

# The investigation of approximate solutions of Deformed Schrödinger Equations for the Hydrogenic Atoms, Heavy Quarkonium Systems $Q\bar{Q}$ ( $Q = b, c$ ) and Diatomic molecule Bound-State Problem under Improved Exponential, Generalized, Anharmonic Cornell potential Model in NCPS symmetries

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## Abstract

Recently Ibekwe *et al.* (Ref [16]) have treated the bound state solution of the radial Schrödinger equation for the exponential generalized Cornell anharmonic potential (EGACp, in short, see Eq. (1)) model within the framework of quantum mechanics known in the literature. According to the generalized Bopp's shift method and standard perturbation theory, we generalize this process by adding multi-variable coupling potentials  $\frac{d \exp(-ar)}{2r^3} \mathbf{L}\Theta$ ,  $\frac{\alpha d \exp(-ar)}{2r^2} \mathbf{L}\Theta$ ,  $\frac{f+l(l+1)}{r^4} \mathbf{L}\Theta$ ,  $-\frac{c\mathbf{L}\Theta}{2r^3}$ ,  $-\frac{b\mathbf{L}\Theta}{2r}$  and  $\left(-a\mathbf{L}\Theta + \frac{\mathbf{L}\Theta}{2\mu}\right)$  together with the EGACp model in three-dimensional nonrelativistic quantum mechanics noncommutative phase space (3DNRQm-NCPS) including the effect of the centrifugal term. The multi-variable coupling potentials appear as a result of the effects of NC properties of space and phase on the EGACp model. We called this newly proposed potential the improved generalized Cornell anharmonic potential (IEGACp, in short, see Eq. (2)) model. We were able to discover a new application of this potential to hydrogen atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$  and  $\text{Be}^+$ , in addition to other applications the heavy quarkonium systems ( $c\bar{c}$  and  $b\bar{b}$ ) and some selected diatomic molecules (CO, NO, CH and  $\text{N}_2$ ) known in the literature within the framework of (3DNRQm-NCPS). The obtained energy is sensitive to discreet atomic quantum numbers ( $j, n, l, s$  and  $m$ ), the noncommutativity parameter  $(\theta, \sigma, \chi)$  due to the topological properties of the self-quantum influence of space-space and phase-phase, in (3DNRQm-NCPS) symmetries, in addition to the discreet atomic quantum numbers ( $n, l$ ) and the parameters  $(a, b, c, d, e, f, \alpha)$  or  $(r_e, D_e, \alpha)$  of the EGACp model that appeared in the literature. Furthermore, we have shown that the corresponding Hamiltonian operator in (3DNRQm-NCPS) symmetry is the sum of the Hamiltonian operator of EGACp and three operators, the first one is the perturbed spin-orbit interaction, the second is the modified Zeeman operator while the last part corresponds to the induced rotational Fermi Hamiltonian. *The new mass spectrum* of the heavy quarkonium systems is carried out by introducing the IEGACp model in (3DNRQm-NCPS) symmetries. The comparison with other special cases of potentials in the literature is motivating. The limiting cases are analyzed for  $(\theta, \sigma, \chi)$  and  $(\theta^c, \sigma^c, \chi^c) \rightarrow (0, 0, 0)$  and are compared with those of literature.

**Keywords:** Schrödinger equation, Generalized exponential Cornell potential model, Noncommutative quantum mechanics, Star product, Generalized Bopp's shift method.

## Resumen

Recientemente Ibekwe et al. (Ref [16]) han tratado la solución de estado ligado de la ecuación radial de Schrödinger para el modelo de potencial anarmónico de Cornell generalizado exponencial (EGACp, en resumen, ver Ec. (1)) dentro del marco de la mecánica cuántica conocida en la literatura. De acuerdo con el método de desplazamiento de Bopp generalizado y la teoría de la perturbación estándar, generalizamos este proceso agregando potenciales de acoplamiento multivariable  $(d \exp(-ar))/(2r^3) \mathbf{L}\Theta$ ,  $(\alpha d \exp(-ar))/(2r^2) \mathbf{L}\Theta$ ,  $(f+l(l+1))/r^4 \mathbf{L}\Theta$ ,  $-c\mathbf{L}\Theta/[(2r)^3]$ ,  $-b\mathbf{L}\Theta/2r$  y  $(-a\mathbf{L}\Theta+(\mathbf{L}\Theta)/2\mu)$  juntos con el modelo EGACp en espacio de fase no conmutativo de mecánica cuántica no relativista tridimensional (3DNRQm-NCPS) incluyendo el efecto del término centrífugo. Los potenciales de acoplamiento multivariable aparecen como resultado de los efectos de las propiedades NC del espacio y la fase en el modelo EGACp. Llamamos a este nuevo potencial propuesto modelo de potencial anarmónico de Cornell generalizado mejorado (IEGACp, en resumen, véase la Ec. (2)). Pudimos descubrir una nueva aplicación de este potencial a los átomos de hidrógeno  $\text{He}^+$ ,  $\text{Li}^{+2}$  y  $\text{Be}^+$ , además de otras aplicaciones los sistemas pesados de quarkonio ( $c\bar{c}$  y  $b\bar{b}$ ) y algunas moléculas diatómicas seleccionadas (CO, NO, CH y  $\text{N}_2$ ) conocidos en la literatura en el marco de (3DNRQm-NCPS). La energía obtenida es sensible a los números cuánticos atómicos discretos ( $j, n, l, s$  y  $m$ ), el parámetro de no conmutatividad  $(\theta, \sigma, \chi)$  debido a las propiedades topológicas de la influencia auto-cuántica del espacio-espacio y fase-fase, en simetrías (3DNRQm-NCPS), además de los números cuánticos atómicos discretos ( $n, l$ ) y los parámetros

(a,b,c,d,e,f, $\alpha$ ) o (r\_e,D\_e, $\alpha$ ) de el modelo EGACp que apareció en la literatura. Además, hemos demostrado que el operador hamiltoniano correspondiente en simetría (3DNRQm-NCPS) es la suma del operador hamiltoniano de EGACp y tres operadores, el primero es la interacción espín-órbita perturbada, el segundo es el operador Zeeman modificado mientras que el La última parte corresponde al hamiltoniano de Fermi rotacional inducido. El nuevo espectro de masas de los sistemas pesados de quarkonium se realiza introduciendo el modelo IEGACp en simetrías (3DNRQm-NCPS). La comparación con otros casos especiales de potenciales en la literatura es motivadora. Se analizan los casos límite para  $(\Theta, \sigma, \chi)$  y  $(\Theta^c, \sigma^c, \chi^c) \rightarrow (0, 0, 0)$  y se comparan con los de la literatura.

**Palabras clave:** Ecuación de Schrödinger, Modelo de potencial exponencial generalizado de Cornell, Mecánica cuántica no conmutativa, Producto de estrella, Método de desplazamiento de Bopp generalizado.

## I. INTRODUCTION

The nonrelativistic Schrödinger equation NRSE has been a major tool for researchers since the early years of quantum mechanics, researchers have shown a great deal of interest in solving this equation for different types of spherical symmetric potentials. The confining potentials have received in-depth attention from many researchers in recent years. The search of solutions of the NRSE for the quark-antiquark system is a good example of the Cornell type (the Coulomb potential with linear terms). Kumar and Chand (2012) obtained the energy eigenvalues and normalized eigenfunctions of the radial Schrödinger equation RSE in N-dimensional Hilbert space for the quark-antiquark interaction potential using the power series technique via a suitable ansatz to the wave function [1] and in the next year, analytically obtained the complete energy spectra by computing and adding energy eigenvalues for the ground state, for large and a small distance and normalized eigenfunctions of the RSE in N-dimensional space using the power series technique via a suitable Ansatz to the wave function [2]. Gupta *et al.* [3] (2012) studied heavy quark systems  $Q\bar{Q}$  ( $Q = b, c$ ) in the nonrelativistic framework using the energy-dependent interquark potential of the form harmonic oscillator with a small linear term as energy-dependent as perturbation  $1/2mw^2(1 + \gamma E_{nl})r^2$  plus an inverse-square potential  $\frac{g}{r^2}$ . Abu-Shady calculated masses Heavy Quarkonia ( $b\bar{c}$  and  $c\bar{b}$ ) mesons within the framework of the N-dimensional RSE and obtained the energy eigenvalues and the corresponding wave functions using the Nikiforov-Uvarov NU method [4]. Ikhdair and Babatunde applied the parametric NU and asymptotic iteration method to study the approximate analytic bound state eigensolutions of the RSE for the Hellmann potential [5]. Hamzavi *et al.* by using the generalized parametric NU method obtained the approximate analytical solutions of the RSE for the Hellmann potential and calculated the energy eigenvalues and corresponding eigenfunctions in the closed-form [6]. Onate *et al.* (2017) used the supersymmetric approach to study the approximate analytic solutions of the 3D-SE with the Hellmann potential by applying a suitable approximation scheme to the centrifugal term and calculating the Tsallis entropy and Rényi entropy in position and momentum spaces using the integral method [7]. Kher *et al.* studied the mass spectra of the  $B$  and  $B_s$  mesons using a Cornell potential incorporated with a  $O(1/m)$  correction in the potential energy term and expansion of the kinetic energy term up to  $O(p^{10})$  for relativistic correction of the Hamiltonian using a

Gaussian wave function [8]. The heavy-light mesons HLM in the nonrelativistic quark model under the combination of vector and scalar potentials have been derived by Abu-Shady and Khokha (2018) using the Laplace transformation method and obtained the energy eigenvalues and the corresponding eigenfunctions and calculated the masses of the scalar, vector, pseudoscalar, and pseudovector for  $B$ ,  $B_s$ ,  $D$ , and  $D_s$  mesons in the 3D-space [9]. Abu-Shady and Ezz-Alarab (2019) extend the trigonometric Rosen-Morse potential to calculate heavy-meson properties in the free and hot media and consider the case of spectra of heavy and HLM masses and thermodynamic properties and solving analytically the N-RSE using the exact analytical iteration method [10]. Thermodynamic properties of HLM are calculated within the framework of the N-dimensional RSE by Abu-Shady *et al.* under extended Cornell potential [11]. Abu-Shady *et al.* solved the N-dimensional SE analytically by the NU method and apply their search results to find the properties of the heavy quarkonium system such as charmonium  $c\bar{c}$  and bottomonium  $b\bar{b}$  have the quark and antiquark flavor under Cornell potential plus the quadratic potential and the inverse of quadratic potential [12]. Kuchin and Maksimenko studied the spin-averaged mass spectra of heavy quarkonia and  $Bc$  mesons in a Cornell potential within the framework of SE and obtained the energy eigenvalues and eigenfunctions in compact forms for any  $l$ -value using the NU method [13]. Rahmani *et al.* investigated the SE with a potential containing Coulomb, linear, and quadratic terms and wrote the total wave function as perturbed wave functions using the NU technique and report the Isgur-Wise function parameters to obtain the masses, slope, and curvature parameters of some HLM [14]. Moazami *et al.* studied the mass spectrum and decay properties of HLM in the nonrelativistic potential model being carried out introducing a new potential combination containing Cornell, Gaussian, and inverse square terms [15]. Very recently, Ibekwe *et al.* solved the RSE with an exponential, generalized, anharmonic Cornell potential (EGACp, in short) using the series expansion method [16] of the form:

$$V_{gcp}(r) = ar^2 + br - \frac{c}{r} + \frac{d \exp(-ar)}{r} + \frac{f}{r^2} + e. \quad (1)$$

Where  $a, b, c, d$ , and  $f$  are the potential parameters,  $\alpha$  is the screening parameter  $r$  is the distance between two particles. Here in the present work, we modify the EGACp model by adding new terms  $\frac{d \exp(-ar)}{2r^3} \mathbf{L}\mathbf{O}$ ,  $\frac{\alpha d \exp(-ar)}{2r^2} \mathbf{L}\mathbf{O}$ ,  $\frac{f+l(l+1)}{r^4} \mathbf{L}\mathbf{O}$ ,

$-\frac{c\mathbf{L}\Theta}{2r^3}$ ,  $-\frac{b\mathbf{L}\Theta}{2r}$  and  $(-a\mathbf{L}\Theta + \frac{\bar{\mathbf{L}}\Theta}{2\mu})$  due to the topological properties of the self-quantum influence of space-space and phase-phase including the effect of centrifugal term  $\frac{l(l+1)}{r^2}$  which appear in the third additive part  $\frac{l(l+1)}{r^4}\mathbf{L}\Theta$ , the resultant quark-antiquark interaction potential, we call it the improved exponential, generalized, anharmonic Cornell potential model (IEGACp, in short), is written as:

$$V_{nc}^{gcp}(r) = V_{gcp}(r) + \left(\frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a\right)\mathbf{L}\Theta. \quad (2)$$

The two couplings  $\mathbf{L}\Theta$  and  $\bar{\mathbf{L}}\Theta$  will be clearly stated in the third section. The interest of many researchers in recent years has been focused on the study of many physical systems with (3DNRQm-NCPS) this is due to the multiplicity of scientific applications at the microscopic (atomic and subatomic) or macroscopic levels where the dimensions are large and the forces of gravity dominate mainly. This was due to the emergence of some modern theories such as; the non-renormalizable of standard model, string theory, quantum gravity, new approaches have been developed to tackle this difficulty, among them the Weyl Moyal star product based on noncommutative phase space (NCPS) due to its many applications, especially in nonrelativistic quantum mechanics. (3DNRQm-NCPS) symmetries is a particular

case of the NCPS, has attracted much attention [17, 18, 19, 20, 21, 22, 23, 24, 25], etc. On a personal level, we had several notable milestones, we have studied the (Cornell, Gaussian, and inverse square terms), the modified quark-antiquark interaction potential, the nonrelativistic quark-antiquark potential, Cornell plus inverse quadratic potential, and the modified central complex potential in the symmetries of (3DNRQm-NCPS) [26, 27, 28, 29, 30, 31, 32, 33]. It should be noted that the potential referred to in [16] does not only apply to the quark system only but also goes beyond it to apply to diatomic molecules CO, NO, CH, and N<sub>2</sub> [16].

In the present research paper, we want to extend the study in [16] to the case of nonrelativistic (3DNRQm-NCPS) with the possibility of finding other applications and more profound interpretations in the subatomic scale. The nonrelativistic energy levels of the hydrogenic atoms He<sup>+</sup>, Li<sup>+2</sup> and Be<sup>+</sup>, quark-antiquark systems  $Q\bar{Q}$  ( $Q = b, c$ ), and the diatomic molecules CO, NO, CH and N<sub>2</sub> interact with the improved generalized exponential, generalized, Cornell potential (IEGACp, in short) model in the context of the noncommutative space phase, which has not been obtained yet. The main purpose of this paper is to solve the deformed Schrödinger equation DSE with the IEGACp model. The structure of (3DNRQm-NCPS) based on NC canonical commutations relations in Schrödinger, Heisenberg, and interaction pictures (SP, HP, and IP), respectively, as follows (Throughout this paper, the natural units  $c = \hbar = 1$  will be used) [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]:

$$\begin{cases} [x_i^{nc*}, p_j^{nc}] = [x_{Hi}^{nc}(t)*, p_{Hj}^{nc}(t)] = [x_{li}^{nc}(t)*, p_{lj}^{nc}(t)] = i\hbar_{eff}\delta_{ij}, \\ [x_i^{nc*}, x_j^{nc}] = [x_{Hi}^{nc}(t)*, x_{Hj}^{nc}(t)] = [x_{li}^{nc}(t)*, x_{lj}^{nc}(t)] = i\theta_{ij}, \\ [p_i^{nc*}, p_j^{nc}] = [p_{Hi}^{nc}(t)*, p_{Hj}^{nc}(t)] = [p_{li}^{nc}(t)*, p_{lj}^{nc}(t)] = i\bar{\theta}_{ij}. \end{cases} \quad (3)$$

Here  $\hbar_{eff} = \hbar \left(1 + Tr\left(\frac{\theta\bar{\theta}}{4}\right)\right) \approx \hbar$  is the effective Planck constant, the very small two parameters  $(\theta^{ij}, \bar{\theta}^{ij}) = \varepsilon^{ij}(\theta, \bar{\theta})$  (compared to the energy) are the elements of two antisymmetric real matrixes with dimensions of (length)<sup>2</sup> and (momentum)<sup>2</sup>, respectively while  $\varepsilon^{ij}$  is just an antisymmetric

number ( $\varepsilon^{12} = -\varepsilon^{21} = \varepsilon^{13} = -\varepsilon^{31} = \varepsilon^{23} = -\varepsilon^{32} = 1$ ,  $\varepsilon^{ii} = 0$ ). However, the operators  $\zeta_{iH}^{nc}(t) = (x_{Hi}^{nc} \vee p_{Hi}^{nc})(t)$  and  $\zeta_{il}^{nc}(t) = (x_{li}^{nc} \vee p_{lj}^{nc})(t)$  in Hp and IP, respectively depending on the corresponding operator  $\zeta_{is}^{nc} = x_i^{nc} \vee p_i^{nc}$  in SP with the following projections relations:

$$\begin{cases} \zeta_{iH}(t) = \exp(iH_{gcp}T)\zeta_{is}^{nc} \exp(-iH_{gcp}T), \\ \zeta_{il}(t) = \exp(iH_{ogcp}T)\zeta_{is}^{nc} \exp(-iH_{ogcp}T), \end{cases} \Rightarrow \begin{cases} \zeta_{iH}^{nc}(t) = \exp(iH_{nc}^{gcp}T) * \zeta_{is}^{nc} * \exp(-iH_{nc}^{gcp}T), \\ \zeta_{il}^{nc}(t) = \exp(iH_{onc}^{gcp}T) * \zeta_{is}^{nc} * \exp(-iH_{onc}^{gcp}T). \end{cases} \quad (4)$$

Here  $T = t - t_0$ ,  $\zeta_{is} = x_i \vee p_i$ ,  $\zeta_{iH}(t) = (x_{iH} \vee p_{iH})(t)$  and  $\zeta_{il}^{nc}(t) = (x_{li}^{nc} \vee p_{lj}^{nc})(t)$  are the three representations SP, IP, and IP in NRQM, while the dynamics of new systems  $\frac{d\zeta_{iH}^{nc}(t)}{dt}$

described from the following motion equations in (3DNRQm-NCPS):

$$\frac{d\zeta_{iH}(t)}{dt} = [\zeta_{iH}(t), H_{gcp}] + \frac{\partial\zeta_{iH}(t)}{\partial t} \Rightarrow \frac{d\zeta_{iH}^{nc}(t)}{dt} = [\zeta_{iH}^{nc}(t)*, H_{nc}^{gcp}] + \frac{\partial\zeta_{iH}^{nc}(t)}{\partial t} \quad (5)$$

Here  $H_{gcp}(H_{ogcp})$  and  $H_{nc}^{gcp}(H_{onc}^{gcp})$  denote the ordinary and generalized quantum Hamiltonian (free) operators for the IEGACp model in the NRQM and (3DNRQm-NCPS) respectively. Furthermore(\*), denote to the star product, which is generalized between two arbitrary functions

$(f, g)(x, p)$  of the form  $(f_{nc}, g_{nc})(x^{nc}, p^{nc})$  in (3DNRQm-NCPS) symmetry (see, e.g., [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]):

$$(f * g)(x, p) =$$

$$\left( fg - \frac{i}{2} \theta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} - \frac{i}{2} \bar{\theta}^{ij} \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial p^j} \right) (x, p) + O(\theta^2, \bar{\theta}^2). \quad (6)$$

This allows the formation of a scale of two spaces and phase cells with volumes  $l_{ncs}^3 = \theta^{3/2}$  and  $l_{ncp}^3 = \bar{\theta}^{-3/2}$  respectively. On the other hand, Eq. (6) allows us to satisfy the postulated algebra in Eqs. (3a), (3b), and (3c). The second and the third terms in the above equation are the effects of (space-space) and (phase-phase) noncommutativity properties, respectively. It is therefore the aim of this paper to present approximate solutions of the deformed Schrödinger equation DSE with the IEGACp model in 3-dimensions, using the generalized Bopp's shift method and standard perturbation theory. The organization scheme of the recent work is given as follows: In Sect. 2, we briefly review the ordinary SE with the IEGACp model. Sect. 3 is devoted to studying the DSE by applying the generalized Bopp's shift method for the IEGACp model, by applying standard perturbation theory we find the generalized quantum spectrum for perturbed spin-orbital interaction in the framework of the global group (3DNRQM-NCPS), and then, we derive the magnetic and Fermi spectrums under the IEGACp model. In the Sect. 4, we resume the global spectrum and the corresponding NC Hamiltonian operator for the IEGACp model and the corresponding energy levels of the hydrogen atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$  and  $\text{Be}^+$ , the quark-antiquark systems ( $c\bar{c}$  and  $b\bar{b}$ ) and the diatomic molecules CO, NO, CH and  $\text{N}_2$ . Furthermore, we applied our model to generate the new mass spectra of  $Q\bar{Q}$  ( $Q = b, c$ ) in (3DNRQM-NCPS) symmetries and trade some important special cases in the literature. Finally, in Sect. 5, the achieved results are briefly summarized in the last section.

## II. OVERVIEW OF THE EIGENFUNCTIONS AND EIGENVALUES FOE EGACp MODEL IN NRQM

In this section, we shall recall here the time-independent nonrelativistic Schrödinger equation NRSE for the exponential, generalized Cornell potential model EGACp model, which is an important short-range potential that behaves like a Coulomb potential for small values of and decreases exponentially for large values of Eq. (1). If we insert this potential into the NRSE, we obtain its radial part as follows:

$$\begin{aligned} \frac{d^2 U_{nl}(r)}{dr^2} + \frac{2}{r} \frac{dU_{nl}(r)}{dr} + 2\mu \left( E_{nl} - V_{eff}^{gcp}(r) \right) U_{nl}(r) &= 0, \\ \Rightarrow \frac{d^2 R_{nl}(r)}{dr^2} + 2\mu \left( E_{nl} - V_{nc}^{eff}(r) \right) R_{nl}(r) &= 0, \end{aligned} \quad (7)$$

$$E_{nl} = e + (4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f}) \sqrt{\frac{6a - \left[ 2\mu \left( b + \frac{d\alpha^2}{2} \right) / 4\beta \right]^3}{12\mu}} - 2\mu(c - d)^2 (4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f})^{-2}. \quad (11)$$

Here

$$R_{nl}(r) = \frac{U_{nl}(r)}{r}$$

and

$$V_{eff}^{gcp}(r) = ar^2 + br - \frac{c}{r} + \frac{d \exp(-ar)}{r} + \frac{f}{r^2} + e + \frac{l(l+1)}{r^2}.$$

Which present the effective potential,  $\mu$  is the reduced mass, for the quark-antiquark systems  $Q\bar{Q}$  ( $Q = b, c$ ) the reduced mass  $\mu = \frac{m_Q m_{\bar{Q}}}{m_Q + m_{\bar{Q}}}$  in terms of quark mass  $m_Q$  and antiquark mass  $m_{\bar{Q}}$ .  $E_{nl}$  are the eigenvalues of the exponential, generalized, Cornell potential model while  $n$  and  $l$  are the radials and orbital angular momentum quantum numbers. The two terms  $\left( br - \frac{c}{r} \right)$  are known by Cornell potential,  $\frac{f}{r^2}$  makes the EGACp model more singular and produces better confinement compared to Cornell potential, the quadratic potential  $ar^2$  and the inverse quadratic potential  $\frac{f}{r^2}$  are play a vital role in improving quarkonium properties such as in [3, 16]. The complete wave function  $\Psi_{nlm}(r, \theta, \phi) = U_{nl}(r) Y_l^m(\theta, \phi)$ , as follows [16]:

$$\Psi_{0lm} = a_0 r^L \exp(-ar^2 - \beta r) Y_l^m(\theta, \phi), \quad (8)$$

$$\Psi_{1lm} = (a_0 r^L + a_1 r^{L+2}) \exp(-ar^2 - \beta r) Y_l^m(\theta, \phi), \quad (9)$$

and

$$\Psi_{nlm} = \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right) \exp(-ar^2 - \beta r) Y_l^m(\theta, \phi). \quad (10)$$

Here

$$Y_l^m(\theta, \phi) = \frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!} P_l^{|m|}(\cos(\theta)) e^{im\phi}.$$

is the common rotational wave function in terms of associated Legendre polynomial,

$$L = -\frac{1}{2} + \frac{1}{2} \sqrt{(2l + 1)^2 + 8\mu f},$$

$$\alpha = \sqrt{\frac{\mu}{2} \left( a - \frac{d\alpha^3}{6} \right)},$$

$$\beta = \sqrt{\frac{\mu}{16\alpha^2} \left( b + \frac{d\alpha^2}{2} \right)}.$$

While  $a_n$  are the normalization constants. The energy  $E_{nl}$  of the potential in Eq. (1) is given by [16]:

### III. THE FORMALISM OF THE FORMALISM IEGACp MODEL IN (3DNRQm-NCPS) SYMMETRIES

#### A. Physical model

In this subsection, we devote this part to studying the nonrelativistic IEGACp model  $V_{nc}^{gcp}(r)$ , in (3DNRQm-NCPS) symmetries. To perform this task, the physical form of the DSE, it is necessary to replace the ordinary 3D-Hamiltonian operators  $H_{gcp}(p, x)$ , ordinary energy  $E_{nl}$  and

$$\underbrace{H_{gcp}(p, x)\Psi(\vec{r}) = E_{nl}\Psi(\vec{r})}_{\text{NRQM}} \Rightarrow \underbrace{H_{nc}^{gcp}(p_{nc}, x_{nc}) * \Psi(\vec{r}_{nc}) = E_{nc}^{gcp}\Psi(\vec{r}_{nc})}_{\text{NCPS-symmetries}}$$

Allows us to obtain the deformed radial wave the radial functions  $R_{nl}(r) = \frac{U_{nl}(r)}{r}$  function in (3DNRQm-NCPS) symmetries as follows:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu(E_{nl} - V_{eff}^{gcp}(r)) * R_{nl}(r) = 0. \quad (12)$$

It is established extensively in the literature and a basic text that star products can be simplified by Bopp's shift method, the physicist Fritz Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules  $x \rightarrow x - \frac{i}{2} \frac{\partial}{\partial p}$  and  $p \rightarrow p + \frac{i}{2} \frac{\partial}{\partial x}$  instead of the ordinary correspondence  $x \rightarrow x$  and  $p \rightarrow p + \frac{i}{2} \frac{\partial}{\partial x}$  [18, 42, 69, 70, 71]. In physics literature, this is known as Bopp's shift method. This quantization procedure is known as Bopp quantization. It is known to specialists that Bopp's shift method has been applied effectively and has succeeded in

the corresponding complex wave function  $\Psi(\vec{r})$  in the symmetries of NRQM by 3D-Hamiltonian operators  $H_{nc}^{gcp}(p_{nc}, x_{nc})$ , new unknown values  $E_{nc}^{gcp}$  of energy and the corresponding new complex wave function  $\Psi(\vec{r}_{nc})$ , respectively in (3DNRQm-NCPS) symmetries. Besides, to replace the ordinary product with the star product(\*), this allows us to construct the DSE in (3DNRQm-NCPS) symmetries as (see, e.g., [25, 26, 27, 28, 29, 30, 31, 32]):

simplifying the four fundamental equations, the first one is the nonrelativistic deformed Schrödinger equation NRDSE [27, 28, 29, 30, 31, 32, 51], the second is relativistic deformed Klein-Gordon equation RDKGE [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75], the third is relativistic deformed Dirac equation RDDE [76, 77, 78, 79], and the last is the deformed relativistic Duffin-Kemmer-Petiau equation DRDKPE [80] with the notion of star product to the NRSE, RKGE, relativistic Dirac equation RDE and RDKPE with the notion of ordinary product, respectively. Thus, Bopp's shift method is based on reducing second-order linear differential equations of DNRSE, DRKGE, DRDE, and DRDKPE with star products to second-order linear differential equations of NRSE, RKGE, RDE, and RDKPE without star products with simultaneous translation in the space-space. The CNCCRs with star product in Eqs. (3a), (3b), and (3c) become new CNCCRs without the notion of the star product as follows (see, e.g., [27, 28, 29, 30, 31, 32, 39, 51]):

$$\begin{cases} [x_i^{nc}, p_j^{nc}] = [x_{Hi}^{nc}(t), p_{Hj}^{nc}(t)] = [x_{li}^{nc}(t), p_{lj}^{nc}(t)] = i\hbar_{eff}\delta_{ij}, \\ [x_i^{nc}, x_j^{nc}] = [x_{Hi}^{nc}(t), x_{Hj}^{nc}(t)] = [x_{li}^{nc}(t), x_{lj}^{nc}(t)] = i\theta_{ij}, \\ [p_i^{nc}, p_j^{nc}] = [p_{Hi}^{nc}(t), p_{Hj}^{nc}(t)] = [p_{li}^{nc}(t), p_{lj}^{nc}(t)] = i\bar{\theta}_{ij}. \end{cases} \quad (13)$$

The generalized positions and momentum coordinates  $(x_i^{nc}, p_i^{nc})$  in (3DNRQm-NCPS) symmetries depend on the corresponding usual generalized positions and momentum coordinates  $(x_i, p_i)$  in NRQM by the following, respectively (see, e.g., [18, 42, 69, 70, 71]):

$$(x_i, p_i) \rightarrow (x_i^{nc}, p_i^{nc}) = \left(x_i - \frac{\theta_{ij}}{2} p_j, p_i + \frac{\bar{\theta}_{ij}}{2} x_j\right). \quad (14)$$

The above equation allows us to obtain two operators  $r_{nc}^2$  and  $p_{nc}^2$  in (3DNRQm-NCPS) symmetries (see, e.g., [27, 28, 29, 30, 31, 32, 39, 51]):

$$(r^2, p^2) \rightarrow (r_{nc}^2, p_{nc}^2) = (r^2 - \mathbf{L}\Theta, p^2 + \mathbf{L}\bar{\Theta}). \quad (15)$$

The two couplings  $\mathbf{L}\Theta$  and  $\mathbf{L}\bar{\Theta}$  are given by:

$$\mathbf{L}\Theta = L_x \theta_{12} + L_y \theta_{23} + L_z \theta_{13},$$

$$\mathbf{L}\bar{\Theta} = L_x \bar{\theta}_{12} + L_y \bar{\theta}_{23} + L_z \bar{\theta}_{13}.$$

Here( $L_x, L_y$  and  $L_z$ ) are the three components of the angular momentum operator while  $\theta_{ij} = \theta_{ijj}/2$ . Thus, the reduced radial Schrödinger equation (without star product) can be written as:

$$\left[\frac{d^2}{dr^2} + 2\mu(E_{nl} - V_{eff}^{gcp}(r_{nc}))\right] R_{nl}(r) = 0. \quad (16)$$

The new effective potential  $V_{eff}^{gcp}(r_{nc})$  and the Hamiltonian operator  $H_{nc}^{gcp}(p_{nc}, x_{nc})$  for the IEGACp model can be expressed as:

$$V_{eff}^{gcp}(r_{nc}) = V_{nc}^{gcp}(r_{nc}) + \frac{l(l+1)}{r_{nc}^2} \quad (17a)$$

$$H_{nc}^{gcp}(p_{nc}, x_{nc}) =$$

$$H \left( x_i^{nc} = x_i - \frac{\theta_{ij}}{2} p_j, p_i^{nc} = p_i + \frac{\bar{\theta}_{ij}}{2} x_j \right). \quad (17b)$$

Now, we want to find the IEGACp model  $V_{nc}^{gcp}(r_{nc})$  in (3DNRQm-NCPS) symmetries:

$$V_{nc}^{gcp}(r_{nc}) = ar_{nc}^2 + br_{nc} - \frac{c}{r_{nc}} + \frac{d \exp(-ar_{nc})}{r_{nc}} + \frac{f}{r_{nc}^2} + e. \quad (18)$$

After straightforward calculations, we can obtain the important terms  $ar_{nc}^2$ ,  $br_{nc}$ ,  $-\frac{c}{r_{nc}}$ ,  $\frac{d \exp(-ar_{nc})}{r_{nc}}$ ,  $\frac{f}{r_{nc}^2}$  and  $\frac{l(l+1)}{r_{nc}^2}$  which will be used to determine the IEGACp model in Eq. (18) and the new effective potential  $V_{eff}^{gcp}(r_{nc})$  in (3DNRQm-NCPS) symmetries as:

$$\left\{ \begin{array}{l} ar_{nc}^2 = ar^2 - a\mathbf{L}\Theta + O(\Theta^2), \\ br_{nc} = br - \frac{b\mathbf{L}\Theta}{2r} + O(\Theta^2), \\ d \exp(-ar_{nc}) = d \exp(-ar) + \frac{ad}{2r} \exp(-ar)\mathbf{L}\Theta + O(\Theta^2), \\ \frac{l(l+1)}{r_{nc}^2} = \frac{l(l+1)}{r^2} + \frac{l(l+1)}{r^4} \mathbf{L}\Theta + O(\Theta^2). \end{array} \right. \quad (19)$$

For diatomic molecules CO, NO, CH, and N<sub>2</sub> in ordinary quantum mechanics while  $H_{pert}^{gcp}(p, x)$  composed of seven terms proportional to two infinitesimal parameters ( $\theta$  and  $\bar{\theta}$ ) and then we can be considered as perturbation terms  $H_{pert}^{gcp}(p, x)$  in (3DNRQm-NCPS) symmetries. The first part:

$$\left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L}\Theta,$$

in Eq. (23a) describes the influence of the topological properties on the EGACp model related to the noncommutative space NCS while the second part  $\frac{\mathbf{L}\bar{\theta}}{2\mu}$  related to the noncommutative phase NCP which results from the topological effects on the kinematic term of the main Hamiltonian  $H_{gcp}(p, x)$ , thus, the Hamiltonian operator while the generated part  $H_{pert}^{gcp}(p, x)$  appears as a result of the deformation of the noncommutativity space phase. We can

$$H_{So}^{gcp} = \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \mathbf{L}\mathbf{S} \begin{cases} \varepsilon \text{ for the diatomic molecules} \\ g_s \text{ for the Heavy-light mesons} \end{cases} \quad (24)$$

here

$$\theta = (\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2)^{\frac{1}{2}},$$

$$\text{and } \bar{\theta} = \left( \bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2 \right)^{\frac{1}{2}},$$

while  $\mathbf{L}\mathbf{S}$  is the spin-orbit coupling,  $\varepsilon \approx \frac{1}{137}$  is the atomic fine structure constant,  $g_s$  is the strong coupling constant and  $S$

This gives

$$\frac{d \exp(-ar_{nc})}{r_{nc}} = \frac{d \exp(-ar)}{r} + \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} \right) \mathbf{L}\Theta + O(\Theta^2). \quad (20)$$

By making the substitution above Eqs (19a), (19b), (19c), (19d) and (20) into Eqs. (17) and (18), we find the global working Hamiltonian operator  $H_{nc}^{gcp}(p_{nc}, x_{nc})$  satisfies the equation in (3DNRQm-NCPS) symmetries:

$$\begin{cases} H_{nc}^{gcp}(p_{nc}, x_{nc}) = H_{gcp}(p, x) + H_{pert}^{gcp}(p, x), \\ V_{eff}^{gcp}(r_{nc}) = V_{eff}^{gcp}(r) + V_{pe-ef}^{gcp}(r). \end{cases} \quad (21)$$

The operators  $H_{gcp}(p, x)$ ,  $H_{pert}^{gcp}(p, x)$  and  $V_{pe-ef}^{gcp}(r)$  are given by:

$$H_{gcp}(p, x) = \frac{p^2}{2\mu} + ar^2 + br - \frac{c}{r} + \frac{d \exp(-ar)}{r} + \frac{f}{r^2} + e, \quad (22)$$

and

$$\begin{cases} H_{pert}^{gcp}(p, x) = \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L}\Theta + \frac{\mathbf{L}\bar{\theta}}{2\mu}, \\ V_{pe-ef}^{gcp}(r) = \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L}\Theta. \end{cases} \quad (23)$$

disregard the Haigh terms on ( $\theta$  and  $\bar{\theta}$ ) because we are interested in the correction of the first order of these parameters. We have included the topological effects of NCS on the centrifugal term  $\frac{l(l+1)}{r^2}$  within the perturbed Hamiltonian  $H_{pert}^{gcp}(p, x)$  to be able to apply perturbation theory to find the energetic corrections.

### B Spin-orbit Hamiltonian operator for hydrogenic atoms, diatomic molecules, and heavy-light mesons under the IEGACp model

In this subsection, we want to derive the physical form of the induced Hamiltonian  $H_{pert}^{gcp}$  due to the effect of space-phase non-commutativity. To achieve this goal, we replace  $\mathbf{L}\Theta$  both and  $\mathbf{L}\bar{\theta}$  by useful physical forms ( $\varepsilon\theta\mathbf{L}\mathbf{S}$  or  $g_s\theta\mathbf{L}\mathbf{S}$ ) and ( $\varepsilon\bar{\theta}\mathbf{L}\mathbf{S}$  or  $g_s\bar{\theta}\mathbf{L}\mathbf{S}$ ), respectively (see, e.g., [27, 28, 29, 30, 31, 32, 39, 51]):

denotes the spin of the diatomic molecules (CO, NO, CH and N<sub>2</sub>) or heavy-light mesons ( $c\bar{c}$  and  $b\bar{b}$ ). Thus, the spin-orbit interactions  $H_{So}^{gcp}$  appear automatically as a result of the deformation of the space phase Now, physically, we can rewrite the quantum spin-orbit  $\mathbf{L}\mathbf{S}$  coupling as follows:

$$J = L + S \Rightarrow 2\mathbf{L}\mathbf{S} = J^2 - L^2 - S^2. \quad (25)$$

Here  $J$  is the total momentum of the diatomic molecule such as CO, NO, CH and N<sub>2</sub>. Our recent study can apply to three cases:

➤ The first case considers  $c = Ze$ ,  $Z$  and  $e$  are the atomic numbers and the charge of the electron, the term  $(-\frac{c}{r})$  becomes an attractive Colombian potential, thus, we can consider the Hamiltonian described hydrogenic atoms He<sup>+</sup>, Li<sup>2+</sup>, and Be<sup>+</sup> under the influence of external fields described by other terms  $(ar^2 + br + \frac{d \exp(-ar)}{r} + \frac{f}{r^2} + e)$  in ordinary QM and its extension (3DNRQm-NCPS) symmetries. which allows us to the eigenvalues of the total operator  $J$  that can be

$$\left\{ \begin{aligned} (H_{so}^{u-gcp})_{11} &= \varepsilon k_+ \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \quad \text{if } j = l + \frac{1}{2} \\ (H_{so}^{u-gcp})_{22} &= \varepsilon k_- \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \\ (H_{so}^{np-gcp})_{33} &= 0 \end{aligned} \right. \quad (27)$$

The non-null diagonal elements  $(H_{so}^{gcp})_{11}$  and  $(H_{so}^{gcp})_{22}$  of the perturbed Hamiltonian operator  $H_{pert}^{gcp}(p, x)$  can be

$$\Delta E_{n-so}^{u-gcp} = \langle \Psi | (H_{pert}^{u-gcp})_{11} | \Psi \rangle \text{ and } \Delta E_{n-so}^{d-gcp} = \langle \Psi | (H_{pert}^{d-gcp})_{22} | \Psi \rangle. \quad (28)$$

The second case for the Heavy-light mesons (HLM) for example scalar, vector, pseudoscalar, and pseudovector for  $(B, B_s, D$  and  $D_s)$  mesons, or the heavy quarkonium systems  $c\bar{c}$  and  $b\bar{b}$ , which are consists of quarks and antiquarks of the same system  $Q\bar{Q}$  ( $Q = b, c$ ), the eigenvalues of the spin-orbit coupling operator  $LS$  are  $k(j, l, s) = j(j + 1) - l(l + 1) -$

$$\left\{ \begin{aligned} (H_{so}^{gcp})_{11} &= g_s k_1 \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \quad \text{if } j = l + 1, \\ (H_{so}^{gcp})_{22} &= g_s k_2 \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \quad \text{if } j = l, \\ (H_{so}^{gcp})_{33} &= g_s k_3 \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \quad \text{if } j = l - 1. \end{aligned} \right. \quad (29)$$

here  $g_s$  denote to the strong constant coupling and  $2(k_1, k_2, k_3) \equiv (l, -2, -2l - 2)$ . The non-null diagonal elements  $(H_{so}^{gcp})_{11}$ ,  $(H_{so}^{gcp})_{22}$  and  $(H_{so}^{gcp})_{33}$  of the perturbed Hamiltonian operator  $H_{pert}^{gcp}(p, x)$  can be influenced by the energy values  $E_{nl}$  by creating three new values:

$$\left\{ \begin{aligned} \Delta E_{n-g}^{gcp} &= \langle \Psi | (H_{so}^{gcp})_{11} | \Psi \rangle, \\ \Delta E_{n-gm}^{gcp} &= \langle \Psi | (H_{so}^{gcp})_{22} | \Psi \rangle, \\ \Delta E_{n-l}^{gcp} &= \langle \Psi | (H_{so}^{gcp})_{33} | \Psi \rangle. \end{aligned} \right. \quad (30)$$

➤ In the third case for the diatomic molecules CO, NO, CH, and N<sub>2</sub>, the eigenvalues of the spin-orbit coupling operator  $LS$  are:

$$k(j, l, s) = j(j + 1) - l(l + 1) - s(s + 1). \\ V_{pe-ef}^{gcp}(r) = \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] LS \begin{cases} \varepsilon \text{ for the diatomic molecules and hydrogenic atoms} \\ g_s \text{ for the Heavy-light mesons.} \end{cases} \quad (33)$$

obtained from the interval  $|l - 1/2| \leq j \leq |l + 1/2|$ . We have an occasion of determining two-sided bounds on the eigenvalues of the operator  $G^2 = J^2 - L^2 - S^2$  as follows:

$$k(j, l, s) = j(j + 1) - l(l + 1) - s(s + 1), \\ = \begin{cases} k_-(j = l - 1/2, l, s) \text{ for spin-down}, \\ k_+(j = l + 1/2, l, s) \text{ for spin-down} \end{cases} \quad (26)$$

➤ A second occasion of determining a diagonal matrix  $H_{so}^{gcp}$  of order  $(3 \times 3)$  with diagonal elements  $(H_{so}^{u-gcp})_{11}$ ,  $(H_{so}^{d-gcp})_{22}$  and  $(H_{so}^{np-gcp})_{33}$  as:

influenced by the energy values  $E_{nl}$  by creating three new values:

$s(s + 1)$  corresponding  $j = l + 1$  (spin great),  $j = l$  (spin middle) and  $j = l - 1$  (spin little), respectively. Then, one can form a diagonal matrix for modified nonrelativistic quark-antiquark potential with the diagonal element, and in (3DNRQm-NCPS) symmetries:

The perturbed Hamiltonian operator  $H_{pert}^{gcp}(p, x)$  can be influenced by the energy values  $E_{nl}$  by creating three new values:

$$\Delta E_{n-mol}^{gcp} = \langle \Psi | H_{pert}^{gcp}(p, x) | \Psi \rangle. \quad (31)$$

After straightforward calculation, the radial functions  $R_{nl}(r) = \frac{U_{nl}(r)}{r}$  satisfy the following differential equation in (3DNRQm-NCPS) symmetry for the hydrogenic atoms and the heavy quarkoniums systems under the IEGACp model:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu \left( E_{nl} - \left( V_{eff}^{gcp}(r) + V_{pe-ef}^{gcp}(r) \right) \right) R_{nl}(r) = 0, \quad (32)$$

with

We introduced the generalized effective potential  $V_{nc-eff}^{gcp}(r)$  in (3DNRQm-NCPS) symmetries. We have seen previously that the induced spin-orbit  $H_{so}^{gcp}(p, x)$  is infinitesimal compared to the principal Hamiltonian operator  $H_{gcp}(p, x)$  in the (3DNRQm-NCPS) symmetries for hydrogenic atoms  $He^+$ ,  $Li^{+2}$ , and  $Be^+$ , the heavy quarkonium systems  $c\bar{c}$  and  $b\bar{b}$ , and the diatomic molecules CO, NO, CH, and  $N_2$  under the IEGACp model. This allows us to apply

$$H_{so}^{gcp} = \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f+l(l+1)}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \mathbf{G}^2 \begin{cases} \varepsilon \text{ for the diatomic molecules and hydrogenic atoms} \\ g_s \text{ for the Heavy-light mesons.} \end{cases} \quad (34)$$

**C Bound State Solutions for the perturbed spin-orbit operator for hydrogenic atoms, heavy quarkonium systems, and some diatomic molecules under the IEGACp model**

From our observation of the new Hamiltonian operator in Eq. (21), it can be said that the EGACp potential is extended by including new radial terms:

$$\left( \frac{d \exp(-ar)}{r^3}, \frac{ad \exp(-ar)}{2r^2}, \frac{f+l(l+1)}{r^4}, \frac{c}{r^3} \text{ and } \frac{b}{r} \right),$$

to become an IEGACp model in (3DNRQm-NCPS) symmetries. In addition, the ordinary kinetic energy term  $\frac{p^2}{2\mu}$  has been changed to add to it a new term  $\frac{\mathbf{L}\bar{\theta}}{2\mu}$  accompanying the perturbed potential term:

standard perturbation theory to determine the nonrelativistic energy corrections at the first order of two infinitesimal parameters  $\theta$  and  $\bar{\theta}$  due to noncommutativity space-phase properties. Thus, the induced perturbed spin-orbit operator can be carried out for the three cases referred to through the substitution of Eq. (25) into Eq. (24) yields:

$$\left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L}\bar{\theta}.$$

We note clearly that the expression of the global additive part  $H_{pert}^{gcp}(p, x)$  (Eq. (23)) of the new Hamiltonian operator  $H_{nc}^{gcp}(p, x)$  is also proportional to the infinitesimal vectors  $\theta$  and  $\bar{\theta}$ . This allows us to consider the additive part  $H_{pert}^{gcp}(p, x)$  as a perturbation potential compared with the main potential  $H_{gcp}(p, x)$  (Eq. (22)) in the symmetries of (3DNRQm-NCPS), that is, the inequality  $H_{pert}^{gcp}(p, x) \ll H_{gcp}(p, x)$  has to become achieved. That is all physical justification for applying the time-independent perturbation theory to become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized  $n^{th}$  excited states. Now, we apply the perturbative theory, in the case of (3DNRQm-NCPS), we find the expectation values of the radial terms:  $\frac{\exp(-ar)}{r^3}$ ,  $\frac{\exp(-ar)}{r^2}$ ,  $\frac{1}{r^4}$ ,  $\frac{1}{r^3}$  and  $\frac{1}{r}$  taking into account the wave function which we have seen previously in Eq. (11). Thus, after straightforward calculations, we obtain the following results:

$$\langle n, l, m | \frac{\exp(-ar)}{r^3} | n, l, m \rangle = \int_0^{+\infty} \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \frac{r^2 \exp(-ar)}{r^3} dr, \quad (35)$$

$$\langle n, l, m | \frac{\exp(-ar)}{r^2} | n, l, m \rangle = \int_0^{+\infty} \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \frac{r^2 \exp(-ar)}{r^2} dr, \quad (36)$$

$$\langle n, l, m | \frac{1}{r^4} | n, l, m \rangle = \int_0^{+\infty} \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \frac{r^2}{r^4} dr, \quad (37)$$

$$\langle n, l, m | \frac{1}{r^3} | n, l, m \rangle = \int_0^{+\infty} \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \frac{r^2}{r^3} dr, \quad (38)$$

$$\langle n, l, m | \frac{1}{r} | n, l, m \rangle = \int_0^{+\infty} \left( \sum_{n=0}^{\infty} a_n r^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \frac{r^2}{r} dr. \quad (39)$$

We have applied the property of the spherical harmonics, which has the form:

$$\iint Y_l^m Y_{l'}^{m'} \sin(\theta) d\theta d\varphi = \delta_{ll'} \delta_{mm'}.$$

$$\left\langle \frac{\exp(-ar)}{r^3} \right\rangle_{(0,l,m)} = a_0 \int_0^{+\infty} r^{2L-1} \exp(-2ar^2 - \eta r) dr, \quad (40)$$

$$\left\langle \frac{\exp(-ar)}{r^2} \right\rangle_{(0,l,m)} = a_0 \int_0^{+\infty} r^{2L} \exp(-2ar^2 - \eta r) dr, \quad (41)$$

For relieving the burden of writing, we will provide useful abbreviations  $\langle n, l, m | K | n, l, m \rangle \equiv \langle K \rangle_{(n,l,m)}$ . For the ground state  $n = 0$ , the above expectation values in Eqs. (35), (36), (37), (38) and (39) reduce to the following simple form:



$$\left\langle \frac{1}{r^4} \right\rangle_{(0,l,m)} = a_0 \int_0^{+\infty} r^{2L-2} \exp(-2\alpha r^2 - 2\beta r) dr, \quad (42)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(0,l,m)} = a_0 \int_0^{+\infty} r^{2L-1} \exp(-2\alpha r^2 - 2\beta r) dr, \quad (43)$$

and

$$\left\langle \frac{1}{r} \right\rangle_{(0,l,m)} = a_0 \int_0^{+\infty} r^{2L+1} \exp(-2\alpha r^2 - 2\beta r) dr. \quad (44)$$

Comparing Eqs. (40), (41), (42), (43) and (44) with the integral of the form [81]:

$$\int_0^{+\infty} x^{\gamma-1} \exp(-\lambda x^2 - \beta x) dx (2\lambda)^{-\frac{\gamma}{2}} \Gamma(\gamma) \exp\left(\frac{\beta^2}{8\lambda}\right) D_{-\gamma}\left(\frac{\beta}{\sqrt{2\lambda}}\right), \quad (37)$$

Where  $D_{-\gamma}\left(\frac{\beta}{\sqrt{2\lambda}}\right)$  and  $\Gamma(\gamma)$  denotes the parabolic cylinder functions and Gamma function, respectively. After straightforward calculations, we can obtain explicit results:

$$\left\langle \frac{\exp(-\alpha r)}{r^3} \right\rangle_{(0,l,m)} = a_0 (4\alpha)^{-L} \Gamma(2L) \exp\left(\frac{\eta^2}{16\alpha}\right) D_{-L}\left(\frac{2\beta+\alpha}{\sqrt{4\alpha}}\right), \quad (45)$$

$$\left\langle \frac{\exp(-\alpha r)}{r^2} \right\rangle_{(0,l,m)} = a_0 (4\alpha)^{-\frac{2L+1}{2}} \Gamma(2L+1) \exp\left(\frac{\eta^2}{16\alpha}\right) D_{-(2L+1)}\left(\frac{2\beta+\alpha}{\sqrt{4\alpha}}\right), \quad (46)$$

$$\left\langle \frac{1}{r^4} \right\rangle_{(0,l,m)} = a_0 (4\alpha)^{-\frac{2L-1}{2}} \Gamma(2L-1) \exp\left(\frac{\beta^2}{4\alpha}\right) D_{-(2L-1)}\left(\frac{\beta}{\sqrt{\alpha}}\right), \quad (47)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(0,l,m)} = a_0 (4\alpha)^{-L} \Gamma(2L) \exp\left(\frac{\beta^2}{4\alpha}\right) D_{-2L}\left(\frac{\beta}{\sqrt{\alpha}}\right), \quad (48)$$

and

$$\left\langle \frac{1}{r} \right\rangle_{(0,l,m)} = a_0 (4\alpha)^{-(L+1)} \Gamma(2L+2) \exp\left(\frac{\beta^2}{4\alpha}\right) D_{-(2L+2)}\left(\frac{\beta}{\sqrt{\alpha}}\right). \quad (49)$$

For the first excited state  $n = 1$ , the expectation values in Eqs. (35), (36), (37), (38) and (39) are reduced to the following simple form:

$$\left\langle \frac{\exp(-\alpha r)}{r^3} \right\rangle_{(1,l,m)} = \int_0^{+\infty} (a_0 r^{2L-1} + a_1 r^{2L+3} + 2a_0 a_1 r^{2L+1}) \exp(-2\alpha r^2 - (2\beta + \alpha)r) dr, \quad (50)$$

$$\left\langle \frac{\exp(-\alpha r)}{r^2} \right\rangle_{(1,l,m)} = \int_0^{+\infty} (a_0 r^{2L} + a_1 r^{2L+4} + 2a_0 a_1 r^{2L+2}) \exp(-2\alpha r^2 - (2\beta + \alpha)r) dr, \quad (51)$$

$$\left\langle \frac{1}{r^4} \right\rangle_{(1,l,m)} = \int_0^{+\infty} (a_0 r^{2L-2} + a_1 r^{2L+2} + 2a_0 a_1 r^{2L}) \exp(-2\alpha r^2 - 2\beta r) dr, \quad (52)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(1,l,m)} = \int_0^{+\infty} (a_0 r^{2L-1} + a_1 r^{2L+3} + 2a_0 a_1 r^{2L+1}) \exp(-2\alpha r^2 - 2\beta r) dr, \quad (53)$$

and

$$\left\langle \frac{1}{r} \right\rangle_{(1,l,m)} = \int_0^{+\infty} (a_0 r^{2L+1} + a_1 r^{2L+5} + 2a_0 a_1 r^{2L+3}) \exp(-2\alpha r^2 - 2\beta r) dr. \quad (54)$$

Comparing Eqs. (51), (52), (53) and (54) with the integral of Eq. (37), we obtain the following results:

$$\left\langle \frac{\exp(-\alpha r)}{r^3} \right\rangle_{(1,l,m)} \exp(-Y) (4\alpha)^L = a_0 \Gamma(2L) D_{-2L}(F) + a_1 (4\alpha)^{-2} \Gamma(2L+4) D_{-(2L+4)}(F) + 2a_0 a_1 (4\alpha)^{-1} \Gamma(2L+2) D_{-(2L+2)}(F), \quad (40a)$$

$$\left\langle \frac{\exp(-\alpha r)}{r^2} \right\rangle_{(1,l,m)} \exp(-Y) (4\alpha)^L = a_0 (4\alpha)^{-\frac{1}{2}} \Gamma(2L+1) D_{-(2L+1)}(F) + a_1 (4\alpha)^{-\frac{5}{2}} \Gamma(2L+5) D_{-(2L+5)}(F) + 2a_0 a_1 (4\alpha)^{-\frac{3}{2}} \Gamma(2L+3) D_{-(2L+3)}(F), \quad (55)$$

$$\left\langle \frac{1}{r^4} \right\rangle_{(1,l,m)} \exp\left(-\frac{\beta^2}{4\alpha}\right) (4\alpha)^L = a_0 (4\alpha)^{\frac{1}{2}} \Gamma(2L-1) D_{-(2L-1)}(\Omega) + a_1 (4\alpha)^{\frac{3}{2}} \Gamma(2L+3) D_{-(2L+3)}(\Omega) + 2a_0 a_1 (4\alpha)^{-\frac{1}{2}} \Gamma(2L+1) D_{-(2L+1)}(\Omega), \quad (56)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(1,l,m)} \exp\left(-\frac{\beta^2}{4\alpha}\right) (4\alpha)^L = a_0 \Gamma(2L) D_{-2L}(\Omega) + a_1 (4\alpha)^{-2} \Gamma(2L+4) D_{-(2L+4)}(\Omega) + 2a_0 a_1 (4\alpha)^{-1} \Gamma(2L+2) D_{-(2L+2)}(\Omega), \quad (57)$$

and

$$\left\langle \frac{1}{r} \right\rangle_{(1,l,m)} \exp\left(-\frac{\beta^2}{4\alpha}\right) (4\alpha)^L = a_0(4\alpha)^{-1} \Gamma(2L+2) D_{-(2L+2)}(\Omega) + a_1(4\alpha)^{-3} \Gamma(2L+6) D_{-(2L+6)}(\Omega) + 2a_0 a_1 (4\alpha)^{-2} \Gamma(2L+4) D_{-(2L+4)}(\Omega). \quad (58)$$

with  $F \equiv \frac{2\beta+\alpha}{\sqrt{4\alpha}}$ ,  $\Omega \equiv \frac{\beta}{\sqrt{\alpha}}$  and  $Y \equiv \frac{(2\beta+\alpha)^2}{16\alpha}$ . The principal goal of this subsection is to determine the energy spectrum

$$\Delta E_{n-so}^{u-gcp}(k_+, n, a, b, c, d, e, f, \alpha, j, l, s) \equiv \Delta E_{n-so}^{u-gcp},$$

And

$$\Delta E_{n-so}^{d-gcp}(k_-, n, a, b, c, d, e, f, \alpha, j, l, s) \equiv \Delta E_{n-so}^{d-gcp}.$$

which produced to  $H_{so}^{gcp}$  correspond to  $j = l + \frac{1}{2}$  and  $j = l - \frac{1}{2}$  at the first order of two parameters  $\theta$  and  $\bar{\theta}$  for hydrogenic atoms for  $(n, l)$  states by applying standard perturbation theory and through the structure constants which specified the dimensionality of the IEGACp model of hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$ . Thus, we obtain the following results for the ground state and first excited state, respectively:

$$\begin{cases} \Delta E_{0-so}^{u-gcp} = \varepsilon k_+ \left( \eta(0, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + \frac{1}{2}, \\ \Delta E_{0-so}^{d-gcp} = \varepsilon k_- \left( \eta(0, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - \frac{1}{2} \end{cases} \quad (59)$$

This allows us to generalize the above results to the case of  $n^{\text{th}}$  excited states in (3DNRQm-NCPS) symmetries for the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$  as follows:

$$\begin{cases} \Delta E_{n-so}^{u-gcp} = \varepsilon k_+ \left( \eta(n, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + \frac{1}{2}, \\ \Delta E_{n-so}^{d-gcp} = \varepsilon k_- \left( \eta(n, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1/2. \end{cases} \quad (60)$$

For the heavy quarkonium systems, such as charmonium  $c\bar{c}$  and bottomonium  $b\bar{b}$ , which quarks and antiquarks of the same system  $Q\bar{Q}$ , the eigenvalues of the spin-orbit coupling, we obtain the following results, for the ground state and first excited state, respectively:

$$\eta(n, a, b, c, f, d) = d \left\langle \frac{\exp(-ar)}{r^3} \right\rangle_{(n,l,m)} + \frac{\alpha d}{2} \left\langle \frac{\exp(-ar)}{r^2} \right\rangle_{(n,l,m)} + (f + l(l+1)) \left\langle \frac{1}{r^4} \right\rangle_{(n,l,m)} - \frac{c}{2} \left\langle \frac{1}{r^3} \right\rangle_{(n,l,m)} - \frac{b}{2} \left\langle \frac{1}{r} \right\rangle_{(n,l,m)} - a. \quad (66)$$

For diatomic molecules CO, NO, CH, and  $\text{N}_2$ , the eigenvalues  $\Delta E_{0-mol}^{gcp}$  and  $\Delta E_{1-mol}^{gcp}$  for the ground state and first excited state, which are produced by the effect of spin-orbit coupling **LS**, we obtain the following results, respectively:

$$\begin{cases} \Delta E_{0-mol}^{gcp} = \varepsilon k(l) \left( \eta(0, r_e, D_e) \theta + \frac{\bar{\theta}}{2\mu} \right), \\ \Delta E_{1-mol}^{gcp} = \varepsilon k(l) \left( \eta(1, r_e, D_e) \theta + \frac{\bar{\theta}}{2\mu} \right). \end{cases} \quad (67)$$

$$\Delta E_{1-so}^{u-gcp} = \varepsilon k_+ \left( \eta(1, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + \frac{1}{2}, \quad (61)$$

and

$$\Delta E_{1-so}^{d-gcp} = \varepsilon k_- \left( \eta(1, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - \frac{1}{2}. \quad (62)$$

This allows us to generalize the above results to the case of  $n^{\text{th}}$  excited states in (3DNRQm-NCPS) symmetries for the heavy quarkonium systems  $Q\bar{Q}$  ( $Q = b, c$ ) as follows:

$$\begin{cases} \Delta E_{0-g}^{gcp} = g_s k_1(l) \left( \eta(0, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \\ \Delta E_{0-m}^{gcp} = g_s k_2(l) \left( \eta(0, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l, \\ \Delta E_{0-l}^{gcp} = g_s k_3(l) \left( \eta(0, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1. \end{cases} \quad (63)$$

$$\begin{cases} \Delta E_{1-g}^{gcp} = g_s k_1(l) \left( \eta(1, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \\ \Delta E_{0-m}^{gcp} = g_s k_2(l) \left( \eta(1, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l, \\ \Delta E_{0-l}^{gcp} = g_s k_3(l) \left( \eta(1, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1. \end{cases} \quad (64)$$

and

$$\begin{cases} \Delta E_{n-g}^{gcp} = g_s k_1(l) \left( \eta(n, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \\ \Delta E_{n-m}^{gcp} = g_s k_2(l) \left( \eta(n, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l, \\ \Delta E_{n-l}^{gcp} = g_s k_3(l) \left( \eta(n, a, b, c, f, d) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1. \end{cases} \quad (65)$$

with

This allows us to generalize the above results to the case of  $n^{\text{th}}$  excited states in (3DNRQm-NCPS) symmetries for diatomic molecules CO, NO, CH, and  $\text{N}_2$  as follows:

$$E_{n-mol}^{gcp} = \epsilon k(l) \left( \eta(n, r_e, D_e) \theta + \frac{\bar{\theta}}{2\mu} \right). \quad (68)$$

Four diatomic molecules CO, NO, CH and N<sub>2</sub>, Rani and Chand were selected and adjusted with the potential parameters as [15,33]:

$$\eta(n, r_e, D_e) = -2D_e r_e \left\langle \frac{\exp(-\alpha r)}{r^3} \right\rangle_{(n,l,m)} - \alpha D_e r_e \left\langle \frac{\exp(-\alpha r)}{r^2} \right\rangle_{(n,l,m)} + (D_e r_e^2 + l(l+1)) \left\langle \frac{1}{r^4} \right\rangle_{(n,l,m)} - \frac{D_e r_e}{2} \left\langle \frac{1}{r^3} \right\rangle_{(n,l,m)} - \frac{b D_e}{2} \left\langle \frac{1}{r} \right\rangle_{(n,l,m)} - a. \quad (70)$$

### D Bound state solution for modified Zeeman effect for IEGACp model

In this subsection, having obtained the energy spectrum ( $\Delta E_{n-so}^{u-gcp}$  and  $\Delta E_{n-so}^{d-gcp}$ ) which produced to  $H_{so}^{gcp}(p, x)$  correspond to  $j = l + 1/2$  and  $j = l - 1/2$  at the first order of two parameters  $\theta$  and  $\bar{\theta}$  for hydrogenic atoms, He<sup>+</sup>, Li<sup>+2</sup>, and Be<sup>+</sup> for  $(n, l)$  states, the degenerated energy ( $\Delta E_{n-g}^{gcp}$ ,  $\Delta E_{n-l}^{gcp}$ ,  $\Delta E_{n-m}^{gcp}$ ) of the heavy quarkonium systems, such as charmonium  $c\bar{c}$  and bottomonium  $b\bar{b}$  and the energy  $\Delta E_{n-mol}^{gcp}$  of the diatomic molecules CO, NO, CH and N<sub>2</sub>. Now, it is possible to obtain the second automatic symmetry for the

$$\left[ \left( \frac{d \exp(-\alpha r)}{2r^3} + \frac{ad \exp(-\alpha r)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \theta + \frac{\bar{\theta}}{2\mu} \right] \mathbf{L} \mathbf{O} \Rightarrow \left[ \left( \frac{d \exp(-\alpha r)}{2r^3} + \frac{ad \exp(-\alpha r)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \sigma + \frac{\bar{\sigma}}{2\mu} \right] \mathbf{K} L_z \quad (72)$$

This allowed us to derive the modified magnetic Hamiltonian operator  $H_z^{gcp}(r, \sigma, \bar{\sigma})$  for previous hydrogenic atoms under

$$H_z^{gcp}(r, \sigma, \bar{\sigma}) = \left[ \left( \frac{d \exp(-\alpha r)}{2r^3} + \frac{ad \exp(-\alpha r)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \sigma + \frac{\bar{\sigma}}{2\mu} \right] H_{mod}^z. \quad (73)$$

Here  $H_{mod}^z = \mathbf{K} \mathbf{J} - H_z$  denote to Zeeman effect in (3DNRQm-NCPS), while  $H_z = -\mathbf{K} \mathbf{S}$  is just the ordinary Zeeman effect. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state and  $n^{th}$  excited states of the hydrogenic atoms He<sup>+</sup>, Li<sup>+2</sup>, and Be<sup>+</sup>, the heavy quarkonium systems, and the diatomic molecules under the IEGACp model ( $\Delta E_{0-m}^{hy-GCp}$ ,  $\Delta E_{1-m}^{hy-GCp}$ ,  $\Delta E_{n-m}^{hy-GCp}$ ), ( $\Delta E_{0-m}^{hlm-GCp}$ ,  $\Delta E_{1-m}^{hlm-GCp}$ ,  $\Delta E_{n-m}^{hlm-GCp}$ ) and ( $\Delta E_{0-m}^{mo-GCp}$ ,  $\Delta E_{1-m}^{mo-GCp}$ ,  $\Delta E_{n-m}^{mo-GCp}$ ) of the diatomic molecules CO, NO, CH and N<sub>2</sub>, we just replace ( $k_+$  or  $k(l)$ ) and ( $\theta, \bar{\theta}$ ) in the Eqs. (41a), (41b), (42a), (42b), (42c), (44a) and (44b), by the following parameters  $m$  and  $(\sigma, \bar{\sigma}) \mathbf{K}$ , respectively:

$$\begin{cases} \Delta E_{0-m}^{hy-gcp} = \epsilon \mathbf{K} \left( \eta(0, a, b, c, f) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m, \\ E_{0-m}^{mo-gcp} = \epsilon \mathbf{K} \left( \eta(0, r_e, D_e) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m, \\ \Delta E_{1-m}^{hy-gcp} = \epsilon \mathbf{K} \left( \eta(1, a, b, c, f) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m, \end{cases} \quad (74)$$

$$\begin{cases} \Delta E_{n-m}^{hy-gcp} = \epsilon \mathbf{K} \left( \eta(n, a, b, c, f) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m, \\ E_{n-m}^{mo-gcp} = \epsilon \mathbf{K} \left( \eta(n, r_e, D_e) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m \\ \Delta E_{n-m}^{hy-gcp} = \epsilon \mathbf{K} \left( \eta(n, a, b, c, f) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m, \end{cases} \quad (75)$$

$$a \rightarrow \frac{D_e}{r_e^2}, b \mapsto D_e, c \rightarrow D_e r_e, d \rightarrow -2D_e r_e, e \rightarrow 2D_e$$

and  $d \rightarrow -2D_e r_e$  and  $f \rightarrow D_e r_e^2$ . (69)

This allows us to be obtained  $\eta(n, r_e, D_e)$  from  $\eta(n, a, b, c, f, d)$  as follows:

IEGACp model. This physical phenomenon is induced automatically from the influence of an external uniform magnetic field  $\mathbf{K}$  if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated:

$$(\theta, \bar{\theta}) \rightarrow (\sigma, \bar{\sigma}) \mathbf{K} \quad (71)$$

Here  $\sigma$  and  $\bar{\sigma}$  are just two infinitesimal real proportional constants, and to simplify the calculations without compromising the physical content we choose the magnetic field parallel to the ( $Oz$ ) axis. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

the IEGACp model in global (3DNRQm-NCPS) symmetries as:

and

$$\begin{cases} \Delta E_{0-m}^{hlm-gcp} = g_s \mathbf{K} \left[ \eta(0, a, b, c, f, d) \sigma + \frac{\bar{\sigma}}{2\mu} \right] m \\ \Delta E_{1-m}^{hlm-gcp} = g_s \mathbf{K} \left[ \eta(1, a, b, c, f, d) \sigma + \frac{\bar{\sigma}}{2\mu} \right] m, \\ E_{n-m}^{hlm-gcp} = g_s \mathbf{K} \left[ \eta(n, a, b, c, f, d) \sigma + \frac{\bar{\sigma}}{2\mu} \right] m. \end{cases} \quad (76)$$

We have ( $-l \leq m \leq +l$ ), which allows us to fix  $(2l + 1)$  values for the discrete number  $m$ .

Now, for our purposes, we are interested in finding a new third automatically important symmetry for the improved generalized Cornell potential model at zero temperature in DSE symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential  $H_{pert}^{gcp}$  or the induced rotational Fermi Hamiltonian which we have seen in Eq. (23a). We discover these important physical phenomena when our studied system consists of non-interacting is considered as Fermi gas, it is formed from all the particles in their gaseous state (CO, NO, CH, and N<sub>2</sub>) undergoing rotation with angular velocity  $\Omega$  if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated:

$$\theta \rightarrow \chi\Omega \text{ and } \bar{\theta} \rightarrow \bar{\chi}\Omega \Rightarrow \mathbf{L}\theta \rightarrow \chi\mathbf{L}\Omega \text{ and } \mathbf{L}\bar{\theta} \rightarrow \bar{\chi}\mathbf{L}\Omega. \quad (77)$$

Here  $\chi$  and  $\bar{\chi}$  are just infinitesimal real proportional constants. We can express the effective potential  $H_{pert}^{gcp-rot}$

$$H_{pert}^{gcp-rot} = \left( \chi \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) + \frac{\bar{\chi}}{2\mu} \right) \mathbf{L}\Omega \begin{cases} \varepsilon \text{ For the hydrogenic atoms and diatomic molecules} \\ g_s \text{ For the Heavy-light mesons.} \end{cases} \quad (78)$$

To simplify the calculations without compromising physical content, we choose the rotational velocity  $\Omega$  parallel to the ( $Oz$ ) axis. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\Lambda(\chi, \bar{\chi}, r)\mathbf{L}\Omega = \Lambda(\chi, \bar{\chi}, r)\Omega L_z, \quad (79)$$

with

$$\Lambda(\chi, \bar{\chi}, r) = \chi \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) + \frac{\bar{\chi}}{2\mu}, \quad (80)$$

$$E_{gcp}^{f-rot}(n, a, b, c, f, d, \chi, \bar{\chi}, m) = \left( \eta(n, a, b, c, f, d)\chi + \frac{\bar{\chi}}{2\mu} \right) \Omega m \begin{cases} \varepsilon \text{ For the hydrogenic atoms and diatomic molecules} \\ g_s \text{ For the heavy-light mesons.} \end{cases} \quad (81)$$

It is worth mentioning that the authors in Ref. [82] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature, but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term  $\Lambda(\chi, \bar{\chi}, r)\mathbf{L}\Omega$  automatically appears due to the large symmetries resulting from the deformation of the space-phase. It should be noted that the results obtained in Eqs. (74), (78), and (81) can find it by direct calculation:

$$\Delta E_{n-m}^{hy-gcp} = \Delta E_{n-m}^{mo-gcp} = \varepsilon \aleph m \int_0^{+\infty} \left( \sum_{n=0}^{\infty} ar^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \sigma + \frac{\bar{\sigma}}{2\mu} \right] dr, \quad (82)$$

$$E_{n-m}^{hl-gcp} = g_s \aleph m \int_0^{+\infty} \left( \sum_{n=0}^{\infty} ar^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \sigma + \frac{\bar{\sigma}}{2\mu} \right] dr, \quad (83)$$

and

$$\Delta E_{gcp}^{f-rot} = \Omega m \int_0^{+\infty} \left( \sum_{n=0}^{\infty} ar^{2n+L} \right)^2 \exp(-2ar^2 - 2\beta r) \left[ \left( \frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a \right) \sigma + \frac{\bar{\sigma}}{2\mu} \right] dr. \begin{cases} \varepsilon \text{ For hydrogenic atoms and diatomic molecules} \\ g_s \text{ For heavy-light mesons} \end{cases} \quad (84)$$

## V. RESULTS AND DISCUSSION

### A Global results

In the previous subsections, we obtained the solution of the deformed Schrödinger equation for the IEGAcP model in (3DNRQm-NCPS) symmetry, which is described by the Hamiltonian operator given in Eq. (21) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalues are calculated in the three-dimensional space phase. The modified eigenenergies for the ground state, the first excited state, and  $n^{th}$  excited states of

which induced the rotational movements of the diatomic molecules as follows:

All of this data allows for the discovery of the new energy shift  $\Delta E_{gcp}^{f-rot}(n, a, b, c, f, d, \chi, \bar{\chi}, m)$  due to the perturbed Fermi gas effect  $V_{pert}^{gcp-rot}(r)$  which is generated automatically by the influence of the modified Mobius squared potential for the  $n^{th}$  excited-state in DSE symmetries as follows:

$$(\Delta E_{n-m}^{hy-gcp}, \Delta E_{n-m}^{mo-gcp}, \Delta E_{n-m}^{hl-gcp}) = \langle \Psi | H_z^{gcp-rot} | \Psi \rangle,$$

and

$$\Delta E_{gcp}^{f-rot}(n, a, b, c, f, d, \chi, \bar{\chi}, m) = \langle \Psi | H_{pert}^{gcp-rot} | \Psi \rangle,$$

that takes the following explicit relation:

the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$  under the IEGAcP model:

$$\begin{aligned} E_{nc-0}^{(u,d)hy}(0, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{nc-0}^{(u,d)hy}, \\ E_{nc-1}^{(u,d)Gcp}(1, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{nc-1}^{(u,d)hy}, \\ E_{nc-n}^{(u,d)hy}(n, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{nc-n}^{(u,d)hy}, \end{aligned}$$

with spin-1/2, the degenerated energy

$$\begin{aligned} E_{n-g}^{hlm}(n, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{n-g}^{hlm}, \\ E_{n-m}^{hlm}(n, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{n-m}^{hlm}, \\ E_{n-l}^{hlm}(n, a, b, c, d, e, f, \alpha, j, l, m, s) &\equiv E_{n-l}^{hlm} \end{aligned}$$

of the heavy quarkonium systems  $Q\bar{Q}$  ( $Q = b, c$ ) and the energy  $E_{nc}^{mol}(n, \alpha, j, l, m, s, r_e, D_e, \theta, \sigma, \chi) \equiv E_n^{mol}$  of the diatomic molecules CO, NO, CH and N<sub>2</sub> are obtained in this paper based on our original results presented in Eqs.

(61), (62), (63), (64), (66), (68) and (81), in addition to the ordinary energy for the exponential, generalized, Cornell potential model presented in Eq. (11) takes the form:

- For the hydrogenic atoms He<sup>+</sup>, Li<sup>2+</sup>, and Be<sup>+</sup>
- 

$$E_{nc-0}^{(u,d)hy} = E_{0l} + \varepsilon \left[ \eta(0, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + \varepsilon \begin{cases} k_+ \left( \eta(0, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l + 1/2, \\ k_- \left( \eta(0, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l - 1/2, \end{cases} \quad (85)$$

$$E_{nc-1}^{(u,d)hy} = E_{1l} + \varepsilon \left[ \eta(1, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + \varepsilon \begin{cases} k_+ \left( \eta(1, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l + 1/2, \\ k_- \left( \eta(1, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l - 1/2. \end{cases} \quad (86)$$

and

$$E_{nc-n}^{(u,d)hy} = E_{nl} + \varepsilon \left[ \eta(n, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + \varepsilon \begin{cases} k_+ \left( \eta(n, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l + 1/2, \\ k_- \left( \eta(n, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) & \text{for } j = l - 1/2. \end{cases} \quad (87)$$

$c\bar{c}$  and bottomonium  $b\bar{b}$ :

- For the Heavy quarkonium systems, charmonium

$$E_{n-g}^{hlm} = e + \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right) \sqrt{\frac{6a - \left[ \frac{2\mu \left( b + \frac{d\alpha^2}{2} \right)^3}{4\beta} \right]}{12\mu}} - 2\mu(c - d)^2 \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right)^{-2} + g_s \left[ \eta(n, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + g_s k_1(l) \left[ \eta(n, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right] \text{ if } j = l + 1, \quad (88)$$

$$E_{n-m}^{hlm} = e + \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right) \sqrt{\frac{6a - \left[ \frac{2\mu \left( b + \frac{d\alpha^2}{2} \right)^3}{4\beta} \right]}{12\mu}} - 2\mu(c - d)^2 \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right)^{-2} + g_s \left[ \eta(n, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + g_s k_2(l) \left[ \eta(n, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right] \text{ if } j = l + 1, \quad (89)$$

$$E_{n-l}^{hlm} = e + \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right) \sqrt{\frac{6a - \left[ \frac{2\mu \left( b + \frac{d\alpha^2}{2} \right)^3}{4\beta} \right]}{12\mu}} - 2\mu(c - d)^2 \left( 4n + 1 + \sqrt{(2l + 1)^2 + 8\mu f} \right)^{-2} + g_s \left[ \eta(n, a, b, c, f, d)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + g_s k_3(l) \left[ \eta(n, a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right] \text{ if } j = l + 1, \quad (90)$$

- For the diatomic molecules CO, NO, CH and N<sub>2</sub>:

$$E_{nc}^{mol}(n, \alpha, j, l, m, s, r_e, D_e, \theta, \sigma, \chi) = E_{nl}(r_e, D_e) + \varepsilon \left[ \eta(n, r_e, D_e)(\aleph\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \aleph\bar{\sigma}}{2\mu} \right] m + \varepsilon k(l) \left( \eta(n, r_e, D_e)\theta + \frac{\bar{\theta}}{2\mu} \right). \quad (91)$$

Where the energy of the ground state  $E_{0l}$  and the first excited state  $E_{1l}$  in the symmetries of quantum mechanics under the exponential, generalized, Cornell potential model:

$$E_{0l} = e + \left(1 + \sqrt{(2l+1)^2 + 8\mu f}\right) \sqrt{\frac{6a - \left[\frac{2\mu\left(b + \frac{d\alpha^2}{2}\right)}{4\beta}\right]^3}{12\mu}} - 2\mu(c-d)^2 \left(1 + \sqrt{(2l+1)^2 + 8\mu f}\right)^{-2}, \quad (92)$$

$$E_{1l} = e + \left(5 + \sqrt{(2l+1)^2 + 8\mu f}\right) \sqrt{\frac{6a - \left[\frac{2\mu\left(b + \frac{d\alpha^2}{2}\right)}{4\beta}\right]^3}{12\mu}} - 2\mu(c-d)^2 \left(5 + \sqrt{(2l+1)^2 + 8\mu f}\right)^{-2}, \quad (93)$$

and

$$E_{nl}(r_e, D_e) = 2D_e + \left(4n+1 + \sqrt{(2l+1)^2 + 8\mu D_e r_e^2}\right) \sqrt{\frac{6\frac{D_e}{r_e^2} - \left[\frac{2\mu\left(D_e \frac{D_e^3}{r_e^3}\right)}{4\beta}\right]^3}{12\mu}} - 18\mu D_e^2 r_e^2 \left(4n+1 + \sqrt{(2l+1)^2 + 8\mu D_e r_e^2}\right)^{-2}. \quad (94)$$

Thus, the total energy  $E_{nc-n}^{(u,d)hy}$ ,  $(E_{n-g}^{hlm}, E_{n-m}^{hlm}, E_{n-l}^{hlm})$  and  $E_n^{mol}$  for the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$ , the heavy quarkonium systems  $Q\bar{Q} (Q = b, c)$  and the diatomic molecules CO, NO, CH, and  $\text{N}_2$ , respectively, under the IEGACp model in (3DNRQm-NCPS) symmetries, is the sum of the ordinary part of the energy  $E_{nl}$  and the three corrections of energy that are produced automatically with the effect of the perturbed spin-orbit, modified Zeeman effect, and the induced rotational Fermi Hamiltonian. This is one of the main objectives of our research. It is useful to refer to the new quantitative results of this work as:

➤ Case 1: For the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$

$$H_{GCP}^{nc}(p, x) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) \begin{cases} E_{nc-n}^{u,hy} & \text{for } j = l + \frac{1}{2}, \\ E_{nc-n}^{d,hy} & \text{for } j = l - \frac{1}{2}, \end{cases} \left(\frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi)\right). \quad (95)$$

➤ Case 2: For the heavy quarkonium systems  $Q\bar{Q} (Q = b, c)$  :

$$H_{GCP}^{nc}(p, x) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) \begin{cases} E_{n-g}^{hlm} & \text{if } j = l + 1, \\ E_{n-m}^{hlm} & \text{if } j = l \\ E_{n-l}^{hlm} & \text{if } j = l - 1. \end{cases} \left(\frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi)\right), \quad (96)$$

➤ Case 3: For the diatomic molecules CO, NO, CH and  $\text{N}_2$ :

$$H_{GCP}^{nc}(p, x) \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi) = E_n^{mol} \frac{R_{nl}(r)}{r} Y_l^m(\theta, \phi). \quad (64)$$

This is one of the main motivations for the topic of this work. It is clear that the obtained energies are real, which allows us

to consider the NC diagonal Hamiltonian  $H_{nc}^{GCP}(r, \theta, \bar{\theta}, \sigma, \bar{\sigma}, \chi, \bar{\chi})$  as a Hermitian operator. In addition, and regarding the previously obtained results (22), (34), (73), and (78), the global Hamiltonian operator, at first order in and with the effect of the IEGACp model for hydrogenic atoms for  $(n, l)$  states takes the form as:

$$H_{GCP}^{nc}(r, \theta, \bar{\theta}, \sigma, \bar{\sigma}, \chi, \bar{\chi}) = \left(-\frac{\Delta}{2\mu} + ar^2 + br - \frac{c}{r} + \frac{d \exp(-ar)}{r} + \frac{f}{r^2} + e\right) + \left(\left(\frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2} + \frac{f'}{r^4} - \frac{c}{2r^3} - \frac{b}{2r} - a\right) \sigma + \frac{\bar{\sigma}}{2\mu}\right) (\mathbf{N} \mathbf{J} + \mathbf{N} \mathbf{S}) + \left\{ \left[ \left(\frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2}\right) \theta + \frac{\bar{\theta}}{2\mu} \right] \mathbf{G}^2 + \left[ \left(\frac{d \exp(-ar)}{2r^3} + \frac{ad \exp(-ar)}{2r^2}\right) \chi + \frac{\bar{\chi}}{2\mu} \right] \mathbf{L} \mathbf{\Omega} \right\} \begin{cases} \varepsilon & \text{for the hydrogenic atoms and diatomic molecules} \\ g_s & \text{for the Heavy-light mesons.} \end{cases} \quad (97)$$

This is the equation for the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$ , the heavy quarkonium systems,  $Q\bar{Q} (Q = c, b)$  and the diatomic molecules CO, NO, CH, and  $\text{N}_2$  under the influence

of the IEGACp model interactions. It should be pointed out that this treatment considers only the first-order terms in either  $\theta$  or  $\bar{\theta}$ . The first part of Eq. (97) presents the Hamiltonian operator  $H_{GCP}(p, x)$  (Eq. (22)) in the ordinary

quantum mechanics for the EGACp model while the second term is the perturbed spin-orbit interaction  $H_{so}^{gcp}$  (Eq. (34)), the third term is the modified Zeeman Hamiltonian operator  $H_z^{gcp}(r, \sigma, \bar{\sigma})$  (Eq. (73)) and the fourth part is the induced rotational Fermi Hamiltonian (Eq. (78)), respectively, present, which are induced automatically by the NC properties of space and phase. It is evident to consider the atomic quantum number  $m$  can take  $(2l + 1)$  values and we have also two values for  $j = l + 1/2$  and  $j = l - 1/2$  corresponding to up and down polarities for the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$  and  $\text{Be}^+$ . For the heavy quarkonium systems,  $Q\bar{Q}$  ( $Q = b, c$ ), we have also three values for  $j = l \pm 1$  and  $j = l$ . For the diatomic molecules CO, NO, CH, and  $\text{N}_2$ , we have also three values  $\underbrace{|l - s| \leq j \leq l + s}_{N: \text{value}}$ . Thus, every

state in the usual 3D-space of energy for the IEGACp model will be  $2(2l + 1)$  substate in (3DNRQm-NCPS). Thus, the total complete degeneracy of the energy level of the IEGACp model is obtained as a sum of all allowed values. Total degeneracy is thus,

$$\sum_{l=1}^{n-1} 2(2l + 1) = 2n^2 \rightarrow \sum_{l=1}^{n-1} 2(2l + 1) = 2n^2 \text{ For the hydrogenic atoms, (98)}$$

$$\sum_{l=1}^{n-1} 2(2l + 1) = 2n^2 \rightarrow \sum_{l=1}^{n-1} 3(2l + 1) = 3n^2 \text{ For heavy quarkonium systems, (99)}$$

and

$$\sum_{l=1}^{n-1} 2(2l + 1) = 2n^2 \rightarrow N \sum_{l=1}^{n-1} (2l + 1) = Nn^2 \text{ For diatomic molecules. (100)}$$

### B New mass spectra of heavy quarkonium systems

This subsection is devoted to deriving the mass spectra of  $Q\bar{Q}$  ( $Q = b, c$ ) charmonium and bottomonium under the IEGACp model. It is well known that the spin of charmonium and bottomonium equal two values (0 or 1), because consist of quark and anti-quark. For spin-1, we have three values of  $j$  are  $(j_1 = l + 1, j_2 = l, j_3 = l - 1)$ , which allows us the corresponding three values  $(k_1, k_2, k_3) = \frac{1}{2}(l, -2, -2l - 2)$  and thus, we obtain three values of energy:

$$E_{n-g}^{hlm} = E_{nl} + g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) + g_s l \left( \eta(a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \quad (101)$$

$$E_{n-m}^{hlm} = E_{nl} + g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) m - g_s \left( \eta(a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l, \quad (102)$$

And

$$E_{n-l}^{hlm} = E_{nl} + g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) - g_s(l + 1) \left( \eta(a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1. \quad (103)$$

In the symmetries of ordinary quantum mechanics, the mass spectra  $Q\bar{Q}$  ( $Q = b, c$ ) obtained by applying the following formula [9, 10, 11]:

$$M = 2m_Q + E_{nl}. \quad (104)$$

Here  $m_Q$  is bare quark masses. Thus, the modified mass  $M_{nc}^{gcp}$  ( $s = 1$ ) with spin-1 of  $Q\bar{Q}$  ( $Q = b, c$ ) charmonium and bottomonium become as follows:

$$M_{nc}^{gcp}(s = 1) = 2m_Q + \frac{1}{3}(E_{n-g}^{hlm} + E_{n-m}^{hlm} + E_{n-l}^{hlm}). \quad (105)$$

The value  $\frac{1}{3}(E_{n-g}^{hlm} + E_{n-m}^{hlm} + E_{n-l}^{hlm})$  represents the physically non-polarized energy (energy independent of spin). After a simple calculation, we obtain  $\delta M$ :

$$\delta M(s = 1) = g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) m - \frac{2}{3} g_s \left( \eta(a, b, c, f, d)\theta + \frac{\bar{\theta}}{2\mu} \right). \quad (106)$$

With  $\delta M(s = 1) \equiv M_{nc}^{lgcp}(s = 1) - M$ , this is the noncommutativity contribution for the mass spectra of  $Q\bar{Q}$  ( $Q = b, c$ ) charmonium and bottomonium, in the IEGACp model. For spin-0, we have only one value of  $j = l$ , which allows us the corresponding values  $k = 0$ , and thus, we obtain the energy:

$$E_{n-g}^{hlm} = E_{nl} + g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) m. \quad (107)$$

Thus, the modified mass  $M_{nc}^{lgcp}(s = 0)$  with spin-1 of  $Q\bar{Q}$  ( $Q = b, c$ ) charmonium and bottomonium become as follows:

$$\delta M(s = 0) \equiv M_{nc}^{gcp}(s = 0) - M = g_s \left( \eta(n, a, b, c, f, d)(\mathfrak{N}\sigma + \Omega\chi) + \frac{\bar{\chi}\Omega + \mathfrak{N}\bar{\sigma}}{2\mu} \right) m. \quad (108)$$

We now look at some special cases and relationships between our recent results and some other existing results in our previous work:

1-When we set  $c = -g$  and  $d = 0, e = \delta$ , and  $f = 0$ , the IEGACp model reduces to the extended Heavy-Light Mesons with the additional condition  $h = 0$ , it is easy to show that Eqs. (21), (88), (89) and (90) are reduced to the modified interaction  $H_{pert}^{hlm}(p, x)$  of a particle in the extended nonrelativistic quark-antiquark potential and corresponding NC spectrum  $(E_{nc}^{ehlm}, E_{n-m}^{ehlm}, E_{n-m}^{ehlm})$ , respectively [30]:

$$H_{pert}^{hlm}(p, x) = \left( \frac{g}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L}\mathbf{O} + \frac{\mathbf{L}\bar{\mathbf{O}}}{2\mu}, \quad (109)$$

and

$$E_{nc}^{ehlm} = E_{nl} + g_s \aleph \left( \mathcal{E}(n, g, b, a) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_1(l) \left( \mathcal{E}(n, g, b, a) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \quad (110)$$

$$E_{n-m}^{ehlm} = E_{nl} + g_s \aleph \left( \mathcal{E}(n, g, b, a) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_2(l) \left( \mathcal{E}(n, g, b, a) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l. \quad (111)$$

$$E_{n-m}^{ehlm} = E_{nl} + g_s \aleph \left( \mathcal{E}(n, g, b, a) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_3(l) \left( \mathcal{E}(n, g, b, a) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1, \quad (112)$$

here

$$\mathcal{E}(n, g, b, a) = \frac{g}{2} \langle \frac{1}{r^3} \rangle_{(n,l,m)} - \frac{b}{2} \langle \frac{1}{r} \rangle_{(n,l,m)} - a,$$

in addition to the condition  $\chi = \bar{\chi} = 0$ .

2-When we set  $d = 0$ ,  $f = 0$ , and  $e = 0$ , the IEGACp model reduces to extended quark-antiquark interaction potential, it is easy to show that Eqs. (21), (88), (89) and (90) are reduced to the modified interaction  $H_{pert}^{hlm}(p, x)$  of a particle in the extended nonrelativistic quark-antiquark potential and corresponding NC spectrum  $E_{nc}^{ehlm}$ ,  $E_{n-m}^{ehlm}$  and  $E_{n-m}^{ehlm}$ , respectively [27]:

$$H_{pert}^{hlm}(p_{nc}, x_{nc}) = \left( -\frac{c}{2r^3} - \frac{b}{2r} - a \right) \mathbf{L} \cdot \boldsymbol{\Theta} + \frac{\mathbf{L} \cdot \bar{\boldsymbol{\theta}}}{2\mu}, \quad (113)$$

and

$$E_{nc}^{ehlm} = E_{nl} + g_s \aleph \left( \Sigma(n, a, b, c) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_1(l) \left( \Sigma(n, a, b, c) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l + 1, \quad (114)$$

$$E_{n-m}^{ehlm} = E_{nl} + g_s \aleph \left( \Sigma(n, a, b, c) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_2(l) \left( \Sigma(n, a, b, c) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l, \quad (115)$$

$$E_{n-m}^{ehlm} = E_{nl} + g_s \aleph \left( \Sigma(n, a, b, c) \sigma + \frac{\bar{\sigma}}{2\mu} \right) m + g_s k_3(l) \left( \Sigma(n, a, b, c) \theta + \frac{\bar{\theta}}{2\mu} \right) \text{ if } j = l - 1, \quad (116)$$

with

$$\Sigma(n, a, b, c) = -\frac{c}{2} \langle \frac{1}{r^3} \rangle_{(n,l,m)} - \frac{b}{2} \langle \frac{1}{r} \rangle_{(n,l,m)} - a,$$

in addition to the condition  $\chi = \bar{\chi} = 0$ .

$$E_{nc}^{mol}(n, \alpha, j, l, m, s, r_e, D_e, \Theta^c, \sigma^c, \chi^c) = E_{nl}(r_e, D_e) + \varepsilon \left[ \eta(n, r_e, D_e) (\aleph \sigma^c + \Omega \chi^c) + \frac{\bar{\chi}^c \Omega + \aleph \bar{\sigma}^c}{2\mu} \right] m + \varepsilon k(l) \left[ \eta(n, r_e, D_e) \theta^c + \frac{\bar{\theta}^c}{2\mu} \right]. \quad (119)$$

At the end of this section, we write down the main result of our research: the Schrödinger equation, known in the literature, as the most well-known nonrelativistic wave

### C. Composite systems cases

Now, considering composite systems such as molecules made of  $N = 2$  particles of masses  $m_n (n = 1, 2)$  in the frame of noncommutative algebra, it is worth taking into account features of descriptions of the systems in space. In NRQM symmetries, it was obtained those composite systems with different masses are described with different noncommutative parameters [54, 55, 56, 57, 58]:

$$\begin{cases} [x_i^{nc*}, x_j^{nc}] = [x_i^{nc}(t)^*, x_j^{nc}(t)] = [x_{ii}^{nc}(t)^*, x_{jj}^{nc}(t)] = i\theta_{ij}^c, \\ [p_i^{nc*}, p_j^{nc}] = [p_i^{nc}(t)^*, p_j^{nc}(t)] = [p_{ii}^{nc}(t)^*, p_{ii}^{nc}(t)] = i\bar{\theta}_{ij}^c, \end{cases} \quad (117)$$

with

$$\begin{aligned} \theta_{ij}^c &= \sum_{n=1}^2 (\mu_n^2 \theta_{ij}^{(n)}), \\ \bar{\theta}_{\mu\nu}^c &= \sum_{n=1}^2 (\mu_n^2 \bar{\theta}_{ij}^{(n)}), \\ \text{and } \mu_n &= \frac{m_n}{\sum_n m_n}. \end{aligned}$$

The indices ( $n = 1, 2$ ) label the particle, and  $\theta_{\mu\nu}^{(n)}$  is the parameter of non-commutativity, corresponding to the particle of mass  $m_n$ . Note that in the case of a system of two particles with the same mass  $m_1 = m_2$  such as the homogeneous  $N_2$  diatomic molecule is the parameter  $\theta_{\mu\nu}^{(n)} = \theta_{\mu\nu}$ . Thus, the three parameters  $\theta$ ,  $\sigma$ , and  $\chi$  which appear in Eq. (91) are changed to the new form:

$$\gamma^{c2} = \left( \sum_{n=1}^2 \mu_n^2 \Upsilon \theta_{12}^{(n)} \right)^2 + \left( \sum_{n=1}^2 \mu_n^2 \Upsilon_{23}^{(n)} \right)^2 + \left( \sum_{n=1}^2 \mu_n^2 \Upsilon_{13}^{(n)} \right)^2, \quad (118)$$

with  $\Upsilon^c$  can take the noncommutativity parameters  $(\theta, \bar{\theta}, \sigma, \bar{\sigma}, \chi, \bar{\chi})$ . As mentioned above, in the case of a system of two particles with the same mass,  $m_1 = m_2$  such as the homogeneous  $N_2$  diatomic molecules:

$$\begin{aligned} \theta_{\mu\nu}^{(n)} &= \theta_{\mu\nu}, \\ \sigma_{\mu\nu}^{(n)} &= \sigma_{\mu\nu}, \\ \text{and } \chi_{\mu\nu}^{(n)} &= \chi_{\mu\nu}. \end{aligned}$$

Finally, we can generalize the nonrelativistic global energy  $E_{nc}^{mol}(n, \alpha, j, l, m, s, r_e, D_e, \Theta^c, \sigma^c, \chi^c)$  under the improved generalized Cornell potential model considering that composite systems with different masses are described with different noncommutative parameters for the diatomic CO, NO, and CH as:



similar to the Dirac equation (Dirac, 1928) [83] for fermionic particles with spin-1/2, it can describe the dynamic state of a particle with spin-1/2 (the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$  and  $\text{Be}^+$ ) or similar to the relativistic Duffin-Kemmer equation (Duffin, 1938; Kemmer, 1938; Petiau, 1936) [84, 85,86] for meson with spin-s (the heavy quarkonium systems  $c\bar{c}$  and  $b\bar{b}$ ), it can describe a dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. The conventional nonrelativistic approach of SE under an improved generalized Cornell potential model involves solving the second-order Klein-Gordon equation for spin-0 and the Proca equation for spin-1 [87]. Worthwhile it is better to mention that for the two simultaneous limits  $(\theta, \sigma, \chi)$  and  $(\theta^c, \sigma^c, \chi^c) \rightarrow (0, 0, 0)$  we recover the results of the in Ref. [15]. It is easy to check the extremely results:

$$\begin{aligned} \lim_{(\theta, \theta) \rightarrow (0,0)} E_{nc-n}^{(u,d)hy} &= E_{nl}, \\ \lim_{(\theta, \theta) \rightarrow (0,0)} (E_{n-g}^{hlm}, E_{n-m}^{hlm}, E_{n-l}^{hlm}) &= E_{nl}, \\ \text{and } \lim_{(\theta, \theta) \rightarrow (0,0)} E_n^{mol} &= E_{nl}(\tau_e, D_e). \end{aligned}$$

It should be noted that the following physical endings in this way give a logical indication of the validity of the results of our research.

## VI. CONCLUSIONS

In this paper three-dimensional RSE has been performed for the IEGACp model by using the generalized Bopp's shift method, and standard perturbation theory including the effect of the centrifugal term in (3DNRQm-NCPS) symmetries, we resume the main results:

➤ The energy eigenvalues  $E_{nc-n}^{(u,d)hy}(n, a, b, c, d, e, f, \alpha, j, l, m, s)$  of the bound states of the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$  under the IEGACp model with spin-1/2 for  $n^{th}$  excited states have been analytically found. The energy eigenvalues depend on  $(a, b, c, d, e, f, \alpha)$  parameters and the discrete atomic quantum numbers  $(j, l, m, s)$  have a finite number of the quantized energy spectrum for the IEGACp model.

➤ The energy eigenvalues  $E_{n-g}^{hlm}, E_{n-m}^{hlm}, E_{n-l}^{hlm}$  of the bound states of the heavy quarkonium systems  $c\bar{c}$  and  $b\bar{b}$  under the IEGACp model with spin-(0,1) for excited states have been analytically found.

➤ The energy eigenvalues  $E_n^{mol}(n, \alpha, j, l, m, s, \tau_e, D_e, \theta^c, \sigma^c, \chi^c)$  of the bound states of the diatomic molecules CO, NO, CH, and  $\text{N}_2$  under the IEGACp model with spins for excited states have been analytically found. The energy eigenvalues depend on  $(\tau_e, D_e, \alpha)$  parameters and the discrete atomic quantum numbers  $(j, l, m, s)$ .

➤ The usual kinetic term  $-\frac{\Delta}{2\mu}$  modified to the new form  $(-\frac{\Delta}{2\mu} - \frac{L\vec{\theta}}{2\mu} - \frac{L\vec{\sigma}}{2\mu} - \frac{L\vec{\chi}}{2\mu})$  under the influence of the IEGACp model in (3DNRQm-NCPS) symmetries,

➤ The Hamiltonian operator in (3DNRQm-NCPS)

symmetry  $H_{gcp}^{nc}(r, \theta, \bar{\theta}, \sigma, \bar{\sigma})$  is the sum of the Hamiltonian operator of the IEGACp model  $H_{gcp}(p, x)$  and three operators, the first one is the modified spin-orbit interaction  $H_{so}^{gcp}(r, \theta, \bar{\theta})$ , the second is the modified Zeeman operator  $H_z^{gcp}(r, \sigma, \bar{\sigma})$  while the third operator  $H_{pert}^{gcp-rot}$  is the perturbed Fermi Hamiltonian for the hydrogenic atoms, the heavy quarkonium systems, and the diatomic molecules.

➤ The ordinary Schrödinger equation under the improved generalized Cornell potential model has a physical behavior similar to the Dirac equation for fermionic particles with spin-1/2, it can describe the dynamic state of a particle with spin-1/2 (the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$  and  $\text{Be}^+$ ).

➤ The most important thing that can be evaluated through this new research is the possibility of upgrading the Schrödinger equation known in the literature to the description of the relativistic Duffin-Kemmer–Petiau equation by describing the state of the boson particles with spin (0 or 1).

It has been shown that the DSE under the improved generalized Cornell potential model presents useful symmetry to standing the hydrogenic atoms  $\text{He}^+$ ,  $\text{Li}^{+2}$ , and  $\text{Be}^+$ , the heavy quarkonium systems ( $c\bar{c}$  and  $b\bar{b}$ ) and the diatomic molecules CO, NO and CH, and  $\text{N}_2$  influenced by the IEGACp model. It should be noted that the results obtained in this research would be identical to corresponding results in ordinary quantum mechanics when the two limits  $(\theta, \sigma, \chi)$  and  $(\theta^c, \sigma^c, \chi^c) \rightarrow (0,0,0)$  are applied simultaneously.

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