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Inequalities for $D-$Synchronous Functions and Related Functionals

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Abstract. We introduce in this paper the concept of quadruple $D-$synchronous functions which generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and we also provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.
Keywords: Synchronous Functions, Lipschitzian functions, Chebyshev inequality, Cauchy-Bunyakovsky-Schwarz inequality.
MSC2010: 26D15; 26D10.

Desigualdades para funciones $D-$sincrónicas y funciones relacionadas

Resumen. Introducimos en este artículo el concepto de funciones $D-$sincrónicas cuádruples, que generaliza el concepto de un par de funciones sincrónicas; estableceremos una desigualdad similar a la desigualdad de Chebyshev y también presentamos algunas desigualdades de tipo Cauchy-Bunyakovsky-Schwarz para un funcional asociado con este cuádruple. Se dan algunas aplicaciones para funciones univariadas de la variable real. También se indican desigualdades discretas.
Palabras clave: Funciones $D-$sincrónicas, funciones Lipschitzianas, desigualdad de Chebyshev, desigualdad de Cauchy-Bunyakovsky-Schwarz.

1. Introduction

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and *positive measure* ν on $\mathcal{A}$ with values in $[0, +\infty]$. For

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a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\nu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\nu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\nu$ instead of $\int_\Omega w(x)\,d\nu(x)$. Assume also that $\int_\Omega wd\nu=1$.

We say that the pair of measurable functions (f,g) are *synchronous* on Ω if

$$(f(x)-f(y))(g(x)-g(y))\geq 0 \tag{1}$$

for ν-a.e. $x,y\in\Omega$. If the inequality reverses in (1), the functions are called *asynchronous* on Ω.

If (f,g) are synchronous on Ω and $f,\ g,\ fg\in L_w(\Omega,\nu)$, then the following inequality, that is known in the literature as *Chebyshev's Inequality,* holds:

$$\int_\Omega wfgd\nu\geq\int_\Omega wfd\nu\int_\Omega wgd\nu, \tag{2}$$

where $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$ and $\int_\Omega wd\nu=1$.

If $f,\ g:\Omega\to\mathbb{R}$ are ν-measurable functions and $f,\ g,\ fg\in L_w(\Omega,\nu)$, then we may consider the *Chebyshev functional*

$$T_w(f,g):=\int_\Omega wfgd\nu-\int_\Omega wfd\nu\int_\Omega wgd\nu.$$

The following result is known in the literature as the *Grüss inequality:*

$$|T_w(f,g)|\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{3}$$

provided

$$-\infty<\gamma\leq f(x)\leq\Gamma<\infty,\qquad -\infty<\delta\leq g(x)\leq\Delta<\infty \tag{4}$$

for ν-a.e. $x\in\Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If $f\in L_w(\Omega,\nu)$, then we may define

$$D_w(f):=\int_\Omega w(x)\left|f(x)-\int_\Omega w(y)f(y)\,d\nu(y)\right|d\nu(x). \tag{5}$$

The following refinement of Grüss inequality in the general setting of measure spaces is due to Cerone & Dragomir [1]:

Theorem 1.1. *Let $w,\ f,\ g:\Omega\to\mathbb{R}$ be ν-measurable functions with $w\geq 0$ ν-a.e. on Ω and $\int_\Omega wd\nu=1$. If $f,\ g,\ fg\in L_w(\Omega,\nu)$ and there exist constants $\delta,\ \Delta$ such that*

$$-\infty<\delta\leq g(x)\leq\Delta<\infty\quad \textit{for}\ \ \nu\textit{-a.e. } x\in\Omega, \tag{6}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Motivated by the above results, we introduce in this paper the concept of quadruple $D-$synchronous functions that generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and also provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.

2. $D-$Synchronous functions

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space and f, g, h, $\ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω.

Definition 2.1. The quadruple (f, g, h, ℓ) is called $D-$Synchronous ($D-$Asynchronous) on Ω if

$$\det \begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det \begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \geq (\leq)\, 0 \tag{8}$$

for ν-a.e. (almost every) x, $y \in \Omega$.

This concept is a generalization of synchronous functions, since for $g = 1$, $\ell = 1$ the quadruple (f, g, h, ℓ) is $D-$Synchronous if, and only if, (f, h) is synchronous on Ω.

If g, $\ell \neq 0$ ν-a.e on Ω, then

$$\begin{aligned} &\det \begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det \begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \\ &\qquad = (f(x)g(y) - g(x)f(y))(h(x)\ell(y) - \ell(x)h(y)) \\ &\qquad = g(x)\ell(x)g(y)\ell(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)\left(\frac{h(x)}{\ell(x)} - \frac{h(y)}{\ell(y)}\right) \end{aligned} \tag{9}$$

for ν-a.e. x, $y \in \Omega$. So, if $g\ell > 0$ ν-a.e on Ω the quadruple (f, g, h, ℓ) is $D-$Synchronous if, and only if, $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ is synchronous on Ω.

Theorem 2.2. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f, g, h, ℓ) is D-Synchronous ($D-$Asynchronous), $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and fh, $g\ell$, gh, $f\ell \in L_w(\Omega, \nu)$. Then,*

$$\det \begin{pmatrix} \int_\Omega wfh d\nu & \int_\Omega wgh d\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu \end{pmatrix} \geq (\leq)\, 0. \tag{10}$$

Proof. Since the quadruple (f,g,h,ℓ) is $D-$Synchronous, then

$$\begin{aligned} 0 \le & \left(f(x)g(y)-g(x)f(y)\right)\left(h(x)\ell(y)-\ell(x)h(y)\right) \\ =& f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y) \\ & -f(x)\ell(x)g(y)h(y)-g(x)h(x)f(y)\ell(y) \end{aligned} \tag{11}$$

for ν-a.e. x, $y\in\Omega$.

This is equivalent to

$$\begin{aligned} & f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y) \\ & \qquad \ge f(x)\ell(x)g(y)h(y)+g(x)h(x)f(y)\ell(y) \end{aligned} \tag{12}$$

for ν-a.e. x, $y\in\Omega$.

Multiply (12) by $w(x)w(y)\ge 0$ to get

$$\begin{aligned} & w(x)f(x)h(x)w(y)g(y)\ell(y)+w(x)g(x)\ell(x)w(y)f(y)h(y) \\ & \qquad \ge w(x)f(x)\ell(x)w(y)g(y)h(y)+w(x)g(x)h(x)w(y)f(y)\ell(y) \end{aligned} \tag{13}$$

for ν-a.e. x, $y\in\Omega$.

If we integrate the inequality (13) over $x\in\Omega$, then we get

$$\begin{aligned} & w(y)g(y)\ell(y)\int_\Omega wfhd\nu+w(y)f(y)h(y)\int_\Omega wg\ell d\nu \\ & \qquad \ge w(y)g(y)h(y)\int_\Omega wf\ell d\nu+w(y)f(y)\ell(y)\int_\Omega wghd\nu \end{aligned} \tag{14}$$

for ν-a.e. $y\in\Omega$.

Finally, if we integrate the inequality (14) over $y\in\Omega$, then we get

$$\begin{aligned} & \int_\Omega wfhd\nu\int_\Omega wg\ell d\nu+\int_\Omega wg\ell d\nu\int_\Omega wfhd\nu \\ & \qquad \ge \int_\Omega wf\ell d\nu\int_\Omega wghd\nu+\int_\Omega wghd\nu\int_\Omega wf\ell d\nu, \end{aligned}$$

which is equivalent to the desired result (10). ☑

Corollary 2.3. *Let f, g, h, $\ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω and such that $g\ell>0$ ν-a.e on Ω, $\left(\frac{f}{g},\frac{h}{\ell}\right)$ is synchronous (asynchronous) on Ω, $w\ge 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and fh, $g\ell$, gh, $f\ell\in L_w(\Omega,\nu)$; then the inequality (10) is valid.*

Let f, g, h, $\ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω , $w\ge 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and fh, $g\ell$, gh, $f\ell\in L_w(\Omega,\nu)$; then we can consider the functionals

$$\begin{aligned} \mathcal{D}(f,g,h,\ell;w,\Omega) :=& \det\begin{pmatrix} \int_\Omega wfhd\nu & \int_\Omega wghd\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu \end{pmatrix} \\ =& \int_\Omega wfhd\nu\int_\Omega wg\ell d\nu-\int_\Omega wf\ell d\nu\int_\Omega wghd\nu, \end{aligned} \tag{15}$$

and, for $(f,g)=(h,\ell)$,

$$\begin{aligned}\mathcal{D}(f,g;w,\Omega) &:= \mathcal{D}(f,g,f,g;w,\Omega) \qquad (16)\\ &= \det\begin{pmatrix} \int_\Omega wf^2 d\nu & \int_\Omega wfgd\nu \\ \int_\Omega wfgd\nu & \int_\Omega wg^2 d\nu \end{pmatrix} \\ &= \int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfgd\nu\right)^2,\end{aligned}$$

provided $f^2,\ g^2 \in L_w(\Omega,\nu)$.

We can improve the inequality (10) as follows:

Theorem 2.4. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f,g,h,ℓ) is $D-$Synchronous, $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell \in L_w(\Omega,\nu)$; then,*

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &\geq \max\{|\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|, \qquad (17)\\ &\qquad |\mathcal{D}(f,g,|h|,|\ell|;w,\Omega)|, |\mathcal{D}(|f|,|g|,|h|,|\ell|;w,\Omega)|\} \\ &\geq 0.\end{aligned}$$

Proof. We use the continuity property of the modulus, namely

$$|a-b| \geq ||a|-|b||,\ a,b\in\mathbb{R}.$$

Since (f,g,h,ℓ) is $D-$Synchronous, then

$$\begin{aligned}&(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y)) \qquad (18)\\ &\quad = |f(x)g(y)-g(x)f(y)||h(x)\ell(y)-\ell(x)h(y)| \\ &\quad \geq \begin{cases} |(|f(x)||g(y)|-|g(x)||f(y)|)(h(x)\ell(y)-\ell(x)h(y))| \\ \\ |(f(x)g(y)-g(x)f(y))(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \\ \\ |(|f(x)||g(y)|-|g(x)||f(y)|)(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \end{cases}\end{aligned}$$

for ν-a.e. $x,\ y\in\Omega$.

As in the proof of Theorem 2.2, we have the identity

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &= \frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)(f(x)g(y)-g(x)f(y)) \qquad (19)\\ &\qquad \times (h(x)\ell(y)-\ell(x)h(y))d\nu(x)d\nu(y).\end{aligned}$$

By using the identity (19) and the first branch in (18) we have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &\geq \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(|f(x)|\,|g(y)|-|g(x)|\,|f(y)|)\\ &\qquad\times(h(x)\,\ell(y)-\ell(x)\,h(y))|\,d\nu(x)\,d\nu(y)\\ &\geq \frac{1}{2}\left|\int_\Omega\int_\Omega w(x)\,w(y)\,(|f(x)|\,|g(y)|-|g(x)|\,|f(y)|)\right.\\ &\qquad\left.\times(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right|\\ &=|\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|,\end{aligned}$$

which proves the first part of (17).

The second and third part of (17) can be proved in a similar way and details are omitted. ☑

3. Further results for the functional $\mathcal{D}$

We have the following Schwarz's type inequality for the functional $\mathcal{D}$:

Theorem 3.1. *Let f, g, h, $\ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω , $w\geq 0$ a.e. on Ω with $\int_\Omega w d\nu=1$ and f^2, g^2, h^2, $\ell^2\in L_w(\Omega,\nu)$. Then,*

$$\mathcal{D}^2(f,g,h,\ell;w,\Omega)\leq\mathcal{D}(f,g;w,\Omega)\,\mathcal{D}(h,\ell;w,\Omega). \tag{20}$$

Proof. As in the proof of Theorem 2.4, we have the identities

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &= \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))\\ &\qquad\times(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y),\end{aligned}$$

$$\mathcal{D}(f,g;w,\Omega)=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))^2\,d\nu(x)\,d\nu(y)$$

and

$$\mathcal{D}(h,\ell;w,\Omega)=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,\ell(y)-\ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y).$$

By the Cauchy-Bunyakovsky-Schwarz double integral inequality we have

$$\begin{aligned}&\left(\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))\,(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right)^2\\ &\qquad\leq\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,g(y)-g(x)\,h(y))^2\,d\nu(x)\,d\nu(y)\\ &\qquad\qquad\times\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,\ell(y)-\ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y),\end{aligned}$$

which produces the desired result (20). ☑

Corollary 3.2. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω with $g^2,\ \ell^2\in L_w(\Omega,\nu)$, $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$, and $a,\ A,\ b,\ B\in\mathbb{R}$ such that $A>a$, $B>b$,*

$$ag\leq f\leq Ag \quad \text{and} \quad b\ell\leq h\leq B\ell \tag{21}$$

ν-a.e. on Ω. Then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)|\leq\frac{1}{4}(A-a)(B-b)\int_\Omega wg^2d\nu\int_\Omega w\ell^2d\nu. \tag{22}$$

Proof. In [2] (see also [4, p. 8]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_\Omega wf^2d\nu\int_\Omega wg^2d\nu-\left(\int_\Omega wfgd\nu\right)^2\leq\frac{1}{4}(A-a)^2\left(\int_\Omega wg^2d\nu\right)^2$$

provided that $ag\leq f\leq Ag$ ν-a.e. on Ω and $g^2\in L_w(\Omega,\nu)$.

Since, we also have

$$\int_\Omega wh^2d\nu\int_\Omega w\ell^2d\nu-\left(\int_\Omega wh\ell d\nu\right)^2\leq\frac{1}{4}(B-b)^2\left(\int_\Omega w\ell^2d\nu\right)^2,$$

provided that $b\ell\leq h\leq B\ell$ ν-a.e. on Ω and $\ell^2\in L_w(\Omega,\nu)$. Then, by (20) we have

$$\mathcal{D}^2(f,g,h,\ell;w,\Omega)\leq\frac{1}{16}(A-a)^2(B-b)^2\left(\int_\Omega wg^2d\nu\right)^2\left(\int_\Omega w\ell^2d\nu\right)^2$$

that is equivalent to the desired result (22). ☑

For positive margins we also have:

Corollary 3.3. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω with $g^2,\ell^2\in L_w(\Omega,\nu)$, $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$, and $a,A,b,B>0$ such that $A>a$, $B>b$,*

$$ag\leq f\leq Ag \text{ and } b\ell\leq h\leq B\ell \tag{23}$$

ν-a.e. on Ω. Then we have

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)|\leq\frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\int_\Omega wfgd\nu\int_\Omega wh\ell d\nu. \tag{24}$$

Proof. In [3] (see also [4, p. 16]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality:

$$\int_\Omega wf^2d\nu\int_\Omega wg^2d\nu-\left(\int_\Omega wfgd\nu\right)^2\leq\frac{(A-a)^2}{4aA}\left(\int_\Omega wfgd\nu\right)^2,$$

whenever $ag\leq f\leq Ag$ ν-a.e. on Ω.

Since

$$\int_\Omega wh^2d\nu\int_\Omega w\ell^2d\nu-\left(\int_\Omega wh\ell d\nu\right)^2\leq\frac{(B-b)^2}{4bB}\left(\int_\Omega wh\ell d\nu\right)^2,$$

provided $b\ell\leq h\leq B\ell$ ν-a.e. on Ω, then by (20) we get the desired result (24). ☑

If bounds for the sum and difference are available, then we have:

Corollary 3.4. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω with g^2, $\ell^2 \in L_w(\Omega, \nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants P_1, Q_1, P_2, Q_2 such that*

$$|g - f| \leq P_1, \quad |g + f| \leq Q_1, \quad |h - \ell| \leq P_2, \quad |h + \ell| \leq Q_2 \tag{25}$$

a.e. on Ω; then,

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \leq \frac{1}{4} P_1 Q_1 P_2 Q_2. \tag{26}$$

Proof. In the recent paper [5] we obtained amongst other the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality:

$$\int_\Omega w f^2 d\nu \int_\Omega w g^2 d\nu - \left(\int_\Omega w f g d\nu \right)^2 \leq \frac{1}{4} P_1^2 Q_1^2,$$

provided $|g - f| \leq P_1$, $|g + f| \leq Q_1$ a.e. on Ω.

Since

$$\int_\Omega w h^2 d\nu \int_\Omega w \ell^2 d\nu - \left(\int_\Omega w h \ell d\nu \right)^2 \leq \frac{1}{4} P_2^2 Q_2^2,$$

if $|h - \ell| \leq P_2$, $|h + \ell| \leq Q_2$ a.e. on Ω, then by (20) we get the desired result (26). ☑

If bounds for each function are available, then we have:

Corollary 3.5. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants a_i, A_i, b_i and B_i with $i \in \{1, 2\}$ such that*

$$0 < a_1 \leq f \leq A_1 < \infty, \qquad 0 < a_2 \leq g \leq A_2 < \infty, \tag{27}$$

and

$$0 < b_1 \leq h \leq B_1 < \infty, \qquad 0 < b_2 \leq \ell \leq B_2 < \infty, \tag{28}$$

a.e. on Ω; then,

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \leq \frac{1}{3} (A_1 A_2 - a_1 a_2)(B_1 B_2 - b_1 b_2). \tag{29}$$

Proof. We use the following Ozeki's type inequality obtained in [6]:

$$\int_\Omega w f^2 d\nu \int_\Omega w g^2 d\nu - \left(\int_\Omega w f g d\nu \right)^2 \leq \frac{1}{3} (A_1 A_2 - a_1 a_2)^2,$$

provided $0 < a_1 \leq f \leq A_1 < \infty$, $0 < a_2 \leq g \leq A_2 < \infty$ a.e. on Ω.

Since

$$\int_\Omega w h^2 d\nu \int_\Omega w \ell^2 d\nu - \left(\int_\Omega w h \ell d\nu \right)^2 \leq \frac{1}{3} (B_1 B_2 - b_1 b_2)^2,$$

when $0 < b_1 \leq h \leq B_1 < \infty$, $0 < b_2 \leq \ell \leq B_2 < \infty$ a.e. on Ω, then by (20) we get the desired result (29). ☑

4. Results for univariate functions

Let $\Omega = [a,b]$ be an interval of real numbers, and assume that $f,\ g,\ h,\ \ell : [a,b] \to \mathbb{R}$ are measurable $D-$Synchronous ($D-$Aynchronous), $w \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$ and $fh,\ g\ell,\ gh,\ f\ell \in L_w([a,b])$; then,

$$\int_a^b w(t) f(t) h(t)\,dt \int_a^b w(t) g(t) \ell(t)\,dt \tag{30}$$
$$\geq (\leq) \int_a^b w(t) g(t) h(t)\,dt \int_a^b w(t) f(t) \ell(t)\,dt.$$

Now, assume that $[a,b] \subset (0,\infty)$ and take $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$ with $p,\ q,\ r,\ s \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = t^{p-q} \quad \text{and} \quad \frac{h(t)}{\ell(t)} = t^{r-s}.$$

If $(p-q)(r-s) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a,b]$ while if $(p-q)(r-s) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a,b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$ that

$$\int_a^b w(t) t^{p+r} dt \int_a^b w(t) t^{q+s} dt \geq (\leq) \int_a^b w(t) t^{q+r} dt \int_a^b w(t) t^{p+s} dt, \tag{31}$$

provided $(p-q)(r-s) > (<) 0$.

Assume that $[a,b] \subset (0,\infty)$ and take $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$, $h(t) = \exp(\gamma t)$ and $\ell(t) = \exp(\delta t)$, with $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = \exp[(\alpha-\beta)t] \quad \text{and} \quad \frac{h(t)}{\ell(t)} = \exp[(\gamma-\delta)t].$$

If $(\alpha-\beta)(\gamma-\delta) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a,b]$, while if $(\alpha-\beta)(\gamma-\delta) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a,b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$ that

$$\int_a^b w(t) \exp[(\alpha+\gamma)t]\,dt \int_a^b w(t) \exp[(\beta+\delta)t]\,dt \tag{32}$$
$$\geq (\leq) \int_a^b w(t) \exp[(\beta+\gamma)t]\,dt \int_a^b w(t) \exp[(\alpha+\delta)t]\,dt,$$

provided $(\alpha-\beta)(\gamma-\delta) > (<) 0$.

Consider the functional

$$\mathcal{D}_{p,q,r,s}(w) := \int_a^b w(t) t^{p+r} dt \int_a^b w(t) t^{q+s} dt \tag{33}$$
$$- \int_a^b w(t) t^{q+r} dt \int_a^b w(t) t^{p+s} dt,$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, and $p, q, r, s \in \mathbb{R}$.

We observe that for $t \in [a,b] \subset (0,\infty)$ we have

$$k_{p,q}(a,b) := \begin{cases} a^{p-q}, & \text{if } p \geq q, \\ b^{p-q}, & \text{if } p < q, \end{cases} \leq \frac{f(t)}{g(t)} = t^{p-q} \tag{34}$$

$$\leq K_{p,q}(a,b) := \begin{cases} b^{p-q}, & \text{if } p \geq q, \\ a^{p-q} & \text{if } p < q, \end{cases}$$

and, similarly,

$$k_{r,s}(a,b) \leq \frac{h(t)}{\ell(t)} = t^{r-s} \leq K_{r,s}(a,b).$$

Using the inequality (22) we have

$$\begin{aligned} |\mathcal{D}_{p,q,r,s}(w)| \leq{} & \frac{1}{4}\left[K_{p,q}(a,b) - k_{p,q}(a,b)\right]\left[K_{r,s}(a,b) - k_{r,s}(a,b)\right] \\ & \times \int_a^b w(t)\,t^{2q}dt \int_a^b w(t)\,t^{2s}dt, \end{aligned} \tag{35}$$

while from (24) we have

$$\begin{aligned} |\mathcal{D}_{p,q,r,s}(w)| \leq{} & \frac{1}{4}\frac{\left[K_{p,q}(a,b) - k_{p,q}(a,b)\right]\left[K_{r,s}(a,b) - k_{r,s}(a,b)\right]}{\sqrt{k_{p,q}(a,b)\,k_{r,s}(a,b)\,K_{p,q}(a,b)\,K_{r,s}(a,b)}} \\ & \times \int_a^b w(t)\,t^{p+q}dt \int_a^b w(t)\,t^{r+s}dt. \end{aligned} \tag{36}$$

We also have for $t \in [a,b] \subset (0,\infty)$ that

$$u_p(a,b) := \begin{cases} a^p, & \text{if } p \geq 0, \\ b^p, & \text{if } p < 0, \end{cases} \leq f(t) = t^p$$

$$\leq U_p(a,b) := \begin{cases} b^p, & \text{if } p \geq 0, \\ a^p, & \text{if } p < 0, \end{cases}$$

and the corresponding bounds for $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$, with $p, q, r, s \in \mathbb{R}$.

Making use of the inequality (29) we get

$$\begin{aligned} |\mathcal{D}_{p,q,r,s}(w)| \leq{} & \frac{1}{3}\left(U_p(a,b)\,U_q(a,b) - u_p(a,b)\,u_q(a,b)\right) \\ & \times \left(U_r(a,b)\,U_s(a,b) - u_r(a,b)\,u_s(a,b)\right). \end{aligned} \tag{37}$$

Similar results may be stated for the functional

$$\begin{aligned} \mathcal{D}_{\alpha,\beta,\gamma,\delta}(w) :={} & \int_a^b w(t)\exp\left[(\alpha+\gamma)t\right]dt \int_a^b w(t)\exp\left[(\beta+\delta)t\right]dt \\ & - \int_a^b w(t)\exp\left[(\beta+\gamma)t\right]dt \int_a^b w(t)\exp\left[(\alpha+\delta)t\right]dt \end{aligned}$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, for $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ and $[a,b] \subset (0,\infty)$. Details are omitted.

We say that the function $\varphi : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ if

$$|\varphi(t) - \varphi(s)| \leq L\,|t-s|$$

for any $t,\ s \in [a,b]$.

Define the functional

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) := &\int_a^b w(t)\,f(t)\,h(t)\,dt \int_a^b w(t)\,g(t)\,\ell(t)\,dt\\ &- \int_a^b w(t)\,g(t)\,h(t)\,dt \int_a^b w(t)\,f(t)\,\ell(t)\,dt.\end{aligned}$$

In the next result we provided two upper bounds in terms of Lipschitzian constants:

Theorem 4.1. *Let $f,\ g,\ h,\ \ell : [a,b] \to \mathbb{R}$ be measurable functions and $w \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$.*

(i) If $g(t),\ell(t) \neq 0$ for any $t \in [a,b]$, and $\frac{f}{g}$ is Lipschitzian with the constant $L > 0$, and $\frac{h}{\ell}$ is Lipschitzian with the constant $K > 0$, and $g\ell,\ g\ell e^2 \in L_w([a,b])$ with $e(t) = t,\ t \in [a,b]$, then

$$\begin{aligned}&|\mathcal{D}(f,g,h,\ell;w,[a,b])|\\ &\quad\leq LK\left[\int_a^b w(s)\,|g(s)|\,|\ell(s)|\,ds \int_a^b w(t)\,|\ell(t)|\,|g(t)|\,t^2 dt\right.\\ &\qquad\left. - \left(\int_a^b w(t)\,|g(t)|\,|\ell(t)|\,tdt\right)^2\right]. \qquad (38)\end{aligned}$$

(ii) If, in addition, we have $wg\ell \in L_\infty[a,b]$ and

$$\|wg\ell\|_\infty = \operatorname{esssup}_{t\in[a,b]} |w(t)\,g(t)\,\ell(t)| < \infty,$$

then

$$|\mathcal{D}(f,g,h,\ell;w,[a,b])| \leq \frac{1}{12}(b-a)^4 LK\,\|wg\ell\|_\infty^2. \qquad (39)$$

Proof. We have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) &= \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s)\,(f(t)\,g(s) - g(t)\,f(s))\\ &\quad\times (h(t)\,\ell(s) - \ell(t)\,h(s))\,dtds\\ &= \frac{1}{2}\int_a^b \int_a^b w(t)\,w(s)\,g(t)\,g(s)\,\ell(t)\,\ell(s)\\ &\quad\times \left(\frac{f(t)}{g(t)} - \frac{f(s)}{g(s)}\right)\left(\frac{h(t)}{\ell(t)} - \frac{h(s)}{\ell(s)}\right) dtds.\end{aligned}$$

By taking modulus in this equality, we get

$$
\begin{aligned}
&\left|\mathcal{D}\left(f,g,h,\ell;w,[a,b]\right)\right| \qquad (40)\\
&\quad\leq \frac{1}{2}\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|\times\left|\frac{f\left(t\right)}{g\left(t\right)}-\frac{f\left(s\right)}{g\left(s\right)}\right|\left|\frac{h\left(t\right)}{\ell\left(t\right)}-\frac{h\left(s\right)}{\ell\left(s\right)}\right|dt\,ds\\
&\quad\leq \frac{1}{2}LK\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|\left(t-s\right)^2 dt\,ds.
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
&\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|\left(t-s\right)^2 dtds \qquad (41)\\
&\quad=\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|\left(t^2-2ts+s^2\right)dtds\\
&\quad=2\left(\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|t^2dtds\right.\\
&\qquad\left.-\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|tsdtds\right)\\
&\quad=2\left[\int_a^b w\left(s\right)\left|g\left(s\right)\right|\left|\ell\left(s\right)\right|ds\int_a^b w\left(t\right)\left|g\left(t\right)\right|\left|\ell\left(t\right)\right|t^2dt\right.\\
&\qquad\left.-\left(\int_a^b w\left(t\right)\left|g\left(t\right)\right|\left|\ell\left(t\right)\right|tdt\right)^2\right].
\end{aligned}
$$

On making use of (40) and (41) we get the desired result (38).

If $wg\ell\in L_\infty\left[a,b\right]$, then

$$
\begin{aligned}
&\int_a^b\int_a^b w\left(t\right)w\left(s\right)\left|g\left(t\right)\right|\left|g\left(s\right)\right|\left|\ell\left(t\right)\right|\left|\ell\left(s\right)\right|\left(t-s\right)^2 dtds\\
&\qquad\leq\left\|wg\ell\right\|_\infty^2\int_a^b\int_a^b\left(t-s\right)^2dtds=\frac{1}{6}\left(b-a\right)^4\left\|wg\ell\right\|_\infty^2. \qquad (42)
\end{aligned}
$$

Therefore, by inequalities (40) and (42) we obtain the desired result (39). ☑

5. Discrete inequalities

Consider the n-tuples of real numbers $x=\left(x_1,...,x_n\right)$, $y=\left(y_1,...,y_n\right)$, $z=\left(z_1,...,z_n\right)$ and $u=\left(u_1,...,u_n\right)$. We say that the quadruple $\left(x,y,z,u\right)$ is $D-$*Synchronous* if

$$
\begin{aligned}
0&\leq\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}\det\begin{pmatrix}z_i & z_j\\ u_i & u_j\end{pmatrix} \qquad (43)\\
&=\left(x_iy_j-x_jy_i\right)\left(z_iu_j-z_ju_i\right)
\end{aligned}
$$

for any $i,\ j \in \{1, ..., n\}$.

If $p = (p_1, ..., p_n)$ is a probability distribution, namely, $p_i \geq 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} p_i = 1$, and the quadruple (x, y, z, u) is $D-$Synchronous, then by (10) we have:

$$\mathcal{D}_n(x, y, z, u; p) := \det \begin{pmatrix} \sum_{i=1}^{n} p_i x_i z_i & \sum_{i=1}^{n} p_i y_i z_i \\ \sum_{i=1}^{n} p_i x_i u_i & \sum_{i=1}^{n} p_i y_i u_i \end{pmatrix} = \sum_{i=1}^{n} p_i x_i z_i \sum_{i=1}^{n} p_i y_i u_i - \sum_{i=1}^{n} p_i x_i u_i \sum_{i=1}^{n} p_i y_i z_i \geq 0. \quad (44)$$

For an n-tuples of real numbers $x = (x_1, ..., x_n)$, we denote by $|x| := (|x_1|, ..., |x_n|)$. On making use of the inequality (17), then for any $D-$Synchronous quadruple (x, y, z, u) and for any probability distribution $p = (p_1, ..., p_n)$ we have

$$\mathcal{D}_n(x, y, z, u; p) \geq \max\{|\mathcal{D}_n(|x|, y, z, u; p)|, |\mathcal{D}_n(x, |y|, z, u; p)|, |\mathcal{D}_n(|x|, |y|, z, u; p)|\} \geq 0. \quad (45)$$

Observe that if we consider

$$\mathcal{D}_n(x, y; p) := \mathcal{D}_n(x, y, x, y; p) = \sum_{i=1}^{n} p_i x_i^2 \sum_{i=1}^{n} p_i y_i^2 - \left(\sum_{i=1}^{n} p_i x_i y_i\right)^2,$$

then by (20) we have

$$|\mathcal{D}_n(x, y, z, u; p)|^2 \leq \mathcal{D}_n(x, y; p)\, \mathcal{D}_n(z, u; p) \quad (46)$$

for any quadruple (x, y, z, u) and any probability distribution $p = (p_1, ..., p_n)$.

If $a,\ A,\ b,\ B \in \mathbb{R}$ and (x, y, z, u) are such that $A > a,\ B > b$,

$$a y_i \leq x_i \leq A y_i \text{ and } b u_i \leq z_i \leq B u_i \quad (47)$$

for any $i \in \{1, ..., n\}$, then by (22) we have

$$|\mathcal{D}_n(x, y, z, u; p)| \leq \frac{1}{4}(A - a)(B - b) \sum_{i=1}^{n} p_i y_i^2 \sum_{i=1}^{n} p_i u_i^2. \quad (48)$$

If $a,\ A,\ b,\ B > 0$ and condition (47) is valid, then by (24) we have

$$|\mathcal{D}_n(x, y, z, u; p)| \leq \frac{1}{4} \frac{(A - a)(B - b)}{\sqrt{aAbB}} \sum_{i=1}^{n} p_i x_i y_i \sum_{i=1}^{n} p_i z_i u_i. \quad (49)$$

Now, if we use the *Klamkin-McLenaghan's inequality*

$$\sum_{i=1}^{n} p_i x_i^2 \sum_{i=1}^{n} p_i y_i^2 - \left(\sum_{i=1}^{n} p_i x_i y_i\right)^2 \leq \left(\sqrt{A} - \sqrt{a}\right)^2 \sum_{i=1}^{n} p_i x_i y_i \sum_{i=1}^{n} p_i x_i^2$$

that holds for x, y satisfying the condition (47) with A, $a > 0$, then by (46) we get

$$\begin{aligned} &\left|\mathcal{D}_n(x,y,z,u;p)\right| \\ &\quad\leq \left(\sqrt{A}-\sqrt{a}\right)\left(\sqrt{B}-\sqrt{b}\right) \\ &\qquad\times\left(\sum_{i=1}^{n} p_i x_i y_i\right)^{1/2}\left(\sum_{i=1}^{n} p_i x_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i z_i u_i\right)^{1/2}\left(\sum_{i=1}^{n} p_i z_i^2\right)^{1/2}, \end{aligned} \tag{50}$$

provided (x, y, z, u) satisfy (47) with a, A, b, $B > 0$.

Now, assume that

$$0 < a_1 \leq x_i \leq A_1 < \infty, \qquad 0 < a_2 \leq y_i \leq A_2 < \infty, \tag{51}$$

and

$$0 < b_1 \leq x_i \leq B_1 < \infty, \qquad 0 < b_2 \leq u_i \leq B_2 < \infty, \tag{52}$$

for any $i \in \{1, ..., n\}$; then by (29) we get

$$\left|\mathcal{D}_n(x,y,z,u;p)\right| \leq \frac{1}{3}\left(A_1A_2 - a_1a_2\right)\left(B_1B_2 - b_1b_2\right), \tag{53}$$

for any probability distribution $p = (p_1, ..., p_n)$.

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References

[1] Cerone P. and Dragomir S.S., “A refinement of the Grüss inequality and applications”, *Tamkang J. Math.* 38 (2007), No. 1, 37-49. doi: 10.5556/j.tkjm.38.2007.92.

[2] Dragomir S.S., “A counterpart of Schwarz’s inequality in inner product spaces”. arXiv: math/0305373.

[3] Dragomir S.S., “A generalization of the Cassels and Greub-Reinboldt inequalities in inner product spaces”. arXiv: math/0306352.

[4] Dragomir S.S., *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers Inc, New York, 2005.

[5] Dragomir S.S., “Reverses of Schwarz inequality in inner product spaces with applications”, *Math. Nachr.* 288 (2015), No. 7, 730-742. doi: 10.1002/mana.201300100.

[6] Izumino S., Mori H. and Seo Y., “On Ozeki’s inequality”, *J. Inequal. Appl.* 2 (1998), No. 3, 235-253. doi: 10.1155/S1025583498000149.