ON A PRODUCTION-INVENTORY SYSTEM WITH DEFECTIVE ITEMS AND LOST SALES

Manikandan R
Department of Mathematics, Central University of Kerala, Kasaragod. Kerala (India).
E-mail:mani@cukerala.ac.in
ORCID:0000-0002-0408-1712

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ABSTRACT

A production-inventory system with the item produced being admitted (added to the inventory) with probability δ as well as an item from the inventory supplied to the customer with probability γ at the end of a service, is considered in this paper. The (s, S) control policy is followed. We obtain the joint distribution of the number of customers and the number of items in the inventory as the product of their marginals under the assumption that customers do not join when the inventory level is zero. Performance measures that impact the system are obtained. A few level-crossing results are derived. In particular optimal pairs (s, S) are obtained through numerical procedures for values of (γ, δ) on the set $\{0.1, 0.2, \ldots, 1\} \times \{0.1, 0.2, \ldots, 1\}$. A comparison of the performance measures for a few (γ, δ) pair values is provided. Finally, we discuss the first emptiness time distribution for the $M/M/1/1$ production-inventory system.

KEYWORDS

Production-inventory, Positive Service Time, Stochastic Decomposition, Defective Items, Lost Sales

1 INTRODUCTION

Sigman and Levi (1992) and Melikov and Molchanov (1992) were the first to introduce inventory with positive service time, and now such systems are popularly known as queueing-inventory systems. They assumed arbitrarily distributed service time and exponentially distributed replenishment lead time with customer arrival forming a Poisson process. Under the condition of stability of the system, they investigate several performance characteristics. In the context of arbitrarily distributed lead time, the reader's attention is invited to a very recent paper by Saffari et al. (2013) where the authors provide a product form solution for system steady-state probability distribution under the assumption that *no customer joins the system when inventory level is zero*.

Schwarz et al. (2006a) requires special mention as the first piece of work to establish asymptotic independence of the number of customers in the system and the number of items in the inventory under exponentially distributed service time as well as lead time and Poisson input of customers. Nevertheless, this is achieved under the assumption that customers do not join when the inventory level is zero (of course, Saffari et al. (2013) is the extension of this to arbitrary distributed lead time). This is despite the strong correlation between the number of customers joining the system and the lead time. Subsequently, several authors made the above assumption in their models to develop product-form solutions, the details of which can be seen below. Their work is subsumed in Krishnamoorthy and Viswanath (2013), wherein the authors have reduced the Schwarz et al. (2006a) model to a production inventory system with a single batch, bulk production for the quantum of inventory required. Krishnamoorthy and Viswanath (2010), Deepak et al. (2008), Schwarz and Daduna (2006b) and Schwarz et al. (2007) are a few other significant contributions to queueing-inventory systems. A detailed survey on queueing-inventory models is given by Krishnamoorthy et al. (2021) to summarise the contributions until 2019.

A classical queue with inventoried items for service is also studied by Saffari et al. (2011). The control policy is (s, Q), and lead time is the mixed exponential distribution. Arrivals, when inventory is out of stock, are lost to the system. This leads to a product-form solution for the system state probability. Schwarz et al. (2007) consider queueing networks with an attached inventory. They consider rerouting the customers served from a particular station when it has zero inventory. Thus no customer is lost to the system. The authors derive the joint stationary distribution of queue length and on-hand inventory at various stations in explicit product form. Using dynamic programming Ning Zhao and Zhanotong Lian (2011) obtained the necessary and sufficient conditions for a priority queueing-inventory system to be stable. A contribution of interest to inventory with positive service time involving a random environment is by Krenzler and Daduna (2014), where again, they establish a stochastic decomposition of the system. They prove a necessary and sufficient condition for a product from steady-state distribution of the joint queueing-environment process to exist. Krenzler and Daduna (2013) investigate inventory with positive service time in a random environment embedded in a Markov chain. They provide a counter-example to show that the steady-state distribution of an $M/G/1/\infty$ system with (s, S) policy and lost sales does not have a product form. Nevertheless, in general, loss systems in a random environment have a product from steady-state distribution.

Apart from the above-mentioned papers, the contributions to production-inventory models with positive service time is worth mentioning; in this context, a recent paper by Yue *et al.* (2019a) considered a production-inventory system with positive service time and vacations of a production facility. Wherein they consider that the system has a single production facility that produces one type of product with (s, S) policy. The production facility takes a vacation of random duration once the inventory level becomes S. The authors obtained the product form of the stationary joint distribution of the queue length and the on-hand inventory level. Again another paper by Yue *et al.* (2019b) discussed a production-inventory system with a service facility, production interruptions, and (s, S) control policy. In this model, also they obtained the product form solution for the stationary joint distribution of the number of customers and the on-hand inventory level. Krishnamoorthy *et al.* (2019) studied an (s, S) production inventory model with lead time involving local purchases to ensure customer satisfaction and goodwill. The problem was modelled as a Continuous Time Markov Chain and obtained stochastic decomposition of system states. Analysis of this model has high social relevance as it helps reduce the total expected cost of the production process, which improves the profit in the cottage industry.

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Otten *et al.* (2019) investigated a class of separable systems consisting of parallel production systems at several locations associated with local inventories under a base-stock policy connected with a supplier network. The production system manufactures according to customers' demands on a make-to-order basis. They studied two lost sales based on local inventory or available inventory. They obtained the product-form steady-state distribution. A recent contribution to a discrete-time production inventory system with positive service time and (s, S) order policy is by Anilkumar and Jose (2021). The authors assumed that the customer arrival follows a Bernoulli process and service time follows a geometric distribution. A supply chain consisting of production-inventory systems at several locations, common supplier couples were considered in Otten (2022). The item routing depends on the inventory to obtain "load balancing."Under a constant review base stock policy, the supplier produces raw materials to replenish local inventories. The service starts immediately If the server is prepared to serve a customer ahead of the line and the inventory has not been depleted. Otherwise, the service begins when the next replenishment arrives at the local inventory. They showed that the stationary distribution is a product of the marginal distributions of the production and inventory-replenishment subsystem. Also, they derived an explicit solution for some special cases for the marginal distribution of the inventory-replenishment subsystem.

Product-form solution for the production-inventory systems is not always possible due to the complexity of the models. However, such problems can be analysed by adopting numerical techniques. Barron (2019) considered a continuous-review storage system under the generalized order-up-to-level policy. They derived the explicit cost components of the resulting costs by taking a simple probability approach and applying stopping time theory to fluid processes and martingales. In another recent work by Beena An (s, S) production inventory model with varying production rates and multiple server vacations was analyzed by Beena, and Jose (2020). The model considered customer arrival as MAP and service time as PH distribution. A production inventory system under (s, S) policy with unit items produced at a time, heterogeneous servers, vacationing servers, and retrial customers is analyzed by Jose and Beena (2020). Another recent work by Jose and Reshmi (2021) discussed a continuous review perishable inventory system with a production unit and retrial facility, where customers arrive in a homogeneous Poisson distribution with (s, S) ordering policy and perishable items. A recent contribution by Noblesse *et al.* (2022) studied a continuous review finite capacity production-inventory system with two products in inventory. The model reflects a supply chain that operates in an environment with high levels of volatility. They considered the production facility a multitype $MMAP[K]/PH[K]/1$ queue. Another recent work by Otten and Daduna (2022) studied a production-inventory system with two classes of customers with different priorities admitted into the system via a flexible admission control scheme. The service time is exponentially distributed with parameter $\mu > 0$ for both types of customers. An arriving demand that finds the inventory depleted is lost because the inventory management follows a base stock policy (lost sales). To find the equilibrium behaviour of the system, they examined the global balance equations of the related Markov process and deduced structural features of the steady-state distribution. The authors derived a sufficient condition for ergodicity for both customer classes in the case of unbounded queues using the Foster-Lyapunov stability criterion.

In all work quoted above, customers are provided with an item from the inventory on completion of service. Nevertheless, there are several situations where a customer may not be served the item with probability one at the end of his service. The service time can be regarded as describing the features of the inventoried item. At the end of this, the customer decides to buy an item with probability γ or leaves the system with complementary probability $1 - \gamma$ without buying the item. In this connection, one may refer to Krishnamoorthy et al. (2015) for some recent developments. In this paper, we analyze such types of situations under Poisson demand, exponentially distributed service time, and the time for producing an item has exponential distribution. We further impose the condition that no customer joins when the on-hand inventory is zero (those who are already present stay back in the system until served). On the production side, a manufactured non-defective item is produced with a certain positive probability δ; hence it goes to the shelf for sale and with complementary probability $1 - \delta$, the item is defective and hence rejected. Thus, this paper further generalizes the work reported by Krishnamoorthy and Vishwanath (2013).

We arrange the presentation in this paper as indicated below: section 2 provides the mathematical

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formulation of the problem under study. The analysis of the system is carried out in section 3. In particular, we derive the long-run stability of the system. Then under this condition, we show that the system steady-state probability distribution can be decomposed: that is to say, we get the system steady-state probability distribution as the product of the marginal distribution of the components. Next, we compute system performance measures that have a significant impact. Further, to construct an appropriate cost function, we compute the expected length of a production cycle in section 4. A few results on up and down crossings of level s during a production cycle are also discussed in that section. Having achieved that, we construct a cost function. Then we look for the optimal pair (s, S) values that would result in cost minimization for different pairs of values of γ and δ and a comparison of the performance measures for a few (γ, δ) pair values is provided. This is reported in section 5. Emptiness time distribution for the $M/M/1/1$ production inventory system is discussed in Section 6. Finally, a few remarks in the conclusion are made. **Notations** used in the sequel are:

- $\mathcal{N}(t)$: number of customers in the system at time t.
- $\mathcal{I}(t)$: inventory level in the system at time t.
- $\mathcal{P}(t)$: status of the production process at time t.
  That is, $\mathcal{P}(t) = \begin{cases} 0, \text{ if production is off at time t.} \\ 1, \text{ if production is on at time t.} \end{cases}$
- $\mathcal{C}(t)$: status of the server is idle/ busy at time t.
  That is, $\mathcal{C}(t) = \begin{cases} 0, \text{ if server is idle at time t.} \\ 1, \text{ if server is busy at time t.} \end{cases}$
- I_k : identity matrix of order k.
- $\boldsymbol{e}$: $(1, 1, ..., 1)'$ a column vector of 1's of appropriate order.
- $\boldsymbol{e_1}$: $(1, 0, ..., 0)$ a row vector having 1 in the first element and 0's of appropriate order.
- CTMC: Continuous-time Markov chain.
- LIQBD: Level independent Quasi birth and death process.

2 DESCRIPTION OF THE MODEL

We consider an (s, S) production inventory system with a single server. Demands by customers for the item occur according to a Poisson process of rate λ. Processing of the customer request requires a random amount of time, which is exponentially distributed with the parameter μ. However, it is not essential that the item from inventory is provided to the customer at the end of a service. More precisely, an item from inventory is provided to a customer with probability γ at the end of his service, and with complement probability $1 - \gamma$, the customer leaves the system empty-handed. When the inventory level depletes to s, the production process is immediately switched on. Each production is of 1 unit, and the production process is kept in the on mode until the inventory level becomes S. To produce an item, it takes a random amount of time which follows exponentially distributed with parameter β. A produced item is not necessarily added to the inventory due to manufacturing defect: with probability δ, it is accepted, and with probability $1 - \delta$, the item is rejected. We assume that no customer is allowed to join the queue when the inventory level is zero; such demands are considered lost. It is assumed that the amount of time for the item produced to reach the retail shop is negligible. Thus the

 https://doi.org/10.17993/3cemp.2022.110250.139-151

system is a CTMC $\{\mathcal{X}(t); t \geq 0\} = \{(\mathcal{N}(t), \mathcal{I}(t), \mathcal{P}(t))\,; t \geq 0\}$. The production process is in on mode if $0 \leq \mathcal{I}(t) \leq s$ and it is in off mode if $\mathcal{I}(t) = S$; but when the inventory level lies between $s+1$ and $S-1$, $\mathcal{P}(t)$ is either 0 or 1 according as the production is in off or in on mode, respectively. Thus to describe the status of the process we need to write $\mathcal{P}(t) = 0$ or 1 only when $\mathcal{I}(t)$ takes values $s+1, \ldots, S-1$. Thus the state space of the CTMC is $\mathbf{\Omega}= \bigcup\limits_{i=0}^{\infty} \mathcal{L}(i)$, where $\mathcal{L}(i)$, called level i of the CTMC, is given by, $\{(i,j,1); 0 \leq j \leq s\} \bigcup \{(i,j,k); s+1 \leq j \leq S-1, k = 0,1\} \bigcup \{(i,S,0)\}, \forall\, i \geq 0$. The number of states (called phases in that level) within i^{th} level is $2S - s$. The infinitesimal generator of this CTMC is,

$$\boldsymbol{\mathcal{Q}} = \begin{bmatrix} B & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \ldots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where B contains transition rates within $\mathcal{L}(0)$; A_0 represents the transition from level i to level $i+1,\ i \geq 0$; A_1 represents the transitions within $\mathcal{L}(i)$ for $i \geq 1$ and A_2 represents transitions from $\mathcal{L}(i)$ to $\mathcal{L}(i-1),\ i \geq 1$. The rates of transitions of the process $\{\mathcal{X}(t); t \geq 0\}$ are

$$[B]_{(j,k)(l,m)} = \begin{cases} -\delta\beta, & \text{for } l=j=0; m=k=1, \\ -(\lambda+\delta\beta), & \text{for } l=j+1; j=0,1,...,S-1; m=k=1, \\ -\lambda, & \text{for } l=j; j=s+1,s+2,...,S; m=k=0, \\ \delta\beta, & \text{for } l=j+1; j=0,1,2,...,s; m=k=1, \\ \delta\beta, & \text{for } l=j+1; j=s+1,s+2,...,S-1; k=0; m=1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_0]_{(j,k)(l,m)} = \begin{cases} \lambda, & \text{for } l=j; j=1,2,...,S; m=k; k=0 \text{ and } 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1]_{(j,k)(l,m)} = \begin{cases} -(\delta\beta+\mu), & \text{for } l=j=0; m=k=1, \\ -(\lambda+\delta\beta+\mu), & \text{for } l=j+1; j=0,1,...,S-1; m=k=1, \\ -(\lambda+\mu), & \text{for } l=j; j=s+1,s+2,...,S; m=k=0, \\ \delta\beta, & \text{for } l=j+1; j=0,1,2,...,s; m=k=1, \\ \delta\beta, & \text{for } l=j+1; j=s+1,s+2,...,S-1; k=0; m=1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_2]_{(j,k)(l,m)} = \begin{cases} \gamma\mu, & \text{for } l=j-1; j=1,2,...,s; m=k=1, \\ \gamma\mu, & \text{for } l=j-1; j=s+1,s+2,...,S; m=k=0, \\ (1-\gamma)\mu & \text{for } l=j; j=1,2,...,S; m=k=1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that all entries (block matrices) in $\boldsymbol{\mathcal{Q}}$ are of the same order, namely, $2S+1$. These matrices contain transition rates within the level (in the case of diagonal entries) and between levels (in the case of off-diagonal entries).

 https://doi.org/10.17993/3cemp.2022.110250.139-151

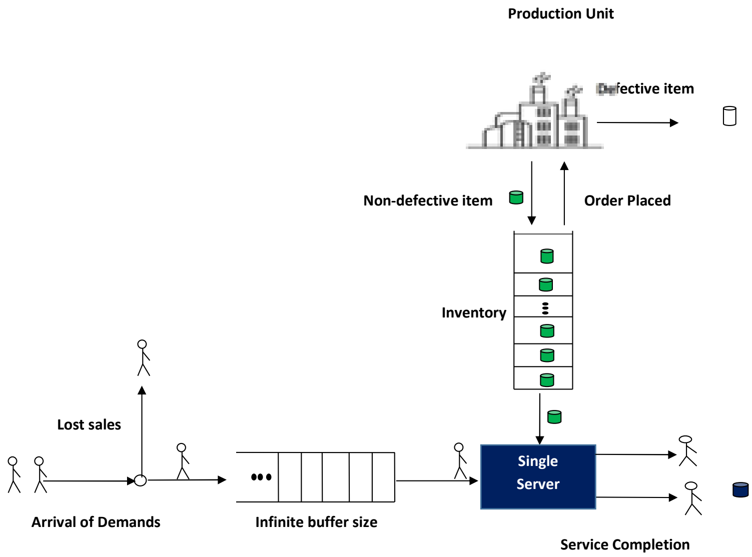

Figure 1 – $M/M/1$ Production-inventory system with defective items and lost sales

3 ANALYSIS OF THE SYSTEM

In this section, we perform the steady-state analysis of the (s, S) production inventory model under study by first establishing the stability condition of the system. Define $A{=}A_0{+}A_1{+}A_2$. This is the infinitesimal generator of the finite state CTMC corresponding to the inventory level $\{(j,1); 0 \leq j \leq s$ with production on$\} \cup \{(j,k); s+1 \leq j \leq S-1, k=0,1\} \cup \{(S,0)$ with production Off$\}$. Let $\boldsymbol{\varphi}$ denote the steady-state probability vector of A. That is $\boldsymbol{\varphi}$ satisfies

$$\boldsymbol{\varphi} A = 0, \;\; \boldsymbol{\varphi} \boldsymbol{e} = 1. \tag{1}$$

For convenience of notation we write $\varphi\,(j,1) = \varphi\,(j)$ for $0 \leq j \leq s$ and $\varphi\,(S,0)$ as$\varphi\,(S)$. By using the above relation (1), we get the components of the probability vector $\boldsymbol{\varphi}$ (note that, $\gamma\mu < \delta\beta$) explicitly as:

$$\varphi\,(s-j) = \varphi\,(S)\,\frac{\gamma\mu}{\delta\beta - \gamma\mu}\left(1 - \left(\frac{\gamma\mu}{\delta\beta}\right)^{S-s}\right)\left(\frac{\gamma\mu}{\delta\beta}\right)^{j}, 0 \leq j \leq s,$$

$$\varphi\,(j,0) = \varphi\,(S)\,, s+1 \leq j \leq S-1,$$

$$\varphi\,(j,1) = \varphi\,(S)\,\frac{\gamma\mu}{\delta\beta - \gamma\mu}\left(1 - \left(\frac{\gamma\mu}{\delta\beta}\right)^{S-j}\right), s+1 \leq j \leq S-1,$$

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and the unknown probability

$$\varphi(S) = \frac{\left(\frac{\gamma\mu}{\delta\beta} - 1\right)^2}{\left(\frac{\gamma\mu}{\delta\beta}\right)^{S+2} - \left(\frac{\gamma\mu}{\delta\beta}\right)^{s+2} - (S-s)\left(\frac{\gamma\mu}{\delta\beta} - 1\right)}.$$

Since the Markov chain under study is an LIQBD process, it is stable if and only if the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\varphi} A_0 \boldsymbol{e} < \boldsymbol{\varphi} A_2 \boldsymbol{e}. \tag{2}$$

We have the following lemma:

Lemma 3.1. *The CTMC* $\{\mathcal{X}(t); t \geq 0\}$ *is stable if and only if* $\lambda < \mu$.

*Proof.*From the well known result in Neuts (1994) on the positive recurrence of A, we have $\boldsymbol{\varphi} A_0 \boldsymbol{e} < \boldsymbol{\varphi} A_2 \boldsymbol{e}$. With a bit of computation, this simplifies to the result $\lambda < \mu$. For future reference we define ρ as

$$\rho = \frac{\lambda}{\mu}. \tag{3}$$

3.1 STEADY-STATE ANALYSIS

For computing the steady-state distribution of the process $\{\mathcal{X}(t); t \geq 0\}$, we first consider a production inventory system with negligible service time where no backlog of customers is allowed (that is when inventory level is zero, no demand joins the system). The rest of the assumptions such as those on the arrival process and lead time are the same as given earlier. Designate the Markov chain so obtained as $\{\widetilde{\mathcal{X}}(t); t \geq 0\}$=$\{(\mathcal{I}(t), \mathcal{P}(t))\,; t \geq 0\}$. Transitions of the generator matrix, $\widetilde{\boldsymbol{\mathcal{Q}}}$ is given by,

$$[\widetilde{\mathcal{Q}}]_{(j,k)(l,m)} = \begin{cases} -\delta\beta, & \text{for } l = j = 0; k = m = 1, \\ -(\gamma\lambda + \delta\beta), & \text{for } l = j+1; j = 0, 1, ..., S-1; k = m = 1, \\ -\gamma\lambda, & \text{for } l = j; j = s+1, s+2, ..., S; k = m = 0, \\ \delta\beta, & \text{for } l = j+1; j = 0, 1, 2, ..., s; k = m = 1, \\ \delta\beta, & \text{for } l = j+1; j = s+1, s+2, ..., S-2; k = 0; m = 1, \\ \delta\beta, & \text{for } l = j+1; j = S-1; k = 1; m = 0, \\ \gamma\lambda, & \text{for } l = j-1; j = s+1; k = 0 \text{ or } 1; m = 1, \\ \gamma\lambda, & \text{for } l = j-1; j = 1, 2, \ldots, s; k = m = 1, \\ \gamma\lambda, & \text{for } l = j-1; j = s+2, s+3, \ldots, S-1; k = m = 0 \text{ or } 1, \\ \gamma\lambda, & \text{for } l = j-1; j = S; k = m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The stationary probability vector $\boldsymbol{\pi}$ of $\widetilde{\boldsymbol{\mathcal{Q}}}$ satisfies

$$\boldsymbol{\pi} \widetilde{\boldsymbol{\mathcal{Q}}} = 0, \;\; \boldsymbol{\pi} \boldsymbol{e} = 1 \tag{4}$$

Thus the components of $\boldsymbol{\pi}$ are given by:

$$\pi\,(s-j) = \pi\,(S)\,\frac{\gamma\lambda}{\delta\beta - \gamma\lambda}\left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right)\left(\frac{\gamma\lambda}{\delta\beta}\right)^{j}, 0 \leq j \leq s,$$

$$\pi\,(j,0) = \pi\,(S)\,, s+1 \leq j \leq S-1,$$

$$\pi\,(j,1) = \pi\,(S)\,\frac{\gamma\lambda}{\delta\beta - \gamma\lambda}\left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-j}\right), s+1 \leq j \leq S-1.$$

and the unknown probability

$$\pi(S) = \frac{\left(\frac{\gamma\lambda}{\delta\beta} - 1\right)^2}{\left(\frac{\gamma\lambda}{\delta\beta}\right)^{S+2} - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{s+2} - (S-s)\left(\frac{\gamma\lambda}{\delta\beta} - 1\right)}.$$

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Using the components of the probability vector $\boldsymbol{\pi}$, we shall find the steady-state probability vector of the CTMC $\{\mathcal{X}(t); t \geq 0\}$. For this, let $\boldsymbol{x}$ be the steady-state probability vector of the original system. Then the steady-state vector must satisfy the set of equations

$$\boldsymbol{x}\boldsymbol{\mathcal{Q}} = 0, \;\; \boldsymbol{x}\boldsymbol{e} = 1. \tag{5}$$

partition $\boldsymbol{x}$ by levels as

$$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots) \tag{6}$$

where the sub-vectors of $\boldsymbol{x}$ are further partitioned as, $\boldsymbol{x_i} = (x_i(0), x_i(1), \dots, x_i(s), x_i(s+1,1), \dots x_i(S-1,1), x_i(s+1,0), \dots x_i(S-1,0), x_i(S)), i \geq 0$. Then the above system of equations reduces to

$$\boldsymbol{x_0} B + \boldsymbol{x_1} A_2 = 0 \tag{7}$$

$$\boldsymbol{x_i} A_0 + \boldsymbol{x_{i+1}} A_1 + \boldsymbol{x_{i+2}} A_2 = 0, i \geq 0 \tag{8}$$

Now we assume that

$$\boldsymbol{x_i} = \xi \left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}, i \geq 0 \tag{9}$$

where ξ is a constant to be determined. It can be easily verified that (7) and (8) are satisfied by (9):

$$\boldsymbol{x_0} B + \boldsymbol{x_1} A_2 = \xi \boldsymbol{\pi} \left(B + \frac{\lambda}{\mu} A_2\right) = \xi \boldsymbol{\pi} \widetilde{\boldsymbol{\mathcal{Q}}} = 0, \tag{10}$$

$$\boldsymbol{x_i} A_0 + \boldsymbol{x_{i+1}} A_1 + \boldsymbol{x_{i+2}} A_2 = \xi \left(\frac{\lambda}{\mu}\right)^{i+1} \boldsymbol{\pi} \left(B + \frac{\lambda}{\mu} A_2\right) = \xi \left(\frac{\lambda}{\mu}\right)^{i+1} \boldsymbol{\pi} \widetilde{\boldsymbol{\mathcal{Q}}} = 0. \tag{11}$$

Now applying the normalizing condition $\boldsymbol{x}\boldsymbol{e}$=1, we get

$$\xi \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \cdots\right] = 1$$

Hence under the condition that $\lambda < \mu$, we have

$$\xi = 1 - \frac{\lambda}{\mu}. \tag{12}$$

We write $\lim_{t \to \infty} \mathcal{X}(t) = \mathcal{X}$. Thus we arrive at the following decomposition of the process $\{\mathcal{X}\}$ in the long run:

Theorem 1. *Under the necessary and sufficient condition $\lambda < \mu$ for stability, the steady-state probability vector of the process $\{\mathcal{X}(t); t \geq 0\}$ has stochastic decomposition: That is, $\boldsymbol{x_i} = (1-\rho)\rho^i \boldsymbol{\pi}, i \geq 0$, where ρ is as defined in* (3) *and $\boldsymbol{\pi}$ is the inventory level probability vector.*

3.2 PERFORMANCE MEASURES

We enumerate below the long-run system performance characteristics that are useful in formulating an optimization problem.

- Mean number of customers in the system, $L_s = \frac{\lambda}{\mu - \lambda}$.
- Mean number of customers waiting in the system during the stock-out period, $W_s = L_s \pi(0)$.
- Mean number of customers waiting in the system when inventory is available, $\widetilde{W_s} = L_s (1 - \pi(0))$.

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- Mean number of items in the inventory,
$E_{inv} = \sum_{i=0}^{s} i\pi(i) + \sum_{i=s+1}^{S-1} i(\pi(i,0) + \pi(i,1))$.
- Mean rate at which the production process is *switched on*,
$E_{on} = \gamma\mu \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \pi(s+1,0) \right)$.
- Expected rate at which items are added to the inventory,
$E_{rp} = \delta\beta \left(\sum_{i=0}^{s} \pi(i) + \sum_{i=s+1}^{S-1} \pi(i,1) \right)$.
- Expected *loss* rate of the *manufactured item* due to rejection,
$M_{loss} = (1-\delta)\beta \left(\sum_{i=0}^{s} \pi(i) + \sum_{i=s+1}^{S-1} \pi(i,1) \right)$.
- Expected *loss* rate of *customers* (customers not joining the system for want of inventory), $C_{loss} = \lambda\pi(0)$.

Following Krishnamoorthy and Viswanath (2013) we have the expected production cycle time as given,

Lemma 3.2. *The expected length of a production cycle is given by,*
$E_{cycle} = \frac{1}{\delta\beta} \left((S-s) \sum_{j=0}^{s} \left(\frac{\gamma\lambda}{\delta\beta}\right)^j + \sum_{j=s+1}^{S-1} (S-j) \left(\frac{\gamma\lambda}{\delta\beta}\right)^j \right) = \frac{1}{\gamma\lambda} \left(\frac{1}{\pi(S)} - (S-s) \right)$.

Corollary 1. *The expected number of production up-crossings of level s is given by,*
$\overline{E} = \left[x_0(s)\frac{\delta\beta}{\lambda+\delta\beta} + \frac{\delta\beta}{\lambda+\mu+\delta\beta} \sum_{i=1}^{\infty} x_i(s) \right] .E_{cycle}$
$= (1-(S-s)\pi(S)) \left(\frac{\delta\beta}{\delta\beta-\gamma\lambda}\right) \left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right) \left(\frac{1-\rho}{\lambda+\delta\beta} + \frac{\rho}{\lambda+\mu+\delta\beta}\right)$.

Corollary 2. *The expected number of production down crossings of level s is given by,*
$\underline{E} = (1-(S-s)\pi(S)) \left(\frac{\gamma\lambda}{(\delta\beta-\gamma\lambda)(\lambda+\gamma\mu+\delta\beta)}\right) \left(1 - \left(\frac{\gamma\lambda}{\delta\beta}\right)^{S-s}\right)$.

Some of the above down and/ up-crossings of s may not go below/above s. The expected number of such crossings are given in the following corollaries

Corollary 3. *The expected number of production down crossings that goes below s in a production cycle,* $P_{down} = \underline{E}$*∗Probability of a service completion before addition of an inventoried item. That is,*

$$P_{down} = \underline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\gamma\mu}{\delta\beta+\gamma\mu}\right) + \xi \int_{t=0}^{\infty} \int_{v=0}^{t} \lambda e^{-\lambda v} \gamma\mu e^{-\mu(t-v)} \delta e^{-\beta t} dv dt \right)$$

$$= \underline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\gamma\mu}{\delta\beta+\gamma\mu}\right) + \frac{\xi\delta\lambda\gamma\mu}{(\lambda+\beta)(\mu+\beta)} \right).$$

Corollary 4. *The expected number of production up-crossings that go above s in a production cycle,* $P_{up} = \overline{E}$*∗Probability of a unit produced before a service completion. That is,*

$$P_{up} = \overline{E}. \left(\begin{array}{l} \sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\delta\beta}{\delta\beta+\gamma\mu}\right) + \left(\frac{\delta\beta}{\delta\beta+\lambda}\right) \xi \\ +\xi \int_{t=0}^{\infty} \int_{v=0}^{t} \lambda e^{-\lambda v} e^{-\mu(t-v)} \delta(1 - e^{-\beta t} - e^{-\beta v}) dv dt \end{array} \right)$$

$$= \overline{E}. \left(\sum_{i=1}^{\infty} \xi \left(\frac{\lambda}{\mu}\right)^i \left(\frac{\delta\beta}{\delta\beta+\gamma\mu}\right) + \left(\frac{\delta\beta}{\delta\beta+\lambda}\right) \xi + \frac{\xi\delta}{(\lambda+\beta)} \left[\frac{\beta(\mu+\beta)+\lambda\mu}{\mu(\mu+\beta)}\right] \right).$$

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Having obtained the expected length of a production cycle we turn to compute the optimal pair (s, S) values and the corresponding minimum costs. Lemma 3.2 provides us the rate at which the production process is switched on in unit time.

4 COMPUTING OPTIMAL (s, S) PAIRS AND THE MINIMUM COST

We look for the optimal values of s (the level, reaching at which the production process is switched on) and the maximum inventory level S of the production inventory model under discussion. Now for checking the optimality of s and S, the following cost function is constructed. Define $\mathcal{F}(s, S)$ as the expected cost per unit time in the long run. Then

$$\mathcal{F}(s, S) = K.E_{on} + h_{inv}.E_{inv} + c_1.C_{loss} + c_2.M_{loss} + c_3.E_{rp} + c_4.W_s + c_5.\widetilde{W}_s$$

where K is the fixed cost for starting a production run, h_{inv} is the cost per unit time per inventory towards holding, c_1 is the cost incurred due to loss per customer when the inventory is out of stock, c_2 is the cost incurred due to rejection per unit manufactured item, c_3 is the cost of production per unit time, c_4 is the waiting cost per unit time per customer during the stock out period and c_5 is the waiting cost per unit time per customer when inventory is available. Though we are not able to compute explicitly the optimal values of s and S, due to the highly complex form of the cost function, we arrive at these using numerical techniques.

For the following input values $\lambda = 2, \mu = 3, \beta = 2.5, K = \$5000, h_{inv} = \$20, c_1 = \$400, c_2 = \$100, c_3 = \$200, c_4 = \$300, c_5 = \$100$ and varying δ and γ we arrive at Table 1. δ and γ are given values from 0.1 to 1 at 0.1 spacing. Note that the case of $\gamma = \delta = 1$ is what is discussed in Krishnamoorthy and Vishwanath (2013). The pair of values given in each cell of Table 1 indicates the optimal (s, S) pair and the value at the bottom of each cell corresponds to the minimum cost (in Dollars). As γ and δ are varied we get distinct optimal pairs of (s, S) and the corresponding minimum cost. We observe that the minimum cost is a decreasing function of δ, or at first decreasing and then starts growing with δ. This can be attributed to the fact that for fixed γ, and for δ increasing, initially the loss of manufactured items get reduced; but subsequently from a point on, the holding cost factor dominates the gain from acceptance of produced item. The optimal (s, S) pair first decreases with δ increasing, comes to a minimum and then starts rising up. Same is the trend shown by the minimum cost values. The explanation for this trend is that with γ increasing, customers are provided the item at the end of their service with increasing probability, so shortage is bound to occur with higher probability. To some extend, increasing δ value can cope with this, since produced items are accepted with higher probability. Nevertheless, increase in δ results in increase in the holding cost. For the given input parameters the "best" among the optimal pair is $(1, 11)$ and the minimum cost is \$461.02 which correspond to $\delta = 1$ and $\gamma = 0.1$.

Now by using the same input values of Table 1 and with $s = 5$ and $S = 11$ we provide a comparison of the performance measures for a few (γ, δ) pair values in Table 2. For example we observe from Table 2 that the production cycle length and loss rate of customers are largest for the (γ, δ) pair values $(1, 0.5)$ and least for $(0.5, 1)$ among the three pairs of values indicated in that table. Similarly expected inventory held is least for (γ, δ) pair value $(1, 0.5)$ and the highest for $(0.5, 1)$.

5 CONCLUSIONS

This paper generalizes a few of the existing works by introducing positive service time in a production-inventory model. It provides a stochastic decomposition of the system's steady state. The expected length of a production cycle is derived. A few level-crossing results are presented. The findings provide the management with the minimum cost for each pair of values of (γ, δ) and the corresponding optimal

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Table 1 – Optimal **(s, S)** values and minimum cost

δ \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	(3,11) 605.4	(1,26) 958.33	(1,12) 1189.3	(1,9) 1309.1	(1,8) 1381.7	(1,7) 1430.3	(1,7) 1465.1	(1,7) 1491.2	(1,6) 1511.6	(1,6) 1527.9
0.2	(1,10) 515.24	(2,13) 649.96	(6,20) 793.76	(1,27) 983.33	(1,15) 1120	(1,13) 1214.3	(1,13) 1282.5	(1,10) 1334.1	(1,9) 1374.4	(1,9) 1406.7
0.3	(1,10) 490.34	(1,12) 610.1	(2,14) 689.76	(4,18) 765.15	(7,25) 804.83	(1,23) 1008.3	(1,16) 1105.2	(1,19) 1180.1	(1,13) 1239.3	((1,12) 1287
0.5	(1,10) 472.89	(1,13) 584.32	(1,15) 664.66	(1,15) 722.9	(1,16) 763.58	2,18) 795.24	(4,21) 838.8	(6,26) 908.47	(1,29) 987.12	(1,24) 1058.3
0.6	(1,10) 468.89	(1,13) 578.74	(1,16) 660.13	(1,16) 721.23	(1,16) 766.65	(1,17) 797.98	(2,18) 821.01	(3,20) 849.05	(5,24) 896.26	(4,29) 959.93
0.7	(1,11) 466.11	(1,14) 574.69	(1,16) 656.36	(1,17) 720.28	(1,17) 769.82	(1,17) 806.51	(1,18) 831.65	(2,18) 849.35	(2,20) 867.81	(4,23) 899.16
0.9	(1,11) 462.32	(1,14) 569.49	(1,16) 651.76	(1,18) 732.47	(1,18) 773.71	(1,19) 818.36	(1,19) 853.53	(1,19) 879.47	(1,19) 896.85	(1,19) 907.9
1	**(1,11) 461.02**	(1,14) 567.74	(1,16) 650.26	(1,18) 717.95	(1,19) 774.64	(1,20) 822.1	(1,20) 860.79	(1,20) 891.35	(1,20) 913.86	(1,20) 928.76

Table 2 – Effect of γ and δ on various performance measures

Performance Measures	$\gamma = 1$ and $\delta = 0.5$	$\gamma = 0.5$ and $\delta = 1$	$\gamma = \delta = 1$
L_s	0.00085731	0.10005	0.038268
W_s	0.75643	0.0013604	0.07402
$\widetilde{W_s}$	1.2436	1.9986	1.926
E_{inv}	1.5852	7.8376	5.9064
E_{rp}	1.2436	0.99932	1.926
E_{cycle}	580.22	3.9955	10.066
C_{loss}	0.75643	0.0013604	0.07402

(s, S) pair. First emptiness time distribution of the inventory is computed for the case when the waiting room capacity is restricted to one. We propose to study the transient behaviour of such a system. In addition, we analyze the case in which the transfer time from the production plant to the retail shop is a positive valued random variable. Also, the case of arbitrarily distributed lead time and/or service time is being investigated.

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REFERENCES

[1] **M.P. Anilkumar, K. P. Jose**, (2021). Stochastic Decomposition of Geo/Geo/1 Production Inventory System, *J. Phys.: Conf. Ser.* 1850 012027, https://doi.org/10.1088/1742-6596/1850/1/012027.

[2] **Ann M. Noblesse, Nikki Sonenberg, Robert N. Boute, Marc R. Lambrecht & Benny Van Houdt.** (2022). A joint replenishment production-inventory model as an MMAP[K]/PH[K]/1 queue, *Stochastic Models*, 38:3, 390-415, https://doi.org/10.1080/15326349.2022.2049822.

[3] **P., Beena., & Jose, K.P.** (2020). A MAP/PH(1), PH(2)/2 Production Inventory Model with Inventory Dependent Production Rate and Multiple Servers. *AIP Conference Proceedings* 2261, 030052; https://doi.org/10.1063/5.0017008.

[4] **T. G. Deepak, A. Krishnamoorthy, Viswanath C. Narayan and K. Vineetha** (2008). Inventory with service time and transfer of customers and/inventory. *Ann. Oper. Res* 160: 191 - 213.

[5] **K. P. Jose, P. Beena.,(2020)**. Investigation of a production inventory model with two servers having multiple vacations, *J. Math. Comput. Sci.* 10, No. 4, 1214-1227 https://doi.org/10.28919/jmcs/4361.

[6] **Jose, K.P., Reshmi, P.S.**(2021). A production inventory model with deteriorating items and retrial demands. *OPSEARCH* 58, 71-82. https://doi.org/10.1007/s12597-020-00471-8.

[7] **A. Krishnamoorthy and Viswanath C. Narayanan** (2010). Production inventory with service time and vacation to the server. *IMA Journal of Management Mathematics* https://doi.org/10.1093/imaman/dpp025.

[8] **A. Krishnamoorthy, B. Lakshmy and R. Manikandan** (2011). A survey on inventory models with positive service time. *OPSEARCH* 48(2):153-169.

[9] **Krishnamoorthy A., Varghese R., Lakshmy B.** (2019) Production Inventory System with Positive Service Time Under Local Purchase. In: Dudin A., Nazarov A., Moiseev A. (eds) *Information Technologies and Mathematical Modelling. Queueing Theory and Applications. ITMM 2019. Communications in Computer and Information Science, vol 1109. Springer, Cham.* https://doi.org/10.1007/978-3-030-33388-1_20.

[10] **A. Krishnamoorthy, R. Manikandan and B. Lakshmy** (2015). A revisit to queueing-inventory system with positive service time. *Ann Oper Res.* 233, 221–236 (2015). https://doi.org/10.1007/s10479-013-1437-x

[11] **A. Krishnamoorthy and Viswanath C Narayanan** (2013). Stochastic Decomposition in Production Inventory with Service Time. *European Journal of Operational Research* 228: 358-366.

[12] **R. Krenzler and H. Daduna** (2014). Loss systems in a random environment - steady state analysis. *Queueing Syst* https://doi.org/10.1007/s11134-014-9426-6.

[13] **R. Krenzler and H. Daduna** (2013). Loss systems in a random environment - embedded Markov chains analysis. http://preprint.math.uni-hamburg.de/public/papers/prst/prst2013-02.pdf.

[14] **M. Saffari, S. Asmussen and R. Haji** (2013). The $M/M/1$ queue with inventory, lost sale, and general lead times, *Queueing Syst* https://doi.org/10. 1007/s11134-012-9337-3.

[15] **Otten, S., Krenzler, R.K., & Daduna, H.** (2019). Separable models for interconnected production-inventory systems. *Stochastic Models*, 36, 48-93. https://doi.org/10.1080/15326349.2019.1692667.

[16] **Otten, S.** (2022). Load balancing in a network of queueing-inventory systems. *Ann Oper Res.* https://doi.org/10.1007/s10479-022-05017-3.

[17] **Otten, S., & Daduna, H.** (2022). Stability of queueing-inventory systems with different priorities. https://doi.org/10.48550/arXiv.2209.08957.

[18] **Ning Zhao and Zhanotong Lian**. (2011). A queueing-inventory system with two classes of customers. *Int. J. Production Economics* 129: 225-231.

[19] **M. F. Neuts** (1994). Matrix-Geometric Solutions in Stochastic Models - An Algorithmic Approach, 2nd ed., Dover Publications, Inc., New York.

[20] **Yonit Barron**. (2019). Critical level policy for a production-inventory model with lost sales, *International Journal of Production Research.* https://doi.org/10.1080/00207543.2018.1504243.

[21] **Yue, D., Qin, Y**. (2019a) A Production Inventory System with Service Time and Production Vacations. *J. Syst. Sci. Syst. Eng.* 28, 168–180. https://doi.org/10.1007/s11518-018-5402-8.

[22] **Yue, D., Wang, S., Zhang, Y.** (2019b). A Production-Inventory System with a Service Facility and Production Interruptions for Perishable Items. In: Li, QL., Wang, J., Yu, HB. (eds) Stochastic Models in Reliability, Network Security and System Safety. JHC80 2019. Communications in Computer and Information Science, vol 1102. Springer, Singapore. https://doi.org/10.1007/978-981-15-0864-6-21.