$\mathcal{Z}$-HYPERRIGIDITY AND $\mathcal{Z}$-BOUNDARY REPRESENTATIONS

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ABSTRACT

In this article, we introduce the notions of $\mathcal{Z}$-finite representations and $\mathcal{Z}$-separation property of representations for operator $\mathcal{Z}$-systems generating C^*-algebras. We use these notions to characterize the $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-systems. We introduce $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-systems. We investigate an analogue version of Saskin's theorem in the setting of operator $\mathcal{Z}$-systems generating C^*-algebras.

KEYWORDS

Completely positive maps, Operator systems, Representations of C^*-algebras, Hilbert C^*-modules.

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1 INTRODUCTION

Let S be a subspace (subalgebra) of $C(X)$, the set of continuous functions on compact metric space X. The *Choquet boundary* of S consists of the points $x \in X$ with the property that there is a unique probability measure μ on X, such that $f(x) = \int_X f d\mu$, $f \in S$. In other words, the points $x \in X$ lie in the Choquet boundary of S if the point evaluation functional $f \mapsto f(x)$, $f \in S$ extends to a unique state on the C^*-algebra $C(X)$. The Choquet boundary is a significant object to study for at least two reasons. The Choquet boundary of S is dense in the *Shilov boundary* of S. Shilov boundary is the smallest closed subset of X on which every function in S attains its maximum modulus. Choquet boundary supplies a tool to identify the "minimal" representations of the elements of S as functions on some compact metric space. For more details, refer to [5].

Korovkin theorem [14] deals with the convergence of positive linear maps on function algebras. The classical Korovkin theorem is as follows: for each $n \in \mathbb{N}$, let $\phi_n : C[0,1] \to C[0,1]$ be a positive linear map. If $\lim_{n\to\infty} ||\phi_n(f) - f||$ for every $f \in \{1, x, x^2\}$, then $\lim_{n\to\infty} ||\phi_n(f) - f||$ for every $f \in C[0,1]$. The set $\{1, x, x^2\}$ is called a Korovkin set in $C[0,1]$. There is a close connection between Korovkin sets and Choquet boundaries. Saskin [5, 23] proved that G is a Korovkin set in $C[0,1]$ if and only if the Choquet boundary of G is $[0,1]$.

Arveson [2] initiated a non-commutative analogue of the Choquet boundary in the context of unital operator algebras and operator systems in C^*-algebra. The central objects in his approach are the so-called boundary representations. Certain unital completely positive linear maps have unique extension property, almost in the spirit of defining property for points to lie in the classical Choquet boundary. The conjecture of Arveson states that every operator system and every unital operator algebra has sufficiently many boundary representations to norm it completely. Hamana [11] constructed the C^*-envelope of the operator system using a different method. Arveson [3] proved the conjecture for separable C^*-algebras. Davidson and Kennedy [8] completely settled conjecture on boundary representations. Fuller, Hartz, and Lupini [10] introduced the notion of boundary representations for operator spaces in ternary rings of operators. They established the natural operator space analogue of Arveson's conjecture on boundary representations. Magajna [17] introduced $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-system generating a C^*-algebra on self-dual Hilbert $\mathcal{Z}$-modules, where $\mathcal{Z}$ is abelian von Neumann algebra. Magajna [17] proved analogue of Arveson's conjecture for $\mathcal{Z}$-boundary representations of C^*-algebra generated by operator $\mathcal{Z}$-systems on self-dual Hilbert $\mathcal{Z}$-modules over abelian von Neumann algebra $\mathcal{Z}$.

Arveson [4] introduced the notion of hyperrigid set, which is a non-commutative analogue of the Korovkin set. Arveson studied hyperrigidity in the setting of operator systems in C^*-algebras, and he tried to prove an analogue version of Saskin's theorem using hyperrigidity and boundary representations. Arveson [4] proved if every operator system is hyperrigid in generating C^*-algebra, then every irreducible representation of C^*-algebra is a boundary representation for the operator system. But he could not be able to prove the converse in generality. The converse of the above result is called *Arveson's hyperrigidity conjecture*. Hyperrigidity conjecture is as follows: for an operator system S and the generated C^*-algebra A, if every irreducible representation of A is a boundary representation for S, then an operator system S is hyperrigid. Arveson [4] showed that the hyperrigidity conjecture is valid for C^*-algebras with a countable spectrum.

Davidson and Kennedy [9] established a dilation-theoretic characterization of the Choquet order on the space of measures on a compact convex set using ideas from the theory of operator algebras. This yields an extension of Cartier's dilation theorem to the non-separable case and a non-separable version of Šaškin's theorem from approximation theory. They showed that a slight variant of this order characterizes the representations of commutative C^*-algebras with the unique extension property relative to a set of generators. This reduces the commutative case of Arveson's hyperrigidity conjecture to whether measures that are maximal concerning the classical Choquet order are also maximal concerning this new order.

Kleski [13] established the hyperrigidity conjecture for all type-I C^*-algebras with additional assumptions on the co-domain. The hyperrigidity conjecture is still open for general C^*-algebras. The hyperrigidity conjecture inspired several studies in recent years [6, 7, 12, 21]. Arunkumar, Shankar, and

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Vijayarajan [1] introduced rectangular hyperrigidity in setting operator spaces in a ternary ring of operators. They established an analogue version of Saskin's theorem in the case of a finite-dimensional ternary ring of operators, and they gave some partial answers analogue to results in the papers [4, 13].

This paper is divided into three sections besides the introduction. In Section 2, we gather the necessary background material and required results. In section 3, we introduce the notions of $\mathcal{Z}$-finite representations for operator $\mathcal{Z}$-systems and $\mathcal{Z}$-separation property of operator $\mathcal{Z}$-systems. These notions are generalizations of finite representation and separating property for representations introduced by Arveson [2]. We use these notions to characterize the $\mathcal{Z}$-boundary representations for operator $\mathcal{Z}$-systems. In section 4, We introduce $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-systems in generating C^*-algebras which is a generalization of hyperrigidity introduced by Arveson [4]. We investigate an analogue version of Saskin's theorem in the setting of operator $\mathcal{Z}$-systems generating C^*-algebra.

2 PRELIMINARIES

A *representation* of a unital C^*-algebra A on a Hilbert space $\mathcal{H}$ makes $\mathcal{H}$ a Hilbert A-module. Let $\mathbb{B}_A(\mathcal{H})$ denote the set of all bounded A-module maps on $\mathcal{H}$. We will denote von Neumann algebras by $\mathcal{A}, \mathcal{B}, ..., \mathcal{Z}$ and general C^*-algebras by $A, B, ...$ Let A be C^*-algebra and A is faithfully represented on a Hilbert space $\mathcal{H}$. $X \subseteq \mathbb{B}(\mathcal{H})$ is said to be a *faithful operator C-system* if X is a norm closed self-adjoint C-subbimodule of $\mathbb{B}(\mathcal{H})$ (for more details and abstract characterization refer [22]).

Let $\mathcal{H}$ be a Hilbert A-module. Let $\mathrm{CCP}_A(X, \mathbb{B}(\mathcal{H}))$ denote the set of all contractive completely positive A-bimodule maps form X into $\mathbb{B}(\mathcal{H})$. Let $\mathrm{UCP}_A(X, \mathbb{B}(\mathcal{H}))$ denote the set of all unital completely positive A-bimodule maps form X into $\mathbb{B}(\mathcal{H})$. Let X be a faithful operator A-system contained in a C^*-algebra B so that A and B have the same unit 1. By the well-known multiplicative domain argument [22, 3.18] any completely positive extension to B of a map $\varphi \in \mathrm{UCP}_A(X, \mathbb{B}(\mathcal{H}))$ must be a A-bimodule map since φ extends the representation $\varphi|_A$.

The motive of this article is to extend the main results of the papers [2], and [4] in the context of Hilbert spaces are replaced by Hilbert C^*-modules over abelian von Neumann algebra $\mathcal{Z}$. For a theory of Hilbert C^*-modules, we refer to [15, 19]. Hilbert C^*-modules over von Neumann algebras $\mathcal{Z}$ are like Hilbert spaces, except that the inner product takes values in $\mathcal{Z}$. Let $\mathcal{E}$ be Hilbert $\mathcal{Z}$-module, we denote $\langle \cdot, \cdot \rangle$ the $\mathcal{Z}$-valued inner product on $\mathcal{E}$ and let $|e| := \sqrt{\langle e, e \rangle}$ the corresponding $\mathcal{Z}$-valued norm. For $e \in \mathcal{E}$, the scalar-valued norm is denoted by $||e|| := ||\langle x, x \rangle||^{\frac{1}{2}}$. A Hilbert $\mathcal{Z}$-module is said to *self-dual* if each $\mathcal{Z}$-module map ϕ from $\mathcal{E}$ to $\mathcal{Z}$ has the form $\phi(e) = \langle e, f \rangle$ for an $f \in \mathcal{E}$. Let $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ denote the set of all bounded $\mathcal{Z}$-module endomorphisms of $\mathcal{E}$. If $\mathcal{E}$ is self-dual then $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is adjointable. If $\mathcal{E} \subseteq \mathcal{F}$ are self-dual C^*-modules over $\mathcal{Z}$ then $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}^{\perp}$.

The following definitions and results are due to Magajna [17]. A map $\psi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ is called $\mathcal{Z}$- dilation of $\varphi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ for self-dual C^*-module $\mathcal{F} \supseteq \mathcal{E}$ over $\mathcal{Z}$ if $p\psi(x)|_{\mathcal{E}} = \varphi(x)\ \forall\ x \in X$, where $p : \mathcal{F} \to \mathcal{E}$ is the orthogonal projection. We write $\psi \succeq_{\mathcal{Z}} \varphi$ if ψ is a $\mathcal{Z}$-dilation of φ. A map $\varphi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to be $\mathcal{Z}$-maximal if every $\psi \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$, where $\mathcal{F}$ is a self-dual C^*-module over $\mathcal{Z}$, satisfying $\psi \succeq_{\mathcal{Z}} \varphi$, decomposes as $\psi = \varphi \oplus \theta$ for some $\theta \in \mathrm{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}^{\perp}))$.

Remark 1. *From [17, Remark 4.12] observe that, if an operator $\mathcal{Z}$-system X is contained in a C^*-algebra B generated by X and containing $\mathcal{Z}$ in its center, any map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ can be extended to a map $\tilde{\varphi} \in UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. An analogue version of Stinespring's dilation theorem for $\tilde{\varphi}$ can be represented as follows:*

$$\tilde{\varphi}(b) = V^*\pi(b)V \quad \forall \quad b \in B,$$

where $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ is a representation on a self-dual C^*-module $\mathcal{F}$ over $\mathcal{Z}$ and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ is an isometry such that $[\pi(B)V\mathcal{E}] = \mathcal{F}$. Observe that $[\pi(B)V\mathcal{E}] = \mathcal{F}$ is the minimality condition for an analogue version of Stinespring's decomposition. For more details see [15, Theorem 5.6] and [20, Corollary 5.3]. Paschke [20, Proposition 5.4] proved the analogue of Arveson's [2, Theorem 1.4.2] affine order isomorphism theorem.

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Definition 1. *[17] A map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to have a $\mathcal{Z}$-unique extension property ($\mathcal{Z}$-u.e.p) if φ has a unique completely positive $\mathcal{Z}$-bimodule extension $\tilde{\varphi} : C^*(X) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ and $\tilde{\varphi}$ is a representation of $C^*(X)$ on $\mathcal{E}$.*

Arveson [3, Proposition 2.4] proved that maximality is equivalent to the notion of unique extension property in the Hilbert space setting. Similar arguments from [3, Proposition 2.4] imply that the idea of $\mathcal{Z}$-maximality is equivalent to the notion of $\mathcal{Z}$-unique extension property in Hilbert $\mathcal{Z}$-module setting.

A representation (i.e., a homomorphism of C^*-algebras) $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is said to be $\mathcal{Z}$-irreducible if $\pi(B)' = \pi(\mathcal{Z})$.

Definition 2. *[17] A map $\varphi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ is said to be $\mathcal{Z}$-pure if every $\psi \in UCP_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$, $\psi \leq \varphi$ implies that $\psi = c\varphi$, where $c \in \mathcal{Z}$.*

Remark 2. *We can observe that an analogue of [2, Corollary 1.4.3] follows from [17, Remark 4.12 and Remark 4.14]. A non zero pure map in $UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ are precisely those of the form $\tilde{\varphi}(b) = V^*\pi(b)V \quad \forall \quad b \in B$, where π is an $\mathcal{Z}$-irreducible representation of B on some self-dual Hilbert C^*-module $\mathcal{F}$ over $\mathcal{Z}$ and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$, $V \neq 0$.*

Definition 3. *[17] A $\mathcal{Z}$-irreducible representation $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ (for some self-dual $\mathcal{E}$) is called $\mathcal{Z}$-boundary representation of B for X if $\pi|_X$ has the $\mathcal{Z}$-unique extension property.*

Magajna [17] proved analogue of Arveson's conjecture on $\mathcal{Z}$-boundary representations as follows:

Theorem 1. *If X is a central operator $\mathcal{Z}$-system generating a C^*-algebra A, then $\mathcal{Z}$-boundary representation of A for X on self-dual Hilbert C^*-modules over $\mathcal{Z}$ completely norm X.*

3 Z-BOUNDARY REPRESENTATION

This section establishes the characterization theorem for $\mathcal{Z}$-boundary representations. This characterization theorem is an analogue version of [2, Theorem 2.4.5]. In general, checking the given representation is $\mathcal{Z}$-boundary representation is not easy. Using this characterization theorem, at least we can detect the representations that are not $\mathcal{Z}$-boundary representations.

Proposition 1. *Let X be a operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. If π is a $\mathcal{Z}$-boundary representation of B for X then $\pi|_X$ is $\mathcal{Z}$-pure.*

Proof. Let $\mathcal{E}$ self-dual Hilbert C^*-module over $\mathcal{Z}$ on which π acts. Let $\varphi_1, \varphi_2 \in \mathrm{CP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ be such that $\pi|_X = \varphi_1 + \varphi_2$. By [17, Remark 4.12] each φ_i can be extended to unital completely positive $\mathcal{Z}$-bimodule map $\tilde{\varphi}_i : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ such that $\tilde{\varphi}_i|_X = \varphi$ for $i = 1, 2$. Observe that $\tilde{\varphi}_1 + \tilde{\varphi}_2 : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$. Since π is a $\mathcal{Z}$-boundary representation for X, thus $\tilde{\varphi}_1(b) + \tilde{\varphi}_2(b) = \pi(b)$ for all $b \in B$. Also, π is an $\mathcal{Z}$-irreducible representation of B so by Remark 2 $\tilde{\varphi}_1 + \tilde{\varphi}_2$ is a $\mathcal{Z}$-pure map in $\mathrm{CP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. Thus, there are $c_i \in \mathcal{Z}$ such that $\tilde{\varphi}_i = c_i\pi$ on B for $i = 1, 2$. Restricting to X we have $\varphi_i = c_i\pi|_X$ for $i = 1, 2$. Hence $\pi|_X$ is $\mathcal{Z}$-pure.

Magajna [16, 18] studied an analogue of C^*-convexity and C^*-extreme points of operators on Hilbert C^*-modules. He introduced A-convexity and A-extreme points as follows: Let $\mathcal{K}$ be a Hilbert module over a C^*-algebra A. A subset $K \subseteq \mathbb{B}_A(\mathcal{K})$ is called *A-convex* if $\sum_{j=1}^n a_j^* y_j a_j \in K$ whenever $y_j \in K$, $a_j \in A$ and $\sum_{j=1}^n a_j^* a_j = 1$. A point x in an A-convex set K is called an *A-extreme point* of K if the condition $x = \sum_{j=1}^n a_j^* y_j a_j$, where $x_j \in K$, $a_j \in A$, $\sum_{j=1}^n a_j^* a_j = 1$ (n finite) and a_j are invertible, implies that there exist unitary elements $u_j \in A$ such that $x_j = u_j^* x u_j$. By [16, Lemma 5.5], it is enough to check the A-extreme point condition for the case $n = 2$.

Proposition 2. *Let X be a operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let π be a $\mathcal{Z}$-irreducible representation of B such that $\mathfrak{K} = \{\varphi \in UCP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$. If π is a $\mathcal{Z}$-boundary representation of B for X then every $\varphi \in \mathfrak{K}$ is a $\mathcal{Z}$-extreme point of $\mathfrak{K}$.*

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Proof. Let φ be in $\mathfrak{K}$. Suppose $\varphi = \sum_{i=1}^{n} V_i^* \varphi_i V_i$, where $\varphi_i \in \mathfrak{K}$, $V_i \in \mathcal{Z}$, $\sum_{i=1}^{n} V_i^* V_i = 1$ (n finite) and V_i are invertible. Since π is a $\mathcal{Z}$-boundary representation of B for X and $\pi|_X = \varphi|_X$, we have $\pi = \varphi$ on B. An analogue version of minimal Stinespring decomposition of φ is trivial. Thus, by [20, Proposition 5.4] the inequality $V_i^* \varphi_i V_i \leq \varphi$ implies that there exist positive contractions $S_i \in \varphi(B)' = \varphi(\mathcal{Z})$ such that $V_i^* \varphi_i V_i = S_i \varphi$. Therefore $\varphi_i = (S_i^{\frac{1}{2}} V_i^{-1})^* \varphi (S_i^{\frac{1}{2}} V_i^{-1})$ and $\varphi_i(1) = \varphi(1) = 1$. Thus $S_i^{\frac{1}{2}} V_i^{-1}$ is an isometry. Again, using an analogue of minimal Stinespring decomposition of φ is trivial. We conclude that φ_i is unitarily equivalent to φ for every i. Hence φ is a $\mathcal{Z}$-extreme point of $\mathcal{K}$.

We introduce a $\mathcal{Z}$-finite representation, an analogue version of finite representation from [2].

Definition 4. *Let X be an operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a representation of B. π is called $\mathcal{Z}$-finite representation for X if for every isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ the condition $V^*\pi(x)V = \pi(x)$ for all $x \in X$, implies that V is unitary.*

Proposition 3. *Let X be an operator $\mathcal{Z}$-system in a C^*-algebra B such that $B = C^*(X)$ and let π be an $\mathcal{Z}$-irreducible representation of B. If π is a $\mathcal{Z}$-boundary representation for X then π is a $\mathcal{Z}$-finite representation for X.*

Proof. Let π acts on the self-dual Hilbert C^*-module $\mathcal{E}$ and let V be an isometry in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ such that $V^*\pi(x)V = \pi(x)$ for all $x \in X$. Then $V^*\pi(\cdot)V : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$. Since π is a $\mathcal{Z}$-boundary representation implies that $V^*\pi(b)V = \pi(b)$ for all $b \in B$. We have V is isometry and $[\pi(B)V\mathcal{E}] = \mathcal{E}$ implies $V\mathcal{E}$ is a reducing subspace for $\pi(B)$. Also, π is $\mathcal{Z}$-irreducible implies $V\mathcal{E} = \mathcal{E}$. Therefore V is unitary. Hence π is a $\mathcal{Z}$-finite representation for X.

We introduce separating operator $\mathcal{Z}$-system, an analogue version of separating operator system from [2].

Definition 5. *Let X be an operator $\mathcal{Z}$-system and B be a C^*-algebra generated by X. Let $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a $\mathcal{Z}$-irreducible representation of B. We say that X $\mathcal{Z}$-separates π if for every $\mathcal{Z}$-irreducible representation σ of B on self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$ and for every isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$, the condition $V^*\sigma(x)V = \pi(x)$ for all $x \in X$ implies that σ and π are unitarily equivalent representations of B.*

Proposition 4. *Let X be an operator $\mathcal{Z}$-system in a C^*-algebra B such that $B = C^*(X)$. If $\pi : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ is a $\mathcal{Z}$-boundary representation of B for X then X $\mathcal{Z}$-separates π.*

Proof. Let $\sigma : B \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ be a $\mathcal{Z}$-irreducible representation of B, where $\mathcal{F}$ is self-dual Hilbert $\mathcal{Z}$-module and $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ is an isometry such that $V^*\sigma(x)V = \pi(x)$ for all $x \in X$. Since π is a $\mathcal{Z}$-boundary representation for X and $V^*\sigma(\cdot)V$ is a completely positive $\mathcal{Z}$-bimodule extension of $\pi|_X$ implies $V^*\sigma(b)V = \pi(b)$ for all $b \in B$. We have V is isometry and $[\pi(B)V\mathcal{E}] = \mathcal{F}$ implies $V\mathcal{E}$ is a reducing subspace for $\sigma(B)$. Also, σ is $\mathcal{Z}$-irreducible implies $V\mathcal{E} = \mathcal{F}$. Thus V is unitary, showing that σ and π are unitarily equivalent representations. Hence X $\mathcal{Z}$-separates π.

The characterization theorem for $\mathcal{Z}$-boundary representations as follows:

Theorem 2. *Let X be an operator $\mathcal{Z}$-systems in a C^*-algebra B such that $B = C^*(X)$. Let π be an $\mathcal{Z}$-irreducible representation of B. Then π is a $\mathcal{Z}$-boundary representation for X if and only if the following conditions are satisfied:*

(i) *π is a $\mathcal{Z}$-finite representation for X.*

(ii) *$\pi|_X$ is $\mathcal{Z}$-pure.*

(iii) *X $\mathcal{Z}$-separates π.*

(iv) *Let $\mathfrak{K} = \{\varphi \in CP_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$ and every φ in $\mathfrak{K}$ is $\mathcal{Z}$-extreme point of $\mathfrak{K}$.*

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Proof. Suppose π is a $\mathcal{Z}$-boundary representation of B for X then conditions (i), (ii),(iii) and (iv) follows from Proposition 3, Proposition 1, Proposition 4 and Proposition 2.

Conversely, Suppose the $\mathcal{Z}$-irreducible representation satisfies all four conditions (i), (ii), (iii), and (iv). Let $\mathfrak{K} = \{\varphi \in \mathrm{CP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E})) : \varphi|_X = \pi|_X\}$. To show π is a $\mathcal{Z}$-boundary representation for X, it is enough to show that $\mathfrak{K}$ is $\{\pi\}$. Using (iv), let φ be $\mathcal{Z}$-extreme point of $\mathfrak{K}$. Now, we prove that φ is a $\mathcal{Z}$-pure in $\mathrm{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$. Let φ_1, φ_2 be in $\mathfrak{K}$ such that $\varphi_1(b) + \varphi_2(b) = \varphi(b)$ for all $b \in B$. In particular, $\varphi_1(x) + \varphi_2(x) = \varphi(x)$ for all $x \in X$. Our assumption, $\varphi|_X = \pi|_X$ is pure implies there exists $c_i \geq 0$ in $\mathcal{Z}$ such that $\varphi_i(x) = c_i\varphi(x)$ for all $x \in X$. If $c_1 = 0$ and $1 \in X$ then $\varphi_1(1) = 0$, thus $\varphi_1 = 0$. This contracts to the choice of φ_1, therefore $c_1 > 0$ and similarly $c_2 > 0$. Using [16, Definition 5.1 and Proposition 5.2], and every $\mathcal{Z}$-extreme points are Choquet $\mathcal{Z}$-points, we have $\varphi_i = (c_i^{\frac{1}{2}})^*\varphi(c_i^{\frac{1}{2}})$ and $(c_1^{\frac{1}{2}})^*c_1^{\frac{1}{2}} + (c_2^{\frac{1}{2}})^*c_2^{\frac{1}{2}} = 1$. Now put $\psi_i = (c_i^{-\frac{1}{2}})^*\varphi_i(c_i^{-\frac{1}{2}})$ for $i = 1, 2$. Then $\psi_i \in \mathfrak{K}$ and $(c_1^{\frac{1}{2}})^*\psi_1(c_1^{\frac{1}{2}}) + (c_2^{\frac{1}{2}})^*\psi_2(c_1^{\frac{1}{2}}) = \varphi$. Since φ is $\mathcal{Z}$-extreme point of $\mathfrak{K}$ then there exists unitary elements $u_i \in \mathcal{Z}$ for $i = 1, 2$ such that $\psi_i = u_i^*\varphi u_i$ for $i = 1, 2$. We have $(c_i^{-\frac{1}{2}})^*\varphi_i(c_i^{-\frac{1}{2}}) = u_i^*\varphi u_i$ for $i = 1, 2$ and $c_i^{-1}\varphi_i = u_i^*\varphi u_i$ for $i = 1, 2$. By [16, Definition 5.1 and Proposition 5.2], thus $\varphi_i = c_i\varphi$ for $i = 1, 2$. Hence φ is a $\mathcal{Z}$-pure in $\mathrm{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$.

By Remark 2, there is a $\mathcal{Z}$-irreducible representation σ of B on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$ and an isometry $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ such that $\varphi = V^*\sigma V$. In particular, $\pi(x) = \varphi(x) = V^*\sigma(x)V$ for all $x \in X$. The assumption (iii), X $\mathcal{Z}$-separates π implies that σ is unitarily equivalent to π. Thus there exists a unitary $U \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$ such that $\sigma = U^*\pi U$. Therefore we have $\pi(x) = (UV)^*\pi(x)UV$ for all $x \in X$. UV is isometry in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$. The assumption (i), π is a $\mathcal{Z}$-finite representation for X implies UV is unitary. Thus $V = U^*UV$ is a unitary in $\mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$. Now $\pi|_X = V^*\sigma V|_X$ becomes $\pi(x) = V^{-1}\sigma(x)V$ for all $x \in X$. $V^{-1}\sigma V$ is a representation of B which agrees with π on X. Therefore $\pi(b) = V^{-1}\sigma(b)V$ for all $b \in B = C^*(X)$. Hence $\varphi = \pi$ on B.

4 Z-HYPERRIGIDITY

In this section, we introduce the notion of $\mathcal{Z}$-hyperrigidity in the operator $\mathcal{Z}$-system. $\mathcal{Z}$-hyperrigidity is an analogue version of Arveson's [4] notion of hyperrigidity. We define $\mathcal{Z}$-hyperrigidity as follows:

Definition 6. *Let A be a C^*-algebra, and let $G \subseteq A$(finite or countably infinite) be a set of generators of A (i.e., $A = C^*(G)$). Then G is said to be $\mathcal{Z}$-hyperrigid if for every faithful representation $A \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ of A on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{E}$ and every sequence of unital completely positive $\mathcal{Z}$-bimodule maps $\varphi_n : \mathbb{B}_{\mathcal{Z}}(\mathcal{E}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$, $n = 1, 2, ...$,*

$$\lim_{n\to\infty} ||\varphi_n(g) - g|| = 0, \ \forall \ g \in G \Longrightarrow \lim_{n\to\infty} ||\varphi_n(a) - a|| = 0, \ \forall \ a \in A. \tag{1}$$

We have lightened notation in the above definition by identifying the C^*-algebra A with its image $\pi(A)$ in a faithful nondegenerate representation $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{E}$. Notably, $\mathcal{Z}$-hyperrigidity of operator $\mathcal{Z}$-system on a self-dual Hilbert $\mathcal{Z}$-module implies not only that (1) should hold for sequences of UCP $\mathcal{Z}$-bimodule maps φ_n defined on $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$, but also that the property should hold for every other faithful representation of A. If $\mathcal{Z} = \mathbb{C}$, then the definition of $\mathcal{Z}$-hyperrigity is the same as the definition of hyperrigidity in [4, definition 1.1].

Proposition 5. *Let A be a C^*-algebra and G be a generating subset of A. Then G is $\mathcal{Z}$-hyperrigid if and only if the operator $\mathcal{Z}$-system generated by G is $\mathcal{Z}$-hyperrigid.*

Proof. The proof follows directly from the definition of $\mathcal{Z}$-hyperrigidity.

Now we prove the characterization theorem for $\mathcal{Z}$-hyperrigid operator $\mathcal{Z}$-systems.

Theorem 3. *Let X be a separable operator $\mathcal{Z}$-system and X generates a C^*-algebra A (i.e., $A = C^*(X)$). The following are equivalent:*

(i) X is $\mathcal{Z}$-hyperrigid.

 https://doi.org/10.17993/3cemp.2022.110250.173-184

(ii) For every nondegenerate representation $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ on a separable self-dual Hilbert $\mathcal{Z}$-module and every sequence $\varphi_n : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ of UCP $\mathcal{Z}$-bimodule maps,

$$\lim_{n\to\infty} ||\varphi_n(x) - \pi(x)|| = 0 \ \forall \ x \in X \Longrightarrow \lim_{n\to\infty} ||\varphi_n(a) - \pi(a)|| = 0 \ \forall \ a \in A.$$

(iii) For every nondegenerate representation $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ on a separable self-dual Hilbert $\mathcal{Z}$-module, $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

(iv) For every unital C^*-algebra B, every unital homomorphism of C^*-algebra $\theta : A \to B$ and every UCP $\mathcal{Z}$-module map $\varphi : B \to B$,

$$\varphi(x) = x \ \forall \ x \in \theta(X) \Longrightarrow \varphi(x) = x \ \forall \ x \in \theta(A).$$

Proof. *(i) $\Longrightarrow$ (ii): Let $\pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a nondegenerate representation on a separable self-dual Hilbert $\mathcal{Z}$-module and let $\varphi_n : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ be a sequence of UCP $\mathcal{Z}$-module maps such that $||\varphi_n(x) - \pi(x)|| \to 0$ for all $x \in X$.*

Let $\sigma : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ be a faithful representation of A on another separable self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$. Then $\sigma \oplus \pi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$ is a faithful representation, so that each of the $\mathcal{Z}$-module maps $\psi_n : (\sigma \oplus \pi)(A) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$

$$\psi_n : \sigma(a) \oplus \pi(a) \mapsto \sigma(a) \oplus \varphi_n(a), \quad a \in A,$$

is unital completely positive $\mathcal{Z}$-module map. By [17, Remark 4.12] ψ_n can be extended to a UCP $\mathcal{Z}$-module map $\tilde{\psi}_n : \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F} \oplus \mathcal{E})$. By our assumption $\varphi_n|_X$ converges to $\pi|_X$ in pointwise norm. Thus $\tilde{\psi}_n$ converges in pointwise norm to the identity map on $(\sigma \oplus \pi)(X)$. Since X is $\mathcal{Z}$-hyperrigid, we have $\tilde{\psi}_n$ converges in pointwise norm to the identity map on $(\sigma \oplus \pi)(A)$. Therefore for every $a \in A$,

$$\begin{aligned} \limsup_{n\to\infty} ||\varphi_n(a) - \pi(a)|| &\leq \limsup_{n\to\infty} ||\sigma(a) \oplus \varphi_n(a) - \sigma(a) \oplus \pi(a)|| \\ &= \lim_{n\to\infty} ||\tilde{\psi}_n(\sigma(a) \oplus \pi(a)) - \sigma(a) \oplus \pi(a)|| = 0. \end{aligned}$$

Hence φ_n converges in pointwise norm to π on A.

(ii) $\Longrightarrow$ (iii): Let $\varphi : A \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ be a unital completely positive $\mathcal{Z}$-module map such that $\varphi|_X = \pi|_X$. Take $\varphi_n(X) = \varphi(X)$ for all $n \in \mathbb{N}$, so by hypothesis (ii), $\varphi(A) = \pi(A)$. Thus $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

(iii) $\Longrightarrow$ (iv): Let ρ be a unital $*$-homomorphism from C^*-algebra A to C^*-algebra B. Let $\varphi : B \to B$ be a UCP $\mathcal{Z}$-module map. φ satisfies $\varphi(\rho(x)) = \rho(x) \ \forall \ x \in X$. We claim that $\varphi(\rho(a)) = \rho(a) \ \forall \ a \in A$.

Let B_0 be the separable C^*-sub algebra of B generated by

$$\rho(A) \cup \varphi(\rho(A)) \cup \varphi^2(\rho(A)) \cup \cdots .$$

Observe that $\varphi(B_0) \subseteq B_0$. Since B_0 is separable, we can faithfully represent $B_0 \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ for some separable self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$. By [17, Remark 4.12], there is a UCP $\mathcal{Z}$-module map $\tilde{\varphi} : \mathbb{B}_{\mathcal{Z}}(\mathcal{F}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ such that $\tilde{\varphi}|_{B_0} = \varphi$ and in particular $\tilde{\varphi}(\rho(x)) = \rho(x)$ for $x \in X$. Since $a \in A \mapsto \rho(a) \in \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ is a representation on a separable self-dual Hilbert $\mathcal{Z}$-module. Our assumption (iii) implies that $\tilde{\varphi}$ must fix $\rho(A)$ elementwise. Therefore $\varphi(\rho(a)) = \tilde{\varphi}(\rho(a)) = \rho(a) \ \forall \ a \in A$.

(iv)$\Longrightarrow$ (i): Suppose that $A \subseteq \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ is faithfully represented on some self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$, and $\varphi_1, \varphi_2, \cdots : \mathbb{B}_{\mathcal{Z}}(\mathcal{F}) \to \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ is a sequence of UCP $\mathcal{Z}$-module maps satisfying $\lim_{n\to\infty} ||\varphi_n(x) - x|| = 0 \ \forall \ x \in X$. We claim that

$$\lim_{n\to\infty} ||\varphi_n(a) - a|| = 0, \quad \forall \ a \in A.$$

Let $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ denote the set of all bounded sequences with components in $\mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ such that $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ is a C^*-algebra. Let $c_0(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ denote the set of all sequences in $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ that converges to zero in norm and $c_0(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ is ideal in $\ell^\infty(\mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$.

 https://doi.org/10.17993/3cemp.2022.110250.173-184

Define the UCP $\mathcal{Z}$-module map $\varphi_0 : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ as follows:

$$\varphi_0(a_1, a_2, a_3, ...) = (\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3), ...).$$

Thus the map φ_0 carries the ideal $c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ into itself. Hence we can define the UCP $\mathcal{Z}$-module map of the quotient $\varphi : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ by

$$\varphi(x + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))) = \varphi_0(x) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})), \quad x \in \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F})).$$

Now consider the natural embedding $\rho : A \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$,

$$\rho(a) = (a, a, a, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})).$$

By our assumption, $||\varphi_n(x) - x|| \to 0$ as $n \to \infty$ for $x \in X$, and thus

$$\varphi(\rho(x)) = (\varphi_1(x), \varphi_2(x), ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) = (x, x, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) = \rho(x).$$

Therefore φ restricts the identity map on $\rho(X)$.

Applying assumption (iv) to the inclusions

$$\rho(X) \subseteq \rho(A) \subseteq \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$$

and the UCP $\mathcal{Z}$-module map $\varphi : \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})) \to \ell^\infty(\mathbb{B}_\mathcal{Z}(\mathcal{F}))/c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$, implies that φ must fix every element of $\rho(A)$. Since $\rho(a) = (a, a, ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$ and

$$\varphi(\rho(a)) = (\varphi_1(a), \varphi_2(a), ...) + c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F})),$$

hence we have $(\varphi_1(a) - a, \varphi_2(a) - a, ...) \in c_0(\mathbb{B}_\mathcal{Z}(\mathcal{F}))$. This proves our claim.

Now we discuss the examples of $\mathcal{Z}$-hyperrigidity. Let $V_1, V_2, ..., V_n$ be an arbitrary set of isometries acting on some self-dual Hilbert $\mathcal{Z}$-module. We exhibit a $\mathcal{Z}$-hyperrigid generator for a C^*-algebra generated by the isometries $V_1, V_2, ..., V_n$.

Theorem 4. *Let $V_1, V_2, ..., V_n$ be a set of isometries on some self-dual Hilbert $\mathcal{Z}$-module and generate a C^*-algebra A. Let*

$$G = \{V_1, V_2, ..., V_n, V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^*\}$$

then G is a $\mathcal{Z}$-hyperrigid generator for A.

Proof. Let X be the operator $\mathcal{Z}$-systems generated by G. By Corollary 5, G is hyperrigid if and only if X is hyperrigid. Using Theorem 3, it is enough to prove that for every nondegenerate representation π of A, $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

Consider a representation $\pi : A \to \mathbb{B}_\mathcal{Z}(\mathcal{E})$ and let $W_1, W_2, ..., W_n$ be isometries such that $W_i = \pi(V_i),\ i = 1, 2, ..., n$. Let $\varphi : A \to \mathbb{B}_\mathcal{Z}(\mathcal{E})$ be a UCP $\mathcal{Z}$-module map satisfying

$$\varphi(V_i) = W_i,\ 1 \le i \le n,$$

and

$$\varphi(V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^*) = W_1W_1^* + W_2W_2^* + \cdots + W_nW_n^*.$$

Thus, $\varphi(x) = \pi(x)\ \forall\ x \in X$. We claim that $\varphi = \pi$ on A.

From Remark 1, Using an analogue version of the Stinespring's dilation theorem. We can express the UCP $\mathcal{Z}$-module map φ as follows:

$$\varphi(a) = W^*\sigma(a)W, \qquad \forall\ a \in A.$$

Where σ is a representation of A on a self-dual Hilbert $\mathcal{Z}$-module $\mathcal{F}$, $W : \mathcal{E} \to \mathcal{F}$ is an isometry, and which is minimal in the sense that span closure of $\sigma(A)W\mathcal{E}$ is $\mathcal{F}$.

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First we prove that $\sigma(V_i)W = WW_i$, $1 \le i \le n$. For $i = 1, 2, ...n$ we have

$$W^*\sigma(V_i)WW^*\sigma(V_i)V = \varphi(V_i)^*\varphi(V_i) = W_i^*W_i = \mathbf{1}_{\mathcal{E}},$$

thus $W^*\sigma(V_i)(\mathbf{1} - WW^*)\sigma(V_i)W = 0$, therefore $W\mathcal{E}$ is invariant under $\sigma(V_i)$. Hence we get $\sigma(V_i)W = WW^*\sigma(V_i)W = W\varphi(V_i) = WW_i$.

Next, since $\sum\limits_{i=1}^{n} W_iW_i^* = \pi(\sum\limits_{i=1}^{n} V_iV_i^*) = \varphi(\sum\limits_{i=1}^{n} V_iV_i^*)$, we get

$$\begin{aligned} \sum_{i=1}^{n} \sigma(V_i)WW^*\sigma(V_i)^* &= \sum_{i=1}^{n} WW_iW_i^*W^* \\ &= W\varphi(\sum_{i=1}^{n} V_iV_i^*)W^* \\ &= WW^* \sum_{i=1}^{n} \sigma(V_iV_i^*)WW^* \\ &= \sum_{i=1}^{n} WW^*\sigma(V_i)\sigma(V_i^*)WW^*. \end{aligned}$$

We know that $\sigma(V_i)W = WW^*\sigma(V_i)W$ for all i. In the above equations, subtract the left side from the right, and we have

$$\sum_{i=1}^{n} WW^*\sigma(V_i)(\mathbf{1}_{\mathcal{F}} - WW^*)\sigma(V_i)^*WW^* = 0.$$

Thus $(\mathbf{1}_{\mathcal{F}} - WW^*)\sigma(V_i)^*WW^* = 0$ for all $i = 1, 2, ..., n$. Therefore $W\mathcal{E}$ is invariant under both $\sigma(V_i)$ and $\sigma(V_i)^*$ for all $i = 1, 2, ..., n$. Since the C^*-algebra A is generated by the V_i, we have $\sigma(A)W\mathcal{E} \subseteq W\mathcal{E}$. By the minimality condition, we have $W\mathcal{E} = \mathcal{F}$. Thus W is unitary. Therefore $\varphi(a) = W^{-1}\sigma(a)W$ is a representation on A. By our assumption, φ agrees with π on a generating set. Hence $\varphi = \pi$ on C^*-algebra A.

The Cuntz algebras $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $V_1, V_2, ..., V_n$ such that $V_1V_1^* + V_2V_2^* + \cdots + V_nV_n^* = \mathbf{1}$. We can discard the identity operator from the generating set G to conclude the above result.

Corollary 1. *The set $G = \{V_1, V_2, ..., V_n\}$ of generators of the Cuntz algebra $\mathcal{O}_n$ is $\mathcal{Z}$-hyperrigid.*

Theorem 5. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A. If X is $\mathcal{Z}$-hyperrigid then every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X.*

Proof. Suppose X is an operator $\mathcal{Z}$-system in a C^*-algebra A. Then by Theorem 3, every nondegenerate representation of A on separable self-dual Hilbert $\mathcal{Z}$-module has the $\mathcal{Z}$-unique extension property when nondegenerate representation restricted to X. Since every $\mathcal{Z}$-irreducible representation of a C^*-algebra A is a nondegenerate representation of A. Therefore, every $\mathcal{Z}$-irreducible representation of A on separable self-dual Hilbert $\mathcal{Z}$-module has the $\mathcal{Z}$-unique extension property when $\mathcal{Z}$-irreducible representation restricted to X. Hence, every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X.

Problem 1. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A. If every $\mathcal{Z}$-irreducible representation of C^*-algebra A is a $\mathcal{Z}$-boundary representation for a separable operator $\mathcal{Z}$-system $X \subseteq A$. Then X is $\mathcal{Z}$-hyperrigid.*

Proposition 6. *Let X be an operator $\mathcal{Z}$-system generating a C^*-algebra $A = C^*(X)$. Let $\pi_i : A \to \mathbb{B}(\mathcal{E}_i)$ be a representation on a self-dual Hilbert $\mathcal{Z}$-module such that $\pi_i|_X$ has the $\mathcal{Z}$-unique extension property for each i in an index set I. Then the direct sum of UCP $\mathcal{Z}$-module maps*

$$\pi = \oplus_{i\in I}\pi_i|_X : X \to \mathbb{B}(\oplus_{i\in I}\mathcal{E}_i)$$

has the $\mathcal{Z}$-unique extension property.

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Proof. Let $\varphi : A \to \mathbb{B}(\oplus_{i\in I}\mathcal{E}_i)$ be a UCP $\mathcal{Z}$-module map such that $\pi|_X = \varphi|_X$. For each $i \in I$, let $\varphi_i : A \to \mathbb{B}(\mathcal{E}_i)$ be the UCP $\mathcal{Z}$-module map such that

$$\varphi_i(a) = P_i\varphi(a)|_{\mathcal{E}_i}, \quad a \in A$$

where P_i is the projection from $\oplus_{i\in I}\mathcal{E}_i$ onto $\mathcal{E}_i$. Observe that $\varphi|_X = \pi|_X$. Our assumption $\pi_i|_X$ has $\mathcal{Z}$-unique extension property implies that $\varphi_i(a) = \pi_i(a)$ for all $a \in A$. Equivalently, we have $P_i\varphi(a)P_i = \pi(a)P_i$ for all $a \in A$. Using the Schwarz inequality of φ, we have

$$\begin{aligned} P_i\varphi(a)^*(\mathbf{1} - P_i)\varphi(a)P_i &= P_i\varphi(a)^*\varphi(a)P_i - P_i\varphi(a)^*P_i\varphi(a)P_i \\ &\leq P_i\varphi(a^*a)P_i - P_i\varphi(a)^*P_i\varphi(a)P_i \\ &= \pi(a^*a)P_i - \pi(a)^*\pi(a)P_i = 0. \end{aligned}$$

Therefore, $|(\mathbf{1} - P_i)\varphi(a)P_i|^2 = 0$. Thus it follows that P_i commutes with the self-adjoint family of operators $\varphi(A)$. Hence for every $a \in A$, we have

$$\varphi(a) = \sum_{i\in I}\varphi(a)P_i = \sum_{i\in I}P_i\varphi(a)P_i = \sum_{i\in I}\pi(a)P_i = \pi(a).$$

Let A be a separable C^*-algebra. The set of unitary equivalence classes of $\mathcal{Z}$-irreducible representations of A is said to be a spectrum of A.

Theorem 6. *Let X be a separable operator $\mathcal{Z}$-system generating a C^*-algebra A and let A have a countable spectrum. If every $\mathcal{Z}$-irreducible representation of A is a $\mathcal{Z}$-boundary representation for X then X is $\mathcal{Z}$-hyperrigid.*

Proof. By the Theorem 3, it is enough to prove that for every representation $\pi : A \to \mathbb{B}(\mathcal{E})$ of A on a separable self-dual Hilbert $\mathcal{Z}$-module, the UCP Z-module map $\pi|_X$ has the $\mathcal{Z}$-unique extension property. Since the spectrum A is countable, A is the type I C^*-algebra. Therefore π can be decomposed uniquely into a direct integral of mutually disjoint type I factor representation. Because the spectrum A is countable, the direct integral must be a countable direct sum. Hence π can be decomposed into a direct sum of subrepresentations $\pi_n : A \to \mathbb{B}(\mathcal{E}_n)$ of A on a separable self-dual Hilbert $\mathcal{Z}$-modules. Thus

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \qquad \pi = \pi_1 \oplus \pi_2 \oplus \cdots$$

With the property that each π_n is unitarily equivalent to a finite or countable direct sum of copies of a single $\mathcal{Z}$-irreducible representations $\sigma_n : A \to \mathbb{B}(\mathcal{F}_n)$ of A on a separable self-dual Hilbert $\mathcal{Z}$-modules.

By our assumption, each UCP $\mathcal{Z}$-module map $\sigma_n|_X$ has the $\mathcal{Z}$-unique extension property. Therefore the above decomposition of $\pi|_X$ can be expressed as a double direct sum of UCP $\mathcal{Z}$-module maps with the $\mathcal{Z}$-unique extension property. Using Proposition 6, we have $\pi|_X$ has the $\mathcal{Z}$-unique extension property.

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