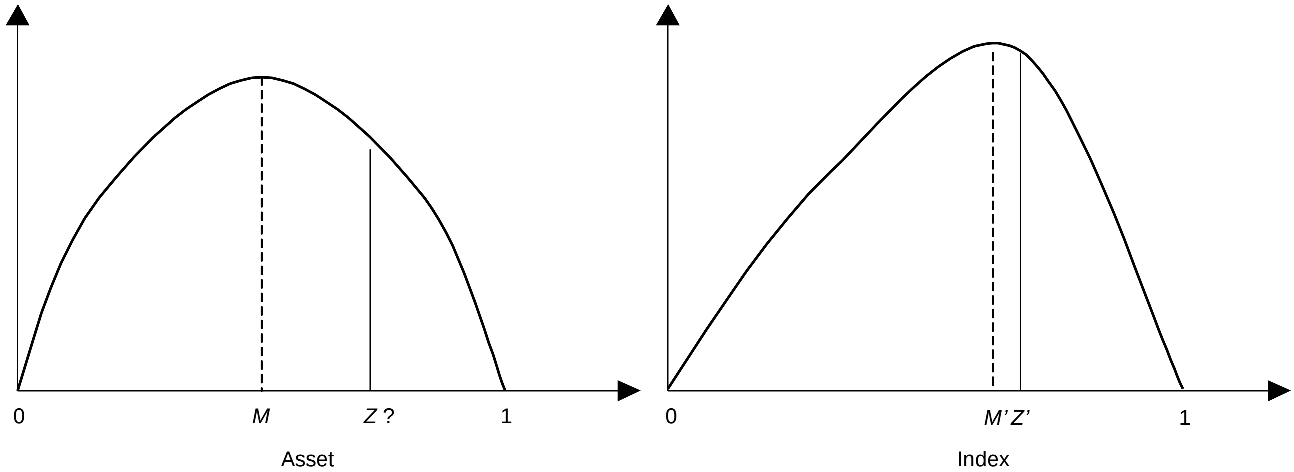


**Figure 1.** Method of two beta distributions.

$$G(\omega) = \int_0^{\omega} \frac{t'^{\,p'-1}(1-t')^{q'-1}}{\beta(p',q')}\,dt' \ .$$

Hence, it would therefore be possible to determine $G(Z')$ and, setting out the problem in the usual way (Fig. 1).

The problem would be reduced to one of solving the integral equation:

$$F(Z) = \int_0^{Z} \frac{t^{\,p-1}(1-t)^{q-1}}{\beta(p,q)}\,dt = G(Z') \ ,$$

in which the unique unknown value is $Z$.

Before continuing our approach, we are going to present the following considerations on how to obtain $G(Z')$. If we wish to obtain tables relating $M'$ and $Z'$ at our disposal, for each of the models, we must proceed as follows[1]: Given $a'$, $m'$, $b'$ and $x'$, we standardise the values and obtain 0, $M'$, 1 and $Z'$. Starting with $M'$, we determine $p'$ and $q'$, depending on the model we want to use (Table 1).

So we would obtain $G(Z')$, starting from $M'$ and $Z'$ as can be observed in Table 2.

On the other hand, starting with the asset values $a$, $m$ and $b$, we obtain 0, $M$ and 1, and, making $F(Z)$ equal to $G(Z')$, we can find the value of $Z$ in the tables, starting from $M$ and $F(Z)$, and later on the value of $x$, starting from the value of $Z$.

Nevertheless, our aim is to solve the previous integral equation through an iterative process, and, at the same time, to automate the choice of the model to be used, following either the maximum variance criterion (MVC) or the minimum variance criterion (mvc). Therefore, let us now see how $p$ and $q$ could be obtained in an automatic way, if previously we had decided to work either with the MVC (Table 3) or with the mvc (Table 4) (see García *et al.*, 2003a, *op. cit.*):

Let us illustrate, in a diagram (Fig. 2), the design of a computer program which will allow us to obtain the value $x$ of the asset, according to the values $a$, $m$ and $b$ and $a'$, $m'$, $b'$ and $x'$, by just deciding on the criterion to be used, i.e. MVC or mvc, both for the asset and the index. Next (Table 5), we standardise the values, by subtracting the smaller one and dividing by the range.

Next we calculate:

$$G(Z') = \int_0^{Z'} \frac{t'^{\,p'-1}(1-t')^{q'-1}}{\beta(p',q')}\,dt' \ ,$$

where $\beta(p',q') = \int_0^1 t'^{\,p'-1}(1-t')^{q'-1}\,dt'$ . Starting from $M$, we repeat all the previous process to obtain the values of $p$ and $q$ corresponding to the asset. Now we need to solve the integral equation:

$$F(Z) = \int_0^{Z} \frac{t^{\,p-1}(1-t)^{q-1}}{\beta(p,q)}\,dt = G(Z') \ ,$$

where $G(Z')$ is a known value. To do this, we have elaborated a computer program using *Matematica 4.0* (see, for example, García *et al.*, 2002a, *op. cit.*). The theoretical foundations of this program are as follows: If $p$ and $q$ are the parameters corresponding to the as-

[1] These tables, in which $M'$ and $Z'$ have perfectly determined their range in the interval [0,1], would cover all the possibilities: it would be enough with fixing the required precision and this does not happen with the tables presented by Caballer (1993) which have been developed according to $p$ and $q$ being unknown their ranges, so they would be always some incomplete tables.

**Table 1.** Obtaining the index distribution function

| **Standardised mode** | **Alternative models** | **Values of *p'* and *q'*** | **Distribution function** |
| --- | --- | --- | --- |
| $M' \Rightarrow$ | Mesokurtic (*k'*) | $k' \to \begin{cases} p' = 1 + k'M' \\ q' = 1 + k'(1 - M') \end{cases}$ | $G(\omega) = \int_0^{\omega} \frac{t'^{\,p'-1}(1-t')^{q'-1}}{\beta(p',q')}\,dt'$ |
| | Caballer (*h'*) | $M' > 1/2 \to \begin{cases} p' = h' + \sqrt{2} \\ q' = h' - \sqrt{2} \end{cases}$ <br> $M' < 1/2 \to \begin{cases} p' = h' - \sqrt{2} \\ q' = h' + \sqrt{2} \end{cases}$ | |
| | Constant variance (*k'*) | $k' \to \begin{cases} p' = 1 + k'M' \\ q' = 1 + k'(1 - M') \end{cases}$ | |
| | Classical | $\begin{cases} p' = 1 + 4M' \\ q' = 1 - 4(1 - M') \end{cases}$ | |

set distribution function, the objective is to solve the integral equation:

$$\int_0^Z \frac{t^{p-1}(1-t)^{q-1}}{\beta(p,q)}\,dt = G(Z') > 0 \ ,$$

where $Z$ is the unknown value. We can calculate $\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$ and set out the integral equation as follows:

$$\int_0^Z t^{p-1}(1-t)^{q-1}dt = G(Z') \cdot \beta(p,q) = H(Z') \ .$$

The iterative process is organised as follows: Starting from $\omega_0 = 0$, we define:

$$A_{\omega_s} = \int_0^{\omega_s} t^{p-1}(1-t)^{q-1}dt \ .$$

and the equation:

$$\omega_k = \omega_{k-1} - \operatorname{Sign}\left(A_{\omega_{k-1}} - H(Z')\right) \cdot \frac{1}{2^k}$$

So $A_{\omega_0} = 0$ and for $k = 1$, we obtain the following:

$$\omega_1 = \omega_0 - \operatorname{Sign}\left(A_{\omega_{1-1}} - H(Z')\right) \cdot \frac{1}{2^1} = \frac{1}{2}$$

and hence

$$A_{\omega_1} = \int_0^{1/2} t^{p-1}(1-t)^{q-1}dt \ .$$

For $k = 2$:

$$\omega_2 = \omega_1 - \operatorname{Sign}\left(A_{\omega_1} - H(Z')\right) \cdot \frac{1}{2^2} = \frac{1}{2} \pm \frac{1}{2^2} \ .$$

Thus, for every value of $k$ ($k = 0, 1, 2, \ldots$) the corresponding value of $\omega_k$ is obtained by adding or subtracting $\frac{1}{2^k}$ from the previous term, $\omega_{k-1}$:

**Table 2.** Obtaining *G*(*Z'*) from *M'* and *Z'*

| ***M'/Z'*** | **0.01** | **0.02** | **...** | | **...** | **0.99** |
| --- | --- | --- | --- | --- | --- | --- |
| **0.01** | | | | | | |
| **0.02** | | | | | | |
| **⋮** | | | | | | |
| | | | | *G*(*Z'*) | | |
| **⋮** | | | | | | |
| **1.00** | | | | | | |

**Table 3.** Maximum variance approach (MVC). Obtaining the parameters *p'* and *q'*

| | | | |
|---|---|---|---|
| 1. | $M' \in (0, 0.14645)$ | Caballer (as $M' < \frac{1}{2}$), $h' = 1 - \frac{\sqrt{2}}{2M'-1}$ | $\begin{cases} p' = h' - \sqrt{2} \\ q' = h' + \sqrt{2} \end{cases}$ |
| 2. | $M' \in (0.14645, 0.85355)$ | Classical | $\begin{cases} p' = 1 + 4M' \\ q' = 1 + 4(1-M') \end{cases}$ |
| 3. | $M' \in (0.85355, 1)$ | Caballer (as $M' > \frac{1}{2}$), $h' = 1 + \frac{\sqrt{2}}{2M'+1}$ | $\begin{cases} p' = h' + \sqrt{2} \\ q' = h' - \sqrt{2} \end{cases}$ |

$$\omega_3 = \left(\frac{1}{2} \pm \frac{1}{2^2}\right) \pm \frac{1}{2^3}, \ldots, \omega_k = \left(\frac{1}{2} \pm \frac{1}{2^2} \pm \ldots \pm \frac{1}{2^{k-1}}\right) \pm \frac{1}{2^k}.$$

Therefore,

$$|\omega_k - \omega_{k-1}| = \frac{1}{2^k}$$

and, taking into account that $\lim_{k \to \infty} \frac{1}{2^k} = 0$, $\{\omega_k\}$ is a Cauchy and, therefore, a convergent sequence, whereby convergence of the iterative process is guaranteed. The number of steps in the iteration would depend on the degree of adjustment, which we try to obtain. If it was a millionth (0.000001), provided that $|A_{\omega_k} - H(Z')|$ is greater than 0.000001, we would go to the following value of $k$. If $|A_{\omega_k} - H(Z')|$ is less than 0.000001, the program stops and gives the last value of $\omega_k$, as a solution for $Z$ and thus $x = a + (b - a)Z$.

## Poly-stage iterative process in the two distribution functions method. Confidence intervals

Let us suppose that we want to value an agricultural property (in general, an asset) taking the level of production per hectare (chosen index) as a reference.

**Table 4.** Minimum variance approach (mvc). Obtaining the parameters *p'* and *q'*

| | | |
|---|---|---|
| 4. | $M' \in (0, 0.2763933)$ | $\begin{cases} \text{Mesokurtic} \to \text{solve}: \\ k'^3(5M'^2 - 5M' + 1) + k'^2(16M'^2 - 16M' + 2) - 5k' - 4 = 0 \\ \text{We take the unique positive solution of } k' \to \begin{cases} p' = 1 + k'M' \\ q' = 1 + k'(1-M') \end{cases} \end{cases}$ |
| 5. | $M' \in \left(0.2763933, \frac{1}{2}\right)$ | $\begin{cases} \text{Caballer} \quad M' < \frac{1}{2} \\ h' = 1 - \frac{\sqrt{2}}{2M'-1} \begin{cases} p' = h' - \sqrt{2} \\ q' = h' + \sqrt{2} \end{cases} \end{cases}$ |
| 6. | $M' \in \left(\frac{1}{2}, 0.7236067\right)$ | $\begin{cases} \text{Caballer} \quad M' > \frac{1}{2} \\ h' = 1 + \frac{\sqrt{2}}{2M'-1} \begin{cases} p' = h' + \sqrt{2} \\ q' = h' - \sqrt{2} \end{cases} \end{cases}$ |
| 7. | $M' \in (0.7236067, 1)$ | $\begin{cases} \text{Mesokurtic} \to \text{solve}: \\ k'^3(5M'^2 - 5M' + 1) + k'^2(16M'^2 - 16M' + 2) - 5k' - 4 = 0 \\ \text{We take the unique positive solution of } k \to \begin{cases} p' = 1 + k'M' \\ q' = 1 + k'(1-M') \end{cases} \end{cases}$ |

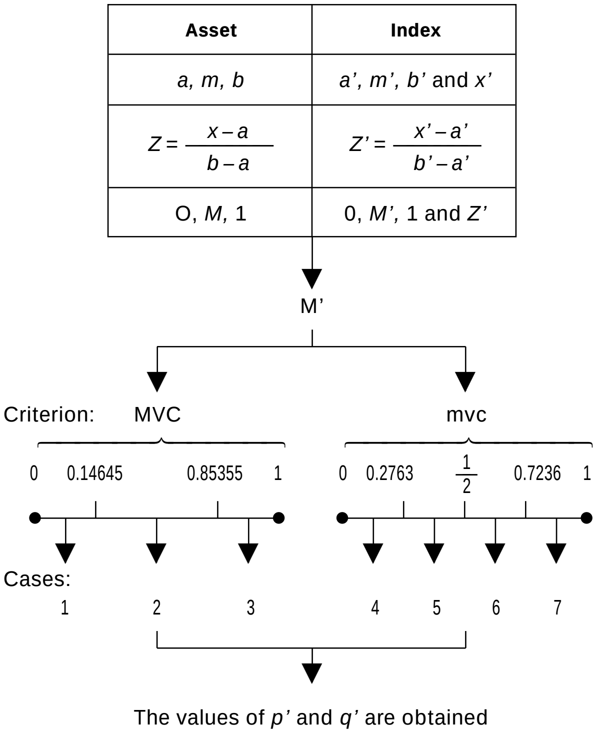


**Figure 2.** Design of a computer program to obtain *p'* and *q'*.

In principle, we will have the values $a$, $m$ and $b$ for the asset and $a'$, $m'$ and $b'$ for the index. Moreover, in this specific case, the index takes the value $x'$ and we want to determine, using the method of the two distribution functions, the value $x$ of the asset in question. It is well-known that, where $G(x')$ is the index distribution function and $F(x)$ the asset one, the value $x$ will be determined by the expression:

$$x = \phi(x'),$$

deduced from the following equality:

$$F(x) = G(x').$$

On the other hand, unless $M$ (the standardised mode) is in the interval (0.27639, 0.723606), one can chose, as a model for the asset distribution function, from the classical, Caballer, of constant variance and mesokurtic models. Analogously, the same is applicable to the index distribution function. Therefore, as we must choose one distribution function for the asset and another for the index, the number of possible choices will be the number of variations with repetition of four elements taken two at a time:

$$VR_4^2 = 4^2 = 16$$

This is illustrated in Table 5.

All in all, the number of possible values for $x$ will be:

**Table 5.** Possible choices for asset and index distribution functions

| Asset/Index | Classical (1) | Caballer (2) | Mesokurtic (3) | Constant variance (4) |
|---|---|---|---|---|
| Classical (1) | (1,1) | (1,2) | (1,3) | (1,4) |
| Caballer (2) | (2,1) | (2,2) | (2,3) | (2,4) |
| Mesokurtic (3) | (3,1) | (3,2) | (3,3) | (3,4) |
| Constant variance (4) | (4,1) | (4,2) | (4,3) | (4,4) |

— 16, when the mesokurtic model can be used either for the asset or for the index.

— 12, when the mesokurtic model can be used for the asset but not for the index.

— 12, when the mesokurtic model can be used for the index but not for the asset.

— 9, when the mesokurtic model cannot be used neither for the asset nor for the index.

Therefore, in a first step of the valuation we could obtain, starting from the values $a$, $m$ and $b$ for the asset and $a'$, $m'$, $b'$ and $x'$ for the index, the values $P_{ij}^1$, $i$, $j = 1,2,3,4$, using each of the procedures described in table 5. Starting from these values (that could be 9, 12 or 16), we calculate again for the asset a minimum value $a_1$, a modal value $m_1$ and a maximum value $b_1$. Taking these values as a starting point, we can apply each of the alternatives in Table 5 to the case ($a_1$, $m_1$, $b_1$) and ($a'$, $m'$, $b'$, $x'$) with which we will obtain, in a second phase of valuation, the values $P_{ij}^2$, $i$, $j = 1,2,3,4$, that, again, can be 9, 12 or 16.

This procedure can be repeated an appropriate number of times in such that, in the *k-th* step of the poly-stage valuation process, the values $P_{ij}^k$, $i$, $j = 1,2,3,4$ could be again 9, 12 or 16, depending on the circumstances.

We should point out that, in steps 2, 3, … of the poly-stage process, it is not possible to determine the mode (although we resort to establishing intervals), in which case we will substitute the mode by the median (see Troutt, 1989), or, according to the empirical relation among the mean, the median and the mode (Spiegel, 2001), for one-modal distributions which are not very asymmetric, the following relation holds:

$$\text{Mean} - \text{mode} = 3(\text{mean} - \text{median}),$$

from which it is deduced that:

$$\text{Mode} = \text{mean} - 3(\text{mean} - \text{median}).$$

All in all, after applying the $k$ steps of the valuation process, we will obtain (Table 6):

**Table 6.** The *k* steps in the valuation process

| Steps | Values | | | | |
|---|---|---|---|---|---|
| Step 1 | $P_{11}^{1}$ | $P_{12}^{1}$ | $P_{13}^{1}$ | ... | $P_{44}^{1}$ |
| Step 2 | $P_{11}^{2}$ | $P_{12}^{2}$ | $P_{13}^{2}$ | ... | $P_{44}^{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| Step *k* | $P_{11}^{k}$ | $P_{12}^{k}$ | $P_{13}^{k}$ | ... | $P_{44}^{k}$ |

For each of these groups of values we can consider the following questions:

1. Is it possible to assume that $P_{ij}^{k}$, $i, j = 1,2,3,4$ follows a normal distribution in spite of the range being finite? Observe that the values $P_{ij}^{k}$, $i, j = 1,2,3,4$ are non-negative, so only the right tail will be fitted and in a finite range.
2. Is it possible to assume that the sample of values $P_{ij}^{k}$, $i, j = 1,2,3,4$ comes from the same normal population as the sample $P_{ij}^{h}$, $i, j = 1,2,3,4$, with $k, h \in Z^{+}$?

To answer the first of these questions, for small samples ($n < 30$), we would recommend the normality test of Shapiro-Wilks (see Peña, 1993). The statistic to be used is:

$$w = \frac{1}{ns^2}\left[\sum_{j=1}^{h} a_{j,n}\left(x_{(n-j+1)} - x_{(j)}\right)\right]^2 = \frac{A^2}{ns^2} ,$$

where:

— $ns^2 = \sum (x_i - \bar{x})^2$ ,

— $h$ is $\frac{n}{2}$ , if $n$ is even, and $\frac{n-1}{2}$ , if $n$ is odd,

— the coefficients $a_{j,n}$ are tabulated, and

— $x_{(j)}$ is the value of the sample in position $j$.

The distribution of $w$ is tabulated.

To answer the second question, let us consider steps 1 and 2 of the valuation process for which we have obtained the samples:

$$x_1, x_2, \ldots, x_{n_1}$$

and

$$x_1, x_2, \ldots, x_{n_2} ,$$

respectively. Denoting:

$$S_n^2 = \frac{1}{n}\sum_{1}^{n}(x_i - \bar{x})^2 ,$$

and

$$S_{n-1}^2 = \frac{1}{n-1}\sum_{1}^{n}(x_i - \bar{x})^2 ,$$

it is verified that, where $x_1, x_2, \ldots, x_{n_1}$ come from a normal population, the random variable:

$$t = \frac{(\bar{x} - \mu)\sqrt{n}}{S_{n-1}} = \frac{(\bar{x} - \mu)\sqrt{n-1}}{S_n}$$

follows a Student's *t*-distribution with $n-1$ degrees of freedom, which allows us to obtain both confidence intervals, with lower and upper endpoints of:

$$I_1 : \bar{x}_1 \pm \frac{t_{n_1-1}^{\varepsilon/2}}{\sqrt{n_1}} S_1$$

and

$$I_2 : \bar{x}_2 \pm \frac{t_{n_2-1}^{\varepsilon/2}}{\sqrt{n_2}} S_2$$

On the other hand, we can check whether both samples come from the same population, i.e. to test the null hypothesis $H_0$ that (supposing that both populations have the same variance):

$$H_0 : \mu_1 = \mu_2$$

at level 0.05, using the statistic

$$t = \frac{\bar{x} - \bar{y}}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} ,$$

where:

$$\hat{\sigma} = \sqrt{\frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}}$$

that follows a Student's *t*-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Finally, considering these confidence intervals for the mean as intervals of probability for the variable, the following reasoning can be followed. After determining both 95 percent confidence intervals $I_1$ and $I_2$, we know that $\Pr(v \in I_1) = 0.95$ and $\Pr(v \in I_2) = 0.95$, and can state that:

$$\Pr(I_1 \cap I_2) = \Pr(I_1) + \Pr(I_2) -$$
$$- \Pr(I_1 \cup I_2) \geq 0.95 + 0.95 - 1 = 0.90,$$

because $\Pr(I_1 \cup I_2)$ is always less than 1.

This two-stage process, therefore, leads to a confidence interval. On the other hand, the poly-stage process is the generalisation of this and should lead to a value, provided that the sequence of the means obtained in steps 1, 2,..., *k*,...:

$$\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k, \ldots$$

converges to a real number. To do this, it must be verified that the variances or the ranges of the intervals

**Table 7.** Maximum, minimum and modal values of an asset and an index

| Values | Asset | Index |
|---|---|---|
| Minimum | $a = 250{,}000$ | $a' = 20{,}000$ |
| Modal | $m = 325{,}000$ | $m' = 32{,}500$ |
| Maximum | $b = 500{,}000$ | $b' = 50{,}000$ |

converge to zero. To conclude this Section we give several definitions that will help to understand the following Section:

**Definition 1 (Stability).** We describe the iterative process as being *stabilised* when, in two consecutive steps of the poly-stage process, we can accept that the two samples (composed of the results from each stage) come from a normal population with the same mean.

**Definition 2 (Convergence).** We can describe the iterative process as *convergent* when the sequence of fitted intervals $[a_i, b_i], i = 1,2,...,n$ is a Cauchy sequence.

**Definition 3 (Validity).** We describe the iterative valuation process as *valid*, for certain initial data, when it is convergent and stable.

## Empirical application

For the practical application, we will take the data of Alonso and Lozano's work (1985) on the valuation of agricultural properties in Valladolid (Spain). Therefore, the initial data are:

— Asset to be valued: Agricultural property (market value, ESP per hectare).

— Index: Income per hectare = 44,010 ESP.

The reference values of both variables are shown in Table 7.

In both cases, we will calculate the standardised modes:

$$M = \frac{m - a}{b - a} \quad \text{and} \quad M' = \frac{m' - a'}{b' - a'},$$

from which we deduce that it is not possible to apply the mesokurtic model either for the asset or for the index, because $M$ and $M'$ belong to the interval (0.27639, 0.723606). We, therefore, obtain the values in Table 8.

By applying the Shapiro-Wilks' test to these data, we obtain the statistic $w$ = 0.859992 which allows us to accept the hypothesis of a normal population, for which the mean, median and mode can be determined.

In the following step, we are going to consider, as reference values for the asset, the ones extracted from the former table:

— $a_2$ = 423,031

— $b_2$ = 481,294

— $m_2$ = 435,814

where the mode has been calculated from the empirical formula:

$$\text{Mode} = \text{mean} - 3(\text{mean} - \text{median}).$$

So $M_2 = \dfrac{m_2 - a_2}{b_2 - a_2} = 0.219418$ and then the mesokurtic model can be used to value the asset and thus another 12 values are obtained in this second step (see Table 9).

By applying the Shapiro-Wilks' test again, $w$ = 0.8831, where the normality hypothesis is accepted at a 5% level. Now let us apply a test to establish whether both populations have the same mean (assuming that both populations have the same variance):

$$\begin{cases} H_0 : \bar{x} - \bar{y} = 0, \\ H_1 : \bar{x} - \bar{y} \neq 0. \end{cases}$$

To do this, we know that the statistic:

$$t = \frac{\bar{x} - \bar{y}}{\hat{\sigma}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},$$

where:

$$\hat{\sigma} = \sqrt{\frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}}$$

**Table 8.** Values obtained in the first step

| $P^1_{11}$ | $P^1_{12}$ | $P^1_{14}$ | $P^1_{21}$ | $P^1_{22}$ | $P^1_{24}$ | $P^1_{41}$ | $P^1_{42}$ | $P^1_{44}$ |
|---|---|---|---|---|---|---|---|---|
| 433,264 | 445,833 | 446,472 | 475,274 | 481,014 | 481,294 | 423,031 | 435,380 | 436,016 |

**Table 9.** Values obtained in the second step

| $P^2_{11}$ | $P^2_{12}$ | $P^2_{14}$ | $P^2_{21}$ | $P^2_{22}$ | $P^2_{24}$ | $P^2_{31}$ | $P^2_{32}$ | $P^2_{34}$ | $P^2_{41}$ | $P^2_{42}$ | $P^2_{44}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 462,674 | 465,903 | 466,074 | 478,122 | 479,103 | 479,150 | 454,268 | 456,942 | 457,169 | 461,181 | 464,367 | 464,538 |

**Table 10.** First test on difference of means

| | |
|---|---|
| Sample 1 size | 9 |
| Sample 2 size | 12 |
| Mean 1 | 449,790.82 |
| Mean 2 | 465,790.9167 |
| Difference | –16,060.0967 |
| Statistic | $A = -2.1919$ |
| $H_0$ is rejected at a 5% level | |

**Table 12.** Second test on difference of means

| | |
|---|---|
| Sample 1 size | 12 |
| Sample 2 size | 9 |
| Mean 1 | 465,790.9167 |
| Mean 2 | 474,263.3389 |
| Difference | –8,472.4222 |
| Statistic | $A = -2.8477$ |
| $H_0$ is rejected at a 5% level | |

follows a Student's $t$-distribution with $n_1 + n_2 - 2$ degrees of freedom. The result is shown in Table 10.

Since both normal populations do not have the same mean, we will consider that the valuation process has still not been stabilised and thus proceed to the next step starting from Table 9 and obtain:

— $a_3 = 454{,}268$
— $b_3 = 479{,}150$
— $m_3 = 461{,}775.67$

from which $M_3 = \dfrac{m_3 - a_3}{b_3 - a_3} = 0.30$ . Now, the mesokurtic model cannot be used to value the asset and so the new process will generate 9 values, as is shown in Table 11.

The Shapiro-Wilks' test gives a value $w = 0.8621$ for the statistic, which allows us to accept the normality hypothesis at a 5% level. We now compare these samples $P_{ij}^2$ and $P_{ij}^3$ to establish whether they come from the same population. The result is shown in Table 12.

The process continues to be unstabilised and, therefore, requires another step. From Table 10, the starting point would be:

— $a_4 = 471{,}511$
— $b_4 = 477{,}261$
— $m_4 = 472{,}838.26$

from which $M_4 = \dfrac{m_4 - a_4}{b_4 - a_4} = 0.230806$, so the mesokurtic beta can be used to value the asset and, therefore, this process will generate 12 new values, i.e. (see Table 13).

The Shapiro-Wilks' test gives $w = 0.90022$ and, therefore, we can accept the normality hypothesis at a 5% level. If now the difference of means between the populations $P_{ij}^3$ and $P_{ij}^4$ are compared, the result is shown in Table 14.

We can say that now the valuation process has been stabilised and proceed to obtain a 95 percent confidence interval for this value. In the first place,

$$S_k = \sqrt{\frac{1}{n_k - 1}\sum_{i=1}^{n_k}(x_i - \bar{x})^2}\ ,$$

from which $S_3 = 2{,}206.94$ and $S_4 = 890.28$. On the other hand,

$$I_k : \bar{x}_k \pm \frac{t_{n_k-1}^{\varepsilon/2}}{\sqrt{n_k}} S_k\ ,$$

and, taking into account that $t_8^{0.975} = 2.31$ and $t_{11}^{0.975} = 2.20$:

$I_3 = (472{,}564;\ 475{,}962.70)$ and $I_4 =$
$= (475{,}138.59;\ 476{,}269.40)$.

The probability that the value we are looking for lies in $I_3$ is 95%:

$$\Pr(v \in I_3) = 0.95$$

the same as for $I_4$:

$$\Pr(v \in I_4) = 0.95$$

Therefore:

$$\Pr(I_3 \cap I_4) = \Pr(I_3) + \Pr(I_4) - \\ - \Pr(I_3 \cup I_4) \geq 0.95 + 0.95 - 1 = 0.90$$

Hence the value we are looking for lies in the interval

(475,138.59; 475,962.70)

with a probability greater than 90%.

Finally, observe the convergence process of maximum, minimum and modal values[2] (Fig. 3).

**Table 11.** Values obtained in the third step

| $P_{11}^3$ | $P_{12}^3$ | $P_{14}^3$ | $P_{21}^3$ | $P_{22}^3$ | $P_{24}^3$ | $P_{41}^3$ | $P_{42}^3$ | $P_{44}^3$ |
|---|---|---|---|---|---|---|---|---|
| 472,535 | 473,788 | 473,846 | 476,658 | 477,233 | 477,261 | 471,51 | 472,738 | 472,801 |

[2] The analytical study of the convergence is not dealt with in this work.

**Table 13.** Values obtained in the fourth step

| $P^4_{11}$ | $P^4_{12}$ | $P^4_{14}$ | $P^4_{21}$ | $P^4_{22}$ | $P^4_{24}$ | $P^4_{31}$ | $P^4_{32}$ | $P^4_{34}$ | $P^4_{41}$ | $P^4_{42}$ | $P^4_{44}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 475,468 | 475,783 | 475,799 | 476,919 | 477,021 | 477,026 | 474,460 | 474,706 | 474,719 | 475,304 | 475,614 | 475,630 |

**Table 14.** Third test on difference of means

| | |
|---|---|
| Sample 1 size | 9 |
| Sample 2 size | 12 |
| Mean 1 | 474,263.3389 |
| Mean 2 | 475,703.9983 |
| Difference | –1,440.6594 |
| Statistic | $A = -2.0623$ |
| $H_0$ is accepted at level 5% | |

As conclusions, the well-known *two betas valuation method* or, in general, the *two distribution functions valuation method*, is an emergent theory in the field of valuation that has arisen in the specific field of agricultural valuation. To apply this method, different works cited in the references have used the distributions suitable for the treatment of risk such as the triangular, trapezoidal or beta distributions.

In this work, we have gone one step further in this valuation method, by including an iterative process that allows all the well-known betas which can be determined starting from the three classical values: maximum, minimum and most likely, to be used simultaneously. In this way, the question of determining which of these betas could best simulate the behaviour of the asset or the index has been overcome, since now it is not necessary to answer this question because the process itself solves this question.

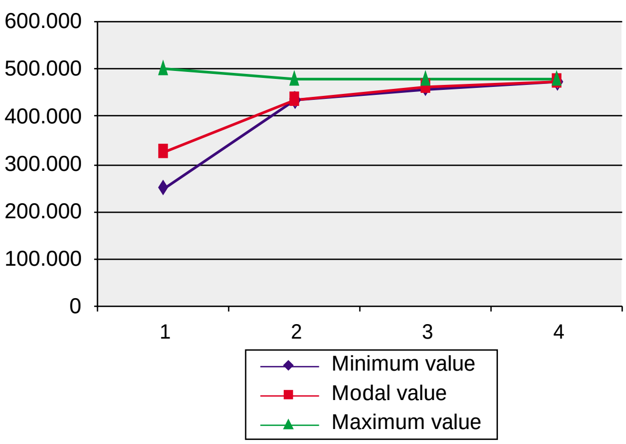


**Figure 3.** Convergence process of maximum, minimum and modal values.

Two questions have been presented in this work: the problem of stability, i.e. starting from the moment when the sets of values obtained from two consecutive processes can be considered to belong to normal populations with the same mean. At that moment, we can say that the process has been stabilised, but the general question remains unanswered, i.e. under what conditions can we verify that the process has been stabilised?

Another point to consider is that of convergence. In the practical case as illustrated in Figure 3, a rapid convergence of the maximum and minimum values from each stage can be observed that suggest a Cauchy sequence; but this question and the conditions that guarantee the convergence of this sequence of fitted intervals has not yet been formally solved. Both questions will be dealt with in a future work.

Finally, another novel contribution of this work is that it can generate either a value for the asset in question (it suffices to take the centre of the last interval) or can produce a confidence interval that, from a formal point of view, in our opinion, is much more appropriate to carry out valuations in situations of uncertainty such as the case we are dealing with here.

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