IDENTIFYING AND DESCRIBING COGNITIVE AND SOCIAL FACTORS INVOLVED IN PROBLEM SOLVING PROCESSES. A CASE STUDY[1]

(Identificación de factores cognitivos y sociales que surgen en los procesos de Resolución de Problemas. Un estudio de caso)

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Dedicado al Profesor Martín M. Socas Robayna con motivo de su jubilación. Sus aportaciones a la investigación en Didáctica de la Matemática siempre nos han resultado extremadamente valiosas.

Resumen

Este trabajo se basa en el análisis de la información recopilada durante dos semanas de clase dedicadas a la introducción de las ecuaciones diferenciales en un primer curso universitario, para el que se diseñó un Módulo de Enseñanza atendiendo a dos propósitos: la introducción del concepto de Ecuación diferencial Ordinaria (EDO) a partir del concepto de derivada de una función, como alternativa al uso de la definición formal, y la creación de un ambiente de resolución de problemas en el que los estudiantes interactuarán entre ellos analizando y discutiendo en torno a las cuestiones planteadas. El objetivo de este trabajo es identificar los procesos cognitivos que los estudiantes mostraron durante el desarrollo del Módulo de Enseñanza (Camacho-Machín, Perdomo-Díaz & Santos-Trigo, 2012), así como la identificación de una serie de factores que permiten detectar y explicar el progreso de los estudiantes durante la implementación del Módulo. Se describen y ejemplifican dichos factores a partir de un conjunto de episodios del proceso de resolución de una de las parejas de estudiantes que participaron en la investigación (con pseudóminos Alexis y Zoraida).

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Abstract

This work is based on the analysis of information gathered during two weeks of class aimed at introducing differential equations in a first year university course, for which a Teaching Module was designed with two main objectives in mind: to introduce the concept of the Ordinary Differential Equation (ODE) based on the concept of the derivative of a function as an alternative to the use of the formal definition and to create a problem solving environment in which students interact with each other, analyzing and discussing the issues in question.

The aim of this study is the identification of certain factors that can be used to detect and explain the progress of students during the development of the Teaching Module. The aforementioned factors to be described will be exemplified by a series of episodes during the resolution process of one of the pairs of students participating in the research: Alexis and Zoraida (pseudonyms).

Keywords: *Problem solving, ODE, Cognitive factor, Social factor*

Introducción

There is a certain amount of consensus among the community of researchers in mathematics education which considers mathematics as a social activity that takes place in an interactive environment between people and therefore depends on the learning environment in which it is developed. In particular, success or failure in the process of mathematical problem solving depends on cognitive aspects, i.e., on the knowledge that students may have about particular mathematical concepts, but it also depends on social issues such as the interpretation of the tasks, the working environment, and others (Cobb & Yackel,1996). The identification of these factors which affect the learning process is of great interest when developing teaching proposals. This paper identifies some of the cognitive and social factors that influenced the problem solving process of a pair of students during the implementation of a teaching module to introduce the ODE concept in a problem-solving situation using technology.

CONCEPTUAL FRAMEWORK

The way students learn a mathematical concept depends not only on the mathematical tools that are available to them and the network of meanings that have been built around them, but also on the environment in which learning takes place. Moreover, the way these mathematical concepts and tools are used to solve problems also depends on the knowledge available and on the environment in which it must be resolved (with pencil and paper, using technology, individually, group ...). Therefore, there is clearly a combination two types of factors, cognitive and social, in the process of mathematical problem solving.

A factor is considered to be of a cognitive type *when we want to emphasize particular conceptions that students engage* and it is considered to be social w*hen we want to emphasize the process of interaction between students, the teacher and the mathematics* (Rasmussen & Ruan, 2008, p.156).

These factors are not independent, and they are presented in an interconnected learning process. The distinction between these factors allows one to identify those factors which are directly related to the mathematical knowledge of students (cognitive factors) and those related to the environment in which learning takes place (social factors). Knowing the former will make it possible to take decisions directed at the design, selection and evaluation of activities, while social factors will provide indicators to develop proposals for the dynamics of classroom work.

This paper describes three cognitive factors (reorganization of mathematical ideas, selection of heuristics and giving meaning to mathematical procedures) and three social factors (familiarity with the terminology, mathematical commitment to a goal and control) which are identified as inter-connected elements. These factors have enabled us to analyze and explain the progress made by a pair of students showed during the resolution of problems used to introduce a new concept of the ODE which was new to them.

METHODOLOGY

Teaching Module

A teaching module was designed to introduce the concept of ODE in the first year of the chemistry degree course whose characteristics are: every problem begins by presenting a situation from which a series of questions will be put to the students for them to answer, such situations are adaptations of three contexts traditionally used in teaching differential equations: the decomposition of chemical elements, the mixture of substances and population dynamics. The aim of the questions included in each of the problems is to guide students in the use of different cognitive processes related to the development of their mathematical proficiency: reflection, representation, interpretation, resolution, generalization, verification, etc. (Kilpatrick et al., 2009). Figure 1 shows an overview of the Teaching Module.

Participants and classroom dynamics

A group of 15 students, who had previously received training about calculus in one and several variables, took part in the Teaching Module. Students were grouped into seven groups (six pairs and one trio) to work on solving the problems of the Module during 10 one hour class sessions. Each session began with a brief sharing of the activities undertaken during the previous session. Then each group was given a Voyage™200 calculator and the set of questions and activities presented in each of the stages of resolution, which were presented separately to the students. During the remainder of the session, students interacted with each other, trying to answer the questions posed. The teacher's role was to clear up the doubts raised by the students and to formalize the mathematical concepts that arose (ODE, order, solution, classification of first order ODEs, initial value problem ...).

	Decay of uranium (1 session)	Mercury Pollution (5 sessions)	Population dynamics (4 sessions)
Context	Context Decomposition of chemical elements	Mixture of substances	Fish Population
Structure	Presentation of the situation. Different issues related to the processes of representation, interpretation and interrelation between the mathematical and a non-mathematical context.	Presentation of the situation. Six stages of resolution including different issues and activities.	Presentation of the situation Five stages of resolution which include different issues and activities.
Description	Different situations of variation and their representations in mathematical language are analyzed. This ends with the introduction of the ODE concept, order and its solution.	The expression of an ODE starting from a situation of variation is obtained. The graphical and algebraic representations of the solution function are analyzed. The situation is generalized.	The expression of an ODE starting from a situation of variation is obtained. The behavior of the solution function using the ODE is analyzed. It generalizes the situation.

Figure 1: Description of the Teaching Module for the introduction of the ODE

DATA ANALYSIS AND DISCUSSION

The data set collected and analyzed includes videotapes of the 15 students' work during each of the ten class sessions and the teacher's interventions, a copy of the written work that students performed during class and a copy of student work in the Voyage^{TM}200 calculator, used as a tool for the development of the proposed activities. At the end of two weeks of class, once all the necessary material had been collected, the data were analyzed by a cyclical process of viewing the tapes, creating transcripts and notes in which the data were interpreted.

As a result of this procedure we identify the cognitive processes that students showed during the development of the Teaching Module (Camacho-Machín, Perdomo-Díaz & Santos-Trigo, 2012a) and several factors that can be used to detect and explain the progress of students during the development of the Module. In this paper we show and describe those factors that will be exemplified by a set of episodes of the resolution process of one of the pairs of students participating in the research: Alexis and Zoraida.

The progress of Alexis and Zoraida as they advanced in the proposed activities in the Teaching Module can be described in terms of the two types of factors described in the conceptual framework: social and cognitive (Rasmussen and Ruan, 2008).

The analysis of this pair’s work in the 10 class sessions was used to identify three cognitive type factors (reorganization of mathematical ideas, selection of heuristics and giving meaning to the mathematical procedures) and three social type factors (familiarity with the terminology, control of response and commitment to a mathematical objective).

Cognitive factors

- Reorganization of mathematical ideas

The solution of the problems posed in the Teaching Module involves the understanding and treatment of different mathematical concepts and ideas (function, derivative, equation, limit, etc.). Alexis and Zoraida, in general, showed the conceptual understanding needed to address the resolution of these problems, but had to rearrange certain mathematical ideas related to the concept of the derivative of a function, the consideration of constants as particular cases of functions and the use of inequations.

The first of these situations takes place during the resolution of problem 1 in the Module (decay of uranium), in relation to the concept of the derivative of a function. Whereas Alexis and Zoraida showed significant limitations in the use of the concept at the beginning of the activity, as they answered the questions raised it became clear that they considered the derivative of a function as an element that allowed them to describe phenomena of variation, arriving at the general conclusion that a positive derivative corresponds to times of increase or growth and a negative derivative refers to phenomena that show a decrease or fall. The following excerpt shows that Zoraida regarded the derivative as a symbol, when she says it is not necessary to indicate the variable with respect to which it is derived ("just derivative"). Alexis, meanwhile, seems to be considering that, although the derivative is related to phenomena of variation ("derivative means variation"), it cannot be used to express situations of no change.

Suppose for the moment that the number of atoms remains unchanged. How would you express this in mathematical terms?
Zoraida: Well, now it is not derivative with respect to time. Alexis: Now there is no variation Zoraida: So just derivative. Alexis: Derivative means variation Zoraida: It's true Alexis: The number of atoms of uranium is equal to a constant Zoraida: OK

Figure 2

As they advance in the activity, this couple relate different situations of variation with different mathematical expressions. The following excerpt shows how they have reorganized the idea that they initially had of the concept of derivative of a function and its link to the phenomena of variation.

What difference would there be between the statement of the problem when $u'(t) = -1$ and $u'(t) = -2$? Hint: Think about in what cases the uranium decreases more rapidly
Z: How? Let's see ... A: If $u'(t) = -2$ decrease more rapidly. Z: Depends. A: It depends on nothing. Z: So in the case $u'(t) = -2$ does it go faster? [Alexis nods]
What difference would there be between the statement of the problem when $u'(t) = -1$ and $u'(t) = -t$?
A: In the $u'(t) = -1$, the reduction of uranium atoms would occur steadily, whereas in the $u'(t) = -t$, the reduction of uranium atoms is variable. Z: OK
In this context, what would $u'(t)$ being positive mean?
A: That would increase the uranium atoms in time. Z: We should also give that answer here, in the case that $u'(t) = t$ it would increase rather than decrease, right? Because it would be positive, not negative. [Zoraida is considering an earlier question: Could it be that $u'(t)$ is equal to t?]

Figure 3

Another mathematical idea that Alexis and Zoraida reviewed during the development of the Module was the consideration of constant values as particular cases of functions. This idea is fundamental when classifying certain ODEs, as is the case of the equation that models the situation in problem 2 $(p' = 0.3 - 0.003p)$. Zoraida quickly identifies this ODE as being linear but her partner finds it hard to understand why it can be considered as linear and as separate variables (a classification pointed out by the teacher). Alexis's difficulty stems from the fact of not considering that constants are particular cases of functions, as is shown in the following fragment of the conversation between Alexis and the teacher.

Alexis: [*after writing the ODE in the form* $p'(t)+3\cdot 10^{-4}p(t)=0.3$]. Now we have the derivative and a function... this is not a function. We have a constant multiplied by a function depending on *t*, equal to another constant. Teacher: Is it linear like this? Alexis: I need terms. Teacher: You cannot say this …? Alexis: Is it *y*? Teacher: No, you can't say this is a function of *t*? Is this constant a function of the independent variable? [*Alexis does not respond*] can a constant be function of an independent variable? Alexis: A constant can be a function of whatever.

Finally, in problem 3 of the Module, Alexis and Zoraida revised their knowledge about inequations in order to use them as a tool in analyzing the behavior of solutions of the ODE $p'=0.19-bp^2$, depending on the initial value of $(p(t)$ after the teacher made them think about the heuristics that were being used to answer.

- Heuristics selection

In the third stage of problem 3 of the Module, corresponding to the particular solution of the problem, the students were asked to analyze for which values the population increases, decreases or remains constant in a model in which the population varies according to the ODE $p'=0.19-bp^2$. The following transcript shows the first attempts of Alexis and Zoraida to answer this question:

Zoraida: If *b* is squared, well ... if it is negative or positive, it will always be positive, and constant as ... Alexis: Eh, eh, eh ... This one here as well Zoraida: Yes, but the squared term wins ... The square is in charge. As the minus is the one in front, with which you put a positive or a negative, it will always be negative. Look, the term that leads is the one that is squared, because it is squared, then it will be much larger, because *b* is negative.[...]

Zoraida: I think that increasing, it would increase. And to lower ... A negative value decreases, and a positive value, decreases. Write it down there [Zoraida refers to the *b* sign].
Alexis: No, a value ... wait and see ... A positive value decreases, a negative value does not decrease
Zoraida: It does here, because the second term would be much higher than ... of course, yes. Come on then, put it. It increases for positive values and decreases for negative values.[...]
Alexis: Not greater or less than zero ... greater or less than 1!
Teacher: What does it matter?
Alexis: If *b* is 10^{-49} this term affects me a little bit
Teacher: Yes, but it affects you, the term is still negative.
Alexis: Yes, but I cannot tell if the total is positive if this is very small.

One can see here that they are using some erroneous arguments, among which is that $bp^2 > 0.19p$ whatever the values of b and p are. Based on this idea, they indicate that it is the sign of b which matters. Alexis then proposes some examples where b is very small, to indicate that the second term could have a very small influence on the sign of the derivative. In short, their heuristics to try to answer the proposed question was that of testing with different values of b and p. With the help of the teacher, they consider using of inequations as an alternative, thereby rearranging this concept which is already known by the students for its use in a new context: the study of solutions of an ODE. This example shows how important the use of heuristics is in problem solving processes (Schoenfeld, 1992).

- Give meaning to mathematical procedures

Normally, once the differential equations are classified, the next step is the use of specific resolution methods for each type of ODE. Camacho-Machín et al. (2012b) show that some students use these solution methods without understanding what the actions that they are performing mean, which often leads to difficulties and errors. During the resolution process Alexis and Zoraida showed that the need for the teacher to intervene, during the development of problem 2 in the Module, to get students to analyse the meaning of mathematical procedures that are being used each time is reconfirmed. This couple had difficulty understanding what "to separate variables" means, as is shown in the picture below where one can see some of their attempts to solve the ODE $p' = 0.3 - 0.003p$.

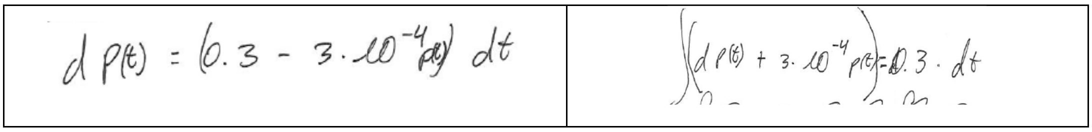

Figure 6

The examples, here, show how the reorganization of mathematical ideas, selection of heuristics and giving meaning to mathematical procedures have influenced the problem solving process in the proposed tasks.

Social Factors

- Familiarization with the terminology

The design of the problems of the Teaching Module helped the students to become familiar with the terminology used in each of them, which has had an influence on the progress of these students. For example, in Problem 1, the evolution coincides with the moment when it is established that $u(t)$ indicates the number of uranium atoms that are in the material at any given moment and raise different issues regarding possible expressions of $u'(t)$ (Figure 3). None of the students' answers was correct before this event. The reorganization of the idea of relationship between the concept of derivative and the phenomena of variation temporally coincides with the beginning of the familiarization process with the terminology and with the fact that the students are asked, explicitly, to justify their answers. Prior to this request, neither Alexis nor Zoraida argued their answers and neither did they ask each other for justifications. This situation confirms the relationships between social and cognitive factors such as we mentioned in the Conceptual Framework.

During the solution process of problem 2 by Alexis and Zoraida, it could be seen that familiarity with the terminology used in every situation was an essential factor for the students' development of strategic competence (Kilpatrick et al., 2009), since the processes of representation and problem solving depend on this. Alexis's sentence: "*It's what it is giving you here is the concentration, i.e., the total concentration, the concentration between the total, the concentration you put in between the volume*", used when Zoraida tells him to use the expression $\frac{p(t)}{10000}$ for the amount of mercury coming out of the tank per minute shows that Alexis is not aware that $p(t)$ refers to the amount of mercury inside the tank at each moment in time. This oversight was crucial in this pair's resolution process, since it prevented them from obtaining the expression of the ODE that modelled problem 2.

- Commitment to a mathematical objective

The fact that Alexis and Zoraida quickly obtained an algebraic expression for the function $p(t)$, show that they have a commitment to a mathematical objective that, in this casse, complicated things for them and, therefore, the resolution of the problem also had an influence on the process of representation of the ODE that modelled problem 2 in the Module. The following excerpt shows this couple tries to find an algebraic expression for $p(t)$ (the amount of mercury that is in the tank at each time t), identifying it with the difference between the amounts of mercury that enter and leave the tank, which corresponds to the variation and not to the amount of mercury.

Let's call $p(t)$ the amount of mercury that is in the tank at any time t. This function is continuous and is also considered to be derivable. How much mercury is there in the tank at one time?
Zoraida: $p(t)$ is equal to ... Alexis: No, no, $p(t)$ " $p(t)$ equal to" no. This says the amount of mercury. Zoraida: [Tries saying something] Alexis: Wait a minute! It would be $p(t)$ ... 3l/min enter, 3 litres is 0.3 grams. It would be 0.3 by t minus differential of ...[...] Zoraida: It would be 3t less ... you do not know how much leaves. You know that liquid comes out but you do not know how much comes out of the amount of mercury. And you know is that this is then equal to 0.7. Do you know? You have a solution of the function, but you do not know how much time has passed. [...] Let's go on to the next.

Figure 7

- Control

At different times during the 10 sessions of the Teaching Module, basically in the stages of the understanding and analysis of the situation of problems 2 and 3, it could be seen that the interaction between Alexis and Zoraida meant that an environment was created in which each of the students controlled the response of their partner, allowing the couple to modify different responses that were initially mistaken such as, for example, in problem 2, the identification of the problem to solve as a chemical problem and not a mathematical problem or the amounts of dissolution going into and coming out of the deposit depending on the time. Some examples of this monitoring process among the students themselves can be seen in the following excerpts from the discussion they had with each other during the stage of understanding problem 2.

Alexis: [...] We are asked to decrease or reduce the concentration to the minimum allowed.

Zoraida: No, we were not asked to reduce. [...] We are asked to find the model to control the concentration, not even to reduce it. So we are being asked to propose an equation or something like that, right?
At what velocity is the mercury solution put into the tank
Alexis: 0.1 g/l No, no, no ...Zoraida: He says velocity Alexis: 3 l/min, or 0.3 per minute Zoraida: Litres? Alexis: Grams, this would be 0.3 g / min. Zoraida: 0.3 per minute, right? Alexis: No, 0.1. Wait
How much mercury enters the tank for every litre of water introduced?
Zoraida: 0.3 per minute, right? Alexis: No, 0.1. Wait Zoraida: Yes, it would be 0.1
How much mercury has been introduced into the tank a minute later
Alexis: 0.1 gr Zoraida: A minute later Alexis: I mean 0.3 gr

Figure 8

Another example of the importance of the control process that is created by the interaction between the students can be seen in the discussion that Alexis and Zoraida have, during the solution process of problem 3, when trying to express the number of fish born in a year in function of $P(t)$, knowing that the birth rate is 410 per thousand per year (Figure 9).

Z: Well, if the number of fish is $P(t)$ ... yes. A: Yes, it would be 410 by ... Z: $P(t)$ Right? ... In 1000 A: The number of fish? Z: The number of fish is 1000

A: The number of fish varies. So ...
Z: It would ... be ... let’s see ... it would be 410 by $P(t)$ divided by 1000
A: And why by $P(t)$?
Z: Because $P(t)$ is the number of fish
A: Ahhhh ... $P(t)$ is the amount of fish there are at any time [...].This would be 410 by $P(t)$ and now it is ...
Z: No, but Alex, if you have 410 and multiply it by 1000, in this case it does not give you 410. Imagine you have 2000 and you multiply by 410? No. It will be that number divided by $1000k$ or $1000n$.
A: No, wait, now we are introducing a variable that does not go. How did you do this?
Z: The problem gives me that number [...]
A: If for every 1000 ... [...] Okay, we are assuming a linear growth. [...] Let's see ... let’s see ... $410P(t)$ divided by 1000!
Z: Divided by 1000? ¿400 divided by 1000?
A: What?
Z: Wait, if it really is 410 by 2000
A: And divided by 1000 ...
Z: By two ... ah, okay. You have to check everything.

Figure 9.

The control process was essential in the early stages of the understanding and analysis of the situation, as well as encouraging the students to reflect on their answers and the arguments used, skills that are a part of the development of mathematical proficiency (Kilpatrick et al., 2009).

SOME CONCLUDING REMARKS

These examples of different stages in Alexis and Zoraida’s process of resolution during the Teaching Module show the presence and importance of social and cognitive factors in the processes of representation and problem solving, as well as in the students’ capacity for reflection and argument, which corresponds to two of the elements that characterize mathematical proficiency: strategic competence and adaptive reasoning (Kilpatrick et al., 2009).

The following diagram (Figure 10) shows the relationship between the design of the Teaching Module and the cognitive and social factors described in this article. One can see the interrelationship that there is between the two types of factors, which appear together throughout the resolution process.

	Problem 1: *Decay of uranium*	**Problem 2:** *Mercury Pollution*			**Problem 3:** *Population dynamics*		
		Stage 1: Understanding	*Stage 2: Analysis*	*Stage 3: Solution (a particular case)*	*Stage 1: Understanding*	*Stage 2: Analysis*	*Stage 3: Solution (a particular case)*
Cognitive Factors	Reorganization of mathematical ideas			Reorganization of mathematical ideas			Reorganization of mathematical ideas
						Selection of heuristics	
				Giving meaning to the mathematical procedures			
Social Factors	Familiarity with the terminology		Familiarity with the terminology			Familiarity with the terminology	
			Commitment to a mathematical objective				
		Control of response			Control of response		

Figure 10.

Furthermore, it seems that there are relationships between the various stages of the problem solving and the identified factors (the reorganization of mathematical ideas seems to be in the solution stage, familiarization with the terminology in the analysis stage...). The data obtained so far do not allow one to come to a definitive conclusion about this, but do show evidence of this possible relationship.

In conclusion, we believe that the identification of cognitive and social factors in learning mathematics and, in particular, the resolution of problems is useful for the design and review of learning trajectories as they can anticipate certain behaviors that students show in the problem solving process.

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