$MAP/PH/1$ QUEUE WITH DISCARDING CUSTOMERS HAVING IMPERFECT SERVICE

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ABSTRACT

In this paper, we consider two queueing models. Model I is on a single-server queueing system in which the arrival process follows MAP with representation $D = (D_0, D_1)$ *of order* m *and service time follows phase-type distribution* $(\boldsymbol{\beta}, S)$ *of order* n*. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has* k *stages. The* p^{th} *stage parameter is* θ_p *for* $1 \leq p \leq k$*. If a customer completes the service in between the realizations of stages* k_1 *and* k_2 $(1 < k_1 < k_2 < k)$ *of the clock, it is a perfect one. On the other hand, if the service gets completed either before the* k_1^{th} *stage realization or after the* k_2^{th} *stage realization, it is discarded because of imperfection. We analyse this model using the matrix-geometric method. We obtain the expected service time and expected waiting time of a tagged customer. Additional performance measures are also computed. We construct a revenue function and numerically analyse it. In Model II, a single server queueing system in which all assumptions are the same as in Model I except the assumption on service time, is considered. Up to stage* k_1 *service time follows phase-type distribution* $(\boldsymbol{\alpha}', T')$ *of order* n_1 *and beyond stage* k_1*, the service time follows phase type distribution* $(\boldsymbol{\beta}', S')$ *of order* n_2*. We compare the values of the revenue function of the two models*

KEYWORDS

Markovian Arrival Process, Phase-type distribution, Erlang Clock, Imperfect Service.

1 INTRODUCTION

Queueing models play an important role in our everyday life. Important application areas of queueing models are production systems, transportation and stocking systems, communication systems, information processing systems, etc. In a manufacturing system, a product goes through several stages to getting processed; the processing time of a product is very important.

Phase type distribution was introduced by Neuts (Neuts,1975) as a generalization of the exponential distribution . Phase type distribution is defined as the distribution of time to absorption of a Markov chain with finite transient states and one absorbing state. Let $\bar{X} = \{X(t) : t \geq 0\}$ denote a continuous time Markov chain with state space $S = \{1, 2, 3, \ldots, m, m+1\}$ where the first m states are transient and the last state is absorbing and with infinitesimal generator matrix

$\tilde{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where T is a square matrix of order m and T^0 is a column vector and $T^0 = -T\boldsymbol{e}$. The initial probability distribution of $\bar{X}$ is $\bar{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \alpha_{m+1})$ where $\boldsymbol{\alpha}$ is a row vector of dimension m and $\alpha_{m+1} = 1 - \boldsymbol{\alpha e}$. Let $Z = inf\{t \geq 0 : X(t) = m+1\}$ be a random variable of time until absorption in state $m+1$. The distribution of Z is called a continuous phase-type distribution (PH distribution) with parameter $(\boldsymbol{\alpha}, T)$. The distribution function of a continuous phase type distribution $PH(\boldsymbol{\alpha}, T)$ is given by $F(t) = 1 - \boldsymbol{\alpha} e^{Tt}\boldsymbol{e}$ for $t \geq 0$ and probability density function is $f(t) = \boldsymbol{\alpha} e^{Tt} T^0$ for $t \geq 0$. The Laplace Stieltjes transform of $PH(\boldsymbol{\alpha}, T)$ is given by $\phi(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1}T^0$ for all $s \in C$ with $Re(s) \geq 0$.

The Markovian Arrival Process(MAP) was introduced by David M. Lucantoni (Lucantoni,1990) as a simpler version of an earlier model proposed by Neuts (Neuts,1979). It is a generalization of the Markov process where arrivals are governed by an underlying m-states Markov chain. A continuous time Markov chain $\{(N(t), J(t)) : t \geq 0\}$ with state space $\{(i, j) : i = 0, 1, 2, \ldots; 1 \leq j \leq m\}$ and infinitesimal generator matrix

$$\bar{Q} = \begin{bmatrix} D_0 & D_1 & & & \\ & D_0 & D_1 & & \\ & & D_0 & D_1 & \\ & & & \ddots & \ddots \end{bmatrix}$$ is called a MAP with matrix representation (D_0, D_1).

D_0 and D_1 are square matrices of order m. $N(t)$ counts the number of arrivals during $(0, t)$ and $J(t)$ represents the phase of the arrival process. D_0 has negative diagonal elements and non-negative off-diagonal elements, and its elements correspond to state transition without an arrival. D_1 is a non-negative matrix whose elements represent state transition with one arrival. Let the matrix D be defined as $D = D_0 + D_1$. Then D is an irreducible infinitesimal generator of the underlying Markov chain $\{J(t)\}$. Let $\boldsymbol{\pi}$ be the invariant probability vector of D, then
$\boldsymbol{\pi} D = 0, \boldsymbol{\pi e} = 1$. The average rate of events in a MAP, which is called the fundamental rate of the MAP, is given by $\lambda = \boldsymbol{\pi} D_1 \boldsymbol{e}$.

The arrival of a negative customer to a queueing system causes the removal of one ordinary customer (called a positive customer) who is present in the queue. But the Negative arrivals have no effect if the system is empty. We can therefore represent a Negative customer as a type of work canceling signal. Queues with negative arrivals were first introduced by Gelenbe (Gelenbe,1991a). So queues with negative arrivals are called G-queues. Those who are interested in a comprehensive analysis of G-queues may refer to Gelenbe et al. (Gelenbe,1991b), Artalejo (Artalejo,2000), and Bocharov and Vishnevskii (Bocharov,2003).

Valentina Klimenok and Alexander Dudin (Klimenok,2012) consider a multi-server queueing system with finite and infinite buffers. The input flow is described by Batch Markovian Arrival Process(BMAP) and the service time has the PH distribution. Besides positive customers, the negative customers arrive according to the Markovian Arrival Process. A negative customer can remove an ordinary customer in

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service if the state service process does not belong to protected phases.

S R Chakravarthy (Chakravarthy,2009) has considered a single server queueing system in which arrivals occur according to a Markovian arrival process. All the customers in the system are lost when the system undergoes disastrous failures. In G-queues a regular customer is pushed out of the system by a negative customer. But here we consider a queueing system in which a customer is discarded if his service completion is not within a stipulated time interval.

The queueing models considered so far in the literature did not look at the possibility of service completion of customers before a threshold or beyond a second threshold. Several real-life situations warrant the completion of services between the lower and upper thresholds. This is necessitated by the fact that the raw material used for the production of a specified item may not get completely processed if completed before time. Similarly, it could get over-processed if the processing completion time gets beyond a threshold. The subject matter of this paper addresses this important aspect in production and manufacturing.

This Queueing model can be applied in various fields in our day-to-day life. For example, in a food manufacturing unit, the correct baking time of a product is a crucial factor. If the baking time exceeds a threshold, the product gets burnt. On the other hand, if the baking time is not sufficient, the product will only be half cooked and will not be acceptable.

Another example is the manufacturing of Nylon wires and films. In the manufacture of nylon, caprolactam (a chemical used as raw material), is melted and the molten caprolactam is catalytically polymerized at previously optimized conditions of temperature, pressure, the concentration of the catalyst, etc. Further, the output of the above process is subjected to another process like extrusion or calendering. Extrusion is used to produce nylon wires, whereas calendering is used to produce nylon films. The condition of this is also an optimized one, in which any variation will cause defective wires and films which will not be suitable for end-use. The condition is optimized based on laboratory and pilot plant situations.

In this paper, we first consider a single-server queueing system in which the arrival process follows MAP and service time follows the continuous phase-type distribution. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has k stages. The p^{th} stage parameter is θ_p for $1 \leq p \leq k$. If a customer completes the service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the clock, the final product is perfect. If it gets completed either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it has to be discarded.

Salient features of this paper are

- it deviates from the classical assumption of merely specifying a service time distribution.
- the lower and upper thresholds for service are the most important additions.
- When a customer enters into service, a generalized Erlang clock is started simultaneously.
- If a customer completes service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the Erlang clock, it is perfect.
- If a customer completes the service either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it is discarded.
- To maximise revenue, in Model II we consider the service time as phase-type distributed with representation $(\boldsymbol{\gamma}', L)$ of order $n = n_1 + n_2$, which is the convolution of the two phase type distributions $(\boldsymbol{\alpha}', T')$ of order n_1 and $(\boldsymbol{\beta}', S')$ of order n_2.

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Notations and abbreviations used

- $LIQBD$: Level independent Quasi-Birth and Death.
- MAP: Markovian Arrival Process.
- $CTMC$: Continuous time Markov chain.
- I_P: Identity matrix of order P.
- $\boldsymbol{e_a}$: Column vector of $1'$s of order a.
- $\boldsymbol{e}$: Column vector of $1'$s of appropriate order.
- $\boldsymbol{x}'$: Transpose of a vector $\boldsymbol{x}$.

The remaining part of this paper is organized as follows. In section 2 the model under study is mathematically formulated. In section 3 we perform the steady-state analysis of the queueing model. Service time analysis and waiting time analysis of a customer are discussed in sections 4 and 5 respectively. Some additional performance measures are provided in section 6. A revenue function is discussed in section 7. Model description and mathematical formulation of model 2 are given in section 8. In section 9 we perform the steady state analysis of model 2. Numerical results are discussed in section 10.

2 Mathematical formulation of Model I

We consider a single-server queueing system in which the arrival process follows MAP with representation $D = (D_0, D_1)$ of order m and service time follows continuous phase-type distribution $(\boldsymbol{\beta}, S)$ of order n. When a customer enters into service, a generalized Erlang clock is started simultaneously. The clock has k stages. The p^{th} stage parameter is θ_p for $1 \leq p \leq k$. If a customer completes the service in between the realizations of stages k_1 and k_2 $(1 < k_1 < k_2 < k)$ of the clock, it is perfect. If a customer completes the service either before the k_1^{th} stage realization or after the k_2^{th} stage realization, it is discarded. The expected service rate is $\mu = [\boldsymbol{\beta}(-S)^{-1}\mathbf{e}]^{-1}$. Let $D = D_0 + D_1$ be the infinitesimal generator matrix of the arrival process and $\boldsymbol{\delta}$ be its stationary probability vector, then $\boldsymbol{\delta} D = 0, \boldsymbol{\delta}\mathbf{e} = 1$. The constant $\lambda = \boldsymbol{\delta} D_1 \mathbf{e}$ referred to as the fundamental rate, gives the expected number of arrivals per unit of time.

2.1 The QBD process

The model described in section 1 can be studied as a LIQBD process. First, we define the following notations:

$N(t)$: number of customers in the system at time t,

$J(t)= j$, if the Erlang clock is in the jth stage at time t, $j = 1, 2, ..., k_2$,

$I_s(t)$: the phase of service process at time t,

$I_a(t)$: the phase of arrival process at time t,

$(N(t), J(t), I_s(t), I_a(t) : t \geq 0\}$ is a LIQBD with state space

$\Omega= \{\{(0, j)/1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, 1 \leq p \leq k_2, 1 \leq i \leq n, 1 \leq j \leq m\}\}$

The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here B_1 is an $m \times m$ matrix which contains the transition within the level 0; B_0 is an $m \times k_2mn$ matrix which contains transitions from level 0 to level 1; B_2 is a $k_2mn \times m$ matrix that contains transitions from level 1 to level 0; A_0 represents transitions from level q to level $q+1$ for $q \geq 1$, A_1 represents transitions within the level q for $q \geq 1$ and A_2 represents transitions from level q to $q-1$ for $q \geq 2$. All these are square matrices of order $k_2mn \times k_2mn$.

$$B_1 = D_0$$

$$B_0 = \begin{bmatrix} \boldsymbol{\beta} \otimes D_1 & \mathbf{0} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} S^0 \otimes I_m \\ S^0 \otimes I_m \\ S^0 \otimes I_m \\ \vdots \\ (S^0 + e_n\theta_{k_2}) \otimes I_m \end{bmatrix}$$

$$A_1 = \begin{bmatrix} C_1 & I_{mn}\theta_1 & & & & & & & \\ & C_2 & I_{mn}\theta_1 & & & & & & \\ & & C_3 & I_{mn}\theta_3 & & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & C_{k_1} & I_{mn}\theta_{k_1} & & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & C_{k_2-1} & I_{mn}\theta_{k_2-1} \\ & & & & & & & & C_{k_2} \end{bmatrix}$$

where $C_h = S \otimes I_m + I_n \otimes D_0 - Imn\theta_h, 1 \leq h \leq k_2$

$$A_2 = \begin{bmatrix} S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ S^0\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & \vdots & \vdots & & \\ (S^0 + e_n\theta_{k_2})\boldsymbol{\beta} \otimes I_m & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{bmatrix}$$

$$A_0 = \begin{bmatrix} I_n \otimes D_1 & & & & & \\ & I_n \otimes D_1 & & & & \\ & & I_n \otimes D_1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & I_n \otimes D_1 \end{bmatrix}$$

3 Steady State Analysis

In this section, we perform the steady state analysis of the queueing model under study by first establishing the stability condition of the queueing system.

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3.1 Stability Condition

The generator matrix $A = A_0 + A_1 + A_2$

$$A=\begin{bmatrix} I_n\otimes D_1+C_1+S^0\boldsymbol{\beta}\otimes I_m & I_{mn}\theta_1 & & & & \\ S^0\boldsymbol{\beta}\otimes I_m & I_n\otimes D_1+C_2 & I_{mn}\theta_2 & & & \\ S^0\boldsymbol{\beta}\otimes I_m & \mathbf{0} & I_n\otimes D_1+C_3 & I_{mn}\theta_3 & & \\ & & & \ddots & \ddots & \\ S^0\boldsymbol{\beta}\otimes I_m & & & & I_n\otimes D_1+C_{k_1} & I_{mn}\theta_{k_1} \\ & & & & \ddots & \ddots \\ (S^0\boldsymbol{\beta}+\mathbf{e}\theta_{k_2}\boldsymbol{\beta})\otimes I_m & & & & & I_n\otimes D_1+C_{k_2} \end{bmatrix}.$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \boldsymbol{\pi_3}, .., \boldsymbol{\pi_{k_1}}, ..., \boldsymbol{\pi_{k_2}})$ denote the steady state probability vector of the generator matrix A.

Here $O(\boldsymbol{\pi}) = 1 \times k_2 mn$ and the $O(\boldsymbol{\pi_r}) = 1 \times nm$ for $1 \leq r \leq k_2$.
Steady state probability vector π satisfying the equations
.

$$\boldsymbol{\pi A} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{1}$$

Using equation (1), we get

$$\boldsymbol{\pi_1}[I_n\otimes D_1+C_1+S^0\boldsymbol{\beta}\otimes I_m]+(\boldsymbol{\pi_2}+\boldsymbol{\pi_3}+\boldsymbol{\pi_4}+.....+\boldsymbol{\pi_{k_1}}+.....+\boldsymbol{\pi_{k_2-1}})[S^0\boldsymbol{\beta}\otimes I_m]+\boldsymbol{\pi_{k_2}}[(S^0\boldsymbol{\beta}+\mathbf{e}\theta_{k_2}\boldsymbol{\beta})\otimes I_m]=\mathbf{0} \tag{2}$$

$$\boldsymbol{\pi_1} I_{mn}\theta_1 + \boldsymbol{\pi_2}[I_n \otimes D_1 + C_2] = \mathbf{0} \tag{3}$$

$$\boldsymbol{\pi_2} I_{mn}\theta_2 + \boldsymbol{\pi_3}[I_n \otimes D_1 + C_3] = \mathbf{0} \tag{4}$$

$$\boldsymbol{\pi_3} I_{mn}\theta_3 + \boldsymbol{\pi_4}[I_n \otimes D_1 + C_4] = \mathbf{0} \tag{5}$$

$$\boldsymbol{\pi_{k_1}} I_{mn}\theta_{k_1} + \boldsymbol{\pi_{k_1+1}}[I_n \otimes D_1 + C_{k_1+1}] = \mathbf{0} \tag{6}$$

$$\boldsymbol{\pi_{k_2-1}} I_{mn}\theta_{k_2-1} + \boldsymbol{\pi_{k_2}}[I_n \otimes D_1 + C_{k_2}] = \mathbf{0} \tag{7}$$

$$\boldsymbol{\pi_1} \times \mathbf{e} + \boldsymbol{\pi_2} \times \mathbf{e} + ... + \boldsymbol{\pi_{k_1}} \times \mathbf{e} + ... + +\boldsymbol{\pi_{k_2}} \times \mathbf{e} = 1 \tag{8}$$

From equation (7);

$$\boldsymbol{\pi_{k_2-1}} = -\boldsymbol{\pi_{k_2}}[I_n \otimes D_1 + C_{k_2}]\frac{1}{\theta_{k_2-1}} I_{mn} \tag{9}$$

By back substitution and using equation (8) we get all the values of $\boldsymbol{\pi_r}'s$. Thus we get the steady-state probability vector of A.

The $LIQBD$ description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi} A_0\mathbf{e} < \boldsymbol{\pi} A_2\mathbf{e}. \tag{10}$$

$$\boldsymbol{\pi} A_0\mathbf{e} = \boldsymbol{\pi_1}(I_n\otimes D_1)\mathbf{e} + \boldsymbol{\pi_2}(I_n\otimes D_1)\mathbf{e} + ... + \boldsymbol{\pi_{k_1}}(I_n\otimes D_1)\mathbf{e} + ... + \boldsymbol{\pi_{k_2}}(I_n\otimes D_1)\mathbf{e} = \sum_{r=1}^{k_2} \pi_r (I_n\otimes D_1)\mathbf{e} \tag{11}$$

$$\begin{aligned}\boldsymbol{\pi} A_2\mathbf{e} &= \boldsymbol{\pi_1}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e} + \boldsymbol{\pi_2}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e} + ... + \boldsymbol{\pi_{k_1}}[S^0\boldsymbol{\beta}\otimes I_m]\mathbf{e} + ... + \boldsymbol{\pi_{k_2}}[(S^0 + e_n\theta_{k_2})\boldsymbol{\beta}\otimes I_m]\mathbf{e} \\ &= \sum_{r=1}^{k_2-1} \boldsymbol{\pi_r}[S^0\boldsymbol{\beta}\otimes I_m] + \boldsymbol{\pi_{k_2}}[(S^0 + e_n\theta_{k_2})\boldsymbol{\beta}\otimes I_m]\mathbf{e}\end{aligned} \tag{12}$$

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Therefore the stability condition is

$$\sum_{r=1}^{k_2} \boldsymbol{\pi_r}(I_n \otimes D_1)\mathbf{e} < \sum_{r=1}^{k_2-1} \boldsymbol{\pi_r}[S^0\boldsymbol{\beta} \otimes I_m] + \boldsymbol{\pi_{k_2}}[(S^0 + e_n\theta_{k_2})\boldsymbol{\beta} \otimes I_m]\mathbf{e} \tag{13}$$

3.2 The Steady State Probability Vector of Q

Let $\boldsymbol{x}$ be the steady state probability vector of Q.

$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2} \ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are each of dimension $1 \times k_2mn$.

Under the stability condition, we have $\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$are obtained by solving the equations

$$\boldsymbol{x_0}B_1 + \boldsymbol{x_1}B_2 = 0 \tag{14}$$

$$\boldsymbol{x_0}B_0 + \boldsymbol{x_1}(A_1 + RA_2) = 0 \tag{15}$$

subject to the normalizing condition

$$\boldsymbol{x_0}\boldsymbol{e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{16}$$

Solving equations (15),(16) and(17), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$. Hence we can find all $\boldsymbol{x_i}$'s.

4 Analysis of Service Time of a Customer

We consider a Markov Process $Y(t) = \{(J(t), I_s(t)) : t \geq 0\}$ where
$J(t)= j$, if the Erlang clock is in the jth stage at time t, $j = 1, 2, ..., k_2$.
$I_s(t)$: the phase of service process at time t

The state space of this process is
$\Omega_1 = \{1, 2, ..k_1, ...k_2\} \times \{1, 2, 3, ..., n\} \bigcup\{\Delta_1\} \bigcup\{\Delta_2\}$,where Δ_1 and Δ_2 denote the absorbing states. Δ_1 denotes the absorption occur due to service completion and Δ_2 denotes absorption occur due to realization of k_2^{th} stage of the Erlang clock.

The infinitesimal generator matrix is

$$Q_1 = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & & & S^0 & \mathbf{0} \\ & S-\theta_2 I & \theta_2 I & & & & & S^0 & \mathbf{0} \\ & & \ddots & \ddots & & & & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & & S-\theta_{k_2} I & S^0 & \mathbf{e}\theta_{k_2} \end{bmatrix}.$$

$$\text{where } S_1 = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & & \\ & S-\theta_2 I & \theta_2 I & & & & \\ & & \ddots & \ddots & & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & & \\ & & & & \ddots & \ddots & \\ & & & & & & S-\theta_{k_2} I \end{bmatrix}.$$

The initial probability vector is $\boldsymbol{\alpha} = (\boldsymbol{\beta}, \mathbf{0}, \mathbf{0}, ..., \mathbf{0})$

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The expected service time of a customer is the time until absorption of the above process which is given by $ES = \boldsymbol{\alpha}(-S_1^{-1})\mathbf{e}$

5 Waiting Time Analysis

To find the expected waiting time of a tagged customer who joins as the rth customer in system, we consider the Markov Processes
$W = \{W(t) : t \geq 0\} = \{(N(t), J(t), I_s(t)) : t \geq 0\}$ where

$N(t)$-Rank of the customer in the system at time t

$J(t)$= j, if the Erlang clockis in the jth stage at time t, $j = 1, 2, ..., k_2$.

$I_s(t)$ - Phase of the service at time t

The rank of the customer decrease by one when a customer ahead of him completes the service. The rank of the customer is assumed to be r if he joins as the rth customer in the system. State-space of $W(t)$ is $\Omega_2 = \{\{r, r-1, r-2, \cdots, 2\} \times \{1, 2, 3, ...k_2\} \times \{1, 2, 3, ..., n\}\} \cup \{\Delta^*\}$
where Δ^* denotes the absorbing state. That is Δ^* denotes the state that the tagged customer selected for service.

The infinitesimal generator is

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & & & & \\ T^0 & T & & & & \\ & T^0\boldsymbol{\beta} & T & & & \\ & & T^0\boldsymbol{\beta} & T & & \\ & & & \ddots & \ddots & \end{bmatrix}.$$

$$\text{where } T = \begin{bmatrix} S-\theta_1 I & \theta_1 I & & & & & \\ & S-\theta_2 I & \theta_2 I & & & & \\ & & & \ddots & \ddots & & \\ & & & S-\theta_{k_1} I & \theta_{k_1} I & & \\ & & & & & \ddots & \ddots \\ & & & & & & S-\theta_{k_2} I \end{bmatrix}.$$

$$T^0 = \begin{bmatrix} S^0 & \mathbf{0} & \mathbf{0} \\ S^0 & \mathbf{0} & \mathbf{0} \\ S^0 & \mathbf{0} & \mathbf{0} \\ \vdots & & \\ S^0 + \mathbf{e}\theta_{k_2} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Let y_{rpi} be the steady-state probability that an arriving customer finds the server in busy with current service phase i, Erlang clock is in p^{th} level and the number of customers in the system including the current arrival tobe r for $1 \leq p \leq k_2$ and $1 \leq i \leq n$

Let $y_r = (y_{r11}, y_{r12}, , ...y_{r1n}, y_{r21}, y_{r22}, ...y_{r2n}, ..., y_{rk_21}, y_{rk_22}, ...y_{rk_2n})$
and $y = (0, y_2, y_3.....y_r)$

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Then $y_r = x_{r-1}(I \otimes \frac{D_1}{\lambda}), r \geq 2$

Waiting time is the time until absorption of the Markov chain is given by Ω_2. Let $W(s)$ denote the Laplace Stieltjes Transform (LST) of waiting time in the queue of an arrival.

Theorem 1. *The LST of the waiting time distribution of an arriving customer is* $W(s) = c\sum_{r=2}^{\infty} y_r(sI - T)^{-1}T^0[\boldsymbol{\beta}(sI - T)^{-1}T^0]^{r-2}, Re(s) \geq 0$, *where the normalising constant* c *is given by* $c = [\sum_{r=2}^{\infty} y_r e]^{-1}$

6 Additional Performance Measures

- probability that the system is empty:
$$P_0 = \boldsymbol{x_0}\mathbf{e}.$$
- Probability that q customers in the system:
$$P_q = \boldsymbol{x_q}\mathbf{e}.$$
- Probability that the server is busy:
$$P_{busy} = \sum_{q=1}^{\infty}\sum_{p=1}^{k_2}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}}.$$
- Expected number of customers in the queue:
$$ECQ = \sum_{q=1}^{\infty}(q-1)\boldsymbol{x_q e}.$$
- Expected number of customers in the system:
$$ECS = \sum_{q=0}^{\infty} q\boldsymbol{x_q e}.$$
.
- Rate at which customers discarded before k_1^{th} stage realization of Erlang clock
$$RK_1 = \sum_{q=1}^{\infty}\sum_{p=1}^{k_1}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}} S^0\mathbf{e}.$$
- Rate at which customers discard after k_2^{th} stage realization of Erlang clock
$$RK_2 = \sum_{q=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qk_2ij}}\theta_{k_2}.$$
- Rate at which customers depart with successful completion of service
$$RP = \sum_{q=1}^{\infty}\sum_{p=k_1+1}^{k_2}\sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{x_{qpij}} S^0 e.$$

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7 Revenue Function

Based on the above performance measures, we construct a revenue function as follows.
CK_1 - Unit time cost of service when customer discarded before the k_1^{th} stage realization of Erlang Clock.

CK_2 - Unit time cost of service when customer discarded after k_2^{th} stage realization of Erlang clock.

RS - Revenue per unit time for successful service.

Then the expected revenue per unit time, $ER = RP \times RS - RK_1 \times CK_1 - RK_2 \times CK_2$.

In this model, customers are discarded when either their service completes before reachig the stage k_1 or goes beyond the stage k_2. To minimise the rate of discarding customers before reaching stage k_1, we have to slow down the service rate up to k_1^{th} stage realization of Erlang clock so as to get the service cross the stage k_1. Similarly to minimse the rate of discarding customers after k_2^{th} stage, we have to increase the service rate beyond k_1^{th} stage realization of the Erlang clock to get the service completed before crossing the boundary k_2. Accordingly, we can reduce the loss to the system due to imperfect service. The extra cost involved while increasing the service rate beyond k_1 gets compensated through slow down of service rate up to the stage k_1, and also through reduced imperfect service.

Next, we proceed to the analysis of Model II.

8 Model II

8.1 Model description and Mathematical Formulation

We consider a single server queueing system in which all assumptions are exactly same as in Model I except the assumption on service time. Upto the stage k_1 service time follows phase-type distribution $(\boldsymbol{\alpha}', T')$ of order n_1 and beyond the stage k_1, the service time follows phase-type distribution $(\boldsymbol{\beta}', S')$ of order n_2. Therefore the entire service time follows phase-type distribution $(\boldsymbol{\gamma}', L)$ of order $n = n_1 + n_2$, which is the convolution of the two phase-type distributions $(\boldsymbol{\alpha}', T')$ of order n_1 and $(\boldsymbol{\beta}', S')$ of order n_2.

Then $\gamma' = (\boldsymbol{\alpha}', \alpha'_{n_1+1}\boldsymbol{\beta}') = (\boldsymbol{\alpha}', \mathbf{0})$, $L = \begin{bmatrix} T' & T'^0\boldsymbol{\beta}' \\ \mathbf{0} & S' \end{bmatrix}$.

Here we take $\alpha'_{n_1+1} = 0$ and $\beta'_{n_2+1} = 0$

The above described model can be studied as a LIQBD process.
Let

$N(t)$: Number of customers in the system at time t,

$J(t)$= p, if the Erlang clock is in the pth stage at time t, $p = 1, 2, ..., k_2$,

$I_s(t)$: the phase of service process at time t,

$I_a(t)$: the phase of arrival process at time t.

$(N(t), J(t), I_s(t), I_a(t) : t \geq 0\}$ is a LIQBD with state space

$\Omega_3 = \{\{(0, j)/1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, 1 \leq p \leq k_1, 1 \leq i \leq n_1, 1 \leq j \leq m\} \bigcup \{(q, p, i, j)/q \geq 1, (k_1 + 1) \leq p \leq k_2, (n_1 + 1) \leq i \leq (n_1 + n_2), 1 \leq j \leq m\}\}$

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The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} B_1^{'} & B_0^{'} & & & \\ B_2^{'} & A_1^{'} & A_0^{'} & & \\ & A_2^{'} & A_1^{'} & A_0^{'} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here $B_1^{'}$ is an $m \times m$ matrix that contains the transition within the level 0; $B_0^{'}$ is an $m \times [k_1 n_1 m + (k_2 - k_1) m n_2]$ matrix which contains transitions from level 0 to level 1; $B_2^{'}$ is a $[k_1 n_1 m + (k_2 - k_1) m n_2] \times m$ matrix which contains transitions from level 1 to level 0; $A_0^{'}$ represents transitions from level q to $q+1$ for $q \geq 1$, $A_1^{'}$ represents transitions within the level q for $q \geq 1$ and $A_2^{'}$ represents transitions from level q to level $q-1$ for $q \geq 2$. All these are square matrices of order $[k_1 n_1 m + (k_2 - k_1) m n_2]$.

$B_1 = B_1^{'} = D_0$

$$B_0^{'} = \begin{bmatrix} \boldsymbol{\alpha}^{'} \otimes D_1 & \mathbf{0} \end{bmatrix}$$

$$B_2^{'} = \begin{bmatrix} T^{'0} \otimes I_m \\ T^{'0} \otimes I_m \\ \vdots \\ T^{'0} \otimes I_m \\ S^{'0} \otimes I_m \\ S^{'0} \otimes I_m \\ \vdots \\ S^{'0} \otimes I_m \\ (S^{'0} + e_{n_2} \theta_{k_2}) \otimes I_m \end{bmatrix}$$

$$A_1^{'} = \begin{bmatrix} F_1 & I_{mn_1}\theta_1 & & & & & & \\ & F_2 & I_{mn_1}\theta_1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & F_{k_1} & \theta_{k_1}\boldsymbol{\beta}^{'} \otimes I_m & & & \\ & & & & E_{k_1+1} & I_{mn_2}\theta_{k_1+2} & & \\ & & & & & E_{k_1+2} & I_{mn_2}\theta_{k_1+1} & \\ & & & & & \ddots & \ddots & \\ & & & & & & & E_{k_2} \end{bmatrix}$$

where $F_t = T^{'} \otimes I_m + I_{n_1} \otimes D_0 - Imn_1\theta_t, 1 \leq t \leq k_1$
$E_r = S^{'} \otimes I_m + I_{n_2} \otimes D_0 - Imn_2\theta_r, k_1 + 1 \leq r \leq k_2$

$$A_2^{'} = \begin{bmatrix} T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \\ S^{'0}\boldsymbol{\alpha}^{'} + e_{n_2}\theta_{k_2}\boldsymbol{\alpha}^{'} \otimes I_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

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$$A_0^{'} = \begin{bmatrix} I_{n_1} \otimes D_1 & & & & & \\ & I_{n_1} \otimes D_1 & & & & \\ & \ddots & \ddots & & & \\ & & I_{n_1} \otimes D_1 & & & \\ & & & I_{n_2} \otimes D_1 & & \\ & & & \ddots & \ddots & \\ & & & & & I_{n_2} \otimes D_1 \end{bmatrix}$$

$$= \begin{bmatrix} I_{k_1} \otimes (I_{n_1} \otimes D_1) & \mathbf{0} \\ \mathbf{0} & I_{(k_2-k_1)} \otimes (I_{n_2} \otimes D_1) \end{bmatrix}$$

9 Steady State Analysis

In this section, we perform the steady state analysis of the queueing model under study by first establishing the stability condition of the queueing system.

9.1 Stability Condition

The generator matrix $A^{'} = A_0^{'} + A_1^{'} + A_2^{'}$

$$A^{'} = \begin{bmatrix} I_{n_1 \otimes D_1} + F_1 + T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & I_{mn_1}\theta_1 & & & & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & I_{n_1 \otimes D_1} + F_2 & I_{mn_1}\theta_2 & & & \\ & & \ddots & \ddots & & \\ T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & & I_{n_1 \otimes D_1} + F_{k_1} & \theta_{k_1}\boldsymbol{\beta}^{'} \otimes I_m & & \\ S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m & & & & I_{n_2} \otimes D_1 + E_{k_1+1} & I_{mn_2}\theta_{k_1+1} \\ & & & & \ddots & \ddots \\ [(S^{'0} + e_{n_2}\theta_{k_2})\boldsymbol{\alpha}^{'}] \otimes I_m & & & & & I_{n_2} \otimes D_1 + E_{k_2} \end{bmatrix}.$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \boldsymbol{\pi_3}, .., \boldsymbol{\pi_{k_1}}, ..., \boldsymbol{\pi_{k_2}})$ denote the steady state probability vector of the generator matrix A

Here $O(\boldsymbol{\pi}) = 1 \times [k_1 n_1 m + (k_2 - k_1) n_2 m]$ and the $O(\boldsymbol{\pi_r}) = 1 \times n_1 m$ for $1 \leq r \leq k_1$ and $O(\boldsymbol{\pi_r}) = 1 \times n_2 m$ for $k_{1+1} \leq r \leq k_2$.

Steady state probability vector π satisfying the equations

.

$$\boldsymbol{\pi} \boldsymbol{A}^{'} = 0, \boldsymbol{\pi}\mathbf{e} = 1. \tag{17}$$

Using equation (18), we get

$$\begin{gathered} \boldsymbol{\pi_1}[I_{n1} \otimes D_1 + F_1 + T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m] + (\boldsymbol{\pi_2} + \boldsymbol{\pi_3} + \boldsymbol{\pi_4} + ... + \boldsymbol{\pi_{k_1}})[T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m] \\ +(\boldsymbol{\pi_{k_1+1}} + ... + \boldsymbol{\pi_{k_2-1}})S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m + \pi_{k_2}(S^{'0} + e_{n_2}\theta_{k_2})\boldsymbol{\alpha}^{'} \otimes I_m) = \mathbf{0} \end{gathered} \tag{18}$$

$$\boldsymbol{\pi_1} I_{mn_1}\theta_1 + \boldsymbol{\pi_2}[I_{n_1} \otimes D_1 + F_2] = \mathbf{0} \tag{19}$$

$$\boldsymbol{\pi_2} I_{mn_1}\theta_2 + \boldsymbol{\pi_3}[I_{n_1} \otimes D_1 + F_3] = \mathbf{0} \tag{20}$$

$$\boldsymbol{\pi_3} I_{mn_1}\theta_3 + \boldsymbol{\pi_4}[I_{n1} \otimes D_1 + F_4] = \mathbf{0} \tag{21}$$

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$$\boldsymbol{\pi_{k1-1}} I_{mn_2}\theta_{k_1-1} + \boldsymbol{\pi_{k_1}}[I_{n1} \otimes D_1 + F_{k_1}] = \mathbf{0} \tag{22}$$

$$\boldsymbol{\pi_{k_1}}(\theta_{k_1}\boldsymbol{\beta}^{'} \otimes I_m) + \boldsymbol{\pi_{k_1+1}}[I_{n_2} \otimes D_1 + E_{k_1+1}] = \mathbf{0} \tag{23}$$

$$\boldsymbol{\pi_{k_2-2}} I_{mn_2}\theta_{k_2-2} + \boldsymbol{\pi_{k_2-1}}[I_{n_2} \otimes D_1 + E_{k_{2-1}}] = \mathbf{0} \tag{24}$$

$$\boldsymbol{\pi_{k_2-1}} I_{mn_2}\theta_{k_2-1} + \boldsymbol{\pi_{k_2}}[I_{n_2} \otimes D_1 + E_{k_2}] = \mathbf{0} \tag{25}$$

$$\boldsymbol{\pi_1} \times \mathbf{e} + \boldsymbol{\pi_2} \times \mathbf{e} + \ldots\ldots\ldots + \boldsymbol{\pi_{k_1}} \times \mathbf{e} + \ldots\ldots\ldots + +\boldsymbol{\pi_{k_2}} \times \mathbf{e} = 1 \tag{26}$$

From equation (25);

$$\boldsymbol{\pi_{k_2-1}} = -\boldsymbol{\pi_{k_2}}[I_{n2} \otimes D_1 + E_{k_2}]\frac{1}{\theta_{k_2-1}} I_{mn_2} \tag{27}$$

By back substitution and using equation (26) we get all the values of $\boldsymbol{\pi_r}'s$.Thus we get the steady-state probability vector of $A^{'}$.

The $LIQBD$ description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of the right drift. That is,

$$\boldsymbol{\pi} A_0^{'}\mathbf{e} < \boldsymbol{\pi} A_2^{'}\mathbf{e}. \tag{28}$$

Therefore the stability condition is

$$\sum_{r=1}^{k_1}\boldsymbol{\pi_r}(I_{n_1} \otimes D_1)\mathbf{e} + \sum_{r=k_1+1}^{k_2}\boldsymbol{\pi_r}(I_{n2} \otimes D_1)\mathbf{e} < \sum_{r=1}^{k_1}\boldsymbol{\pi_r}[T^{'0}\boldsymbol{\alpha}^{'} \otimes I_m]\mathbf{e} + \sum_{r=k_1+1}^{k_2}\boldsymbol{\pi_r}(S^{'0}\boldsymbol{\alpha}^{'} \otimes I_m)\mathbf{e} + \boldsymbol{\pi_{k_2}}(e_{n_2}\theta_{k_2}\boldsymbol{\alpha}^{'} \otimes I_m)\mathbf{e} \tag{29}$$

9.2 The Steady State Probability Vector of Q

Let $\boldsymbol{x}$ be the steady state probability vector of Q.

$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}\ldots)$, where $\boldsymbol{x_0}$ is of dimension $1 \times m$ and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$ are each of dimension $1 \times [k_1mn_1 + (k_2 - k_1)n_2m]$.

Under the stability condition, we have $\boldsymbol{x_i} = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$are obtained by solving the equations

$$\boldsymbol{x_0}B_1 + \boldsymbol{x_1}B_2 = 0 \tag{30}$$

$$\boldsymbol{x_0}B_0 + \boldsymbol{x_1}(A_1 + RA_2) = 0 \tag{31}$$

subject to the normalizing condition

$$\boldsymbol{x_0}\boldsymbol{e} + \boldsymbol{x_1}(I - R)^{-1}\boldsymbol{e} = 1 \tag{32}$$

Solving equations (31), (32) and (33), we get $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$
Hence we can find all $\boldsymbol{x_i}$'s.

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10 Numerical Results

For the arrival process of customers, we consider the following three sets of matrices for D_0 and D_1

MAP with positive correlation (MPC)

$$D_0 = \begin{bmatrix} -1.7615 & 1.7615 & 0 \\ 0 & -1.7615 & 0 \\ 0 & 0 & -11.7054 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.6294 & 0 & 0.1321 \\ 0.1233 & 0 & 11.5821 \end{bmatrix}$$

MAP with negative correlation (MNC)

$$D_0 = \begin{bmatrix} -5 & 5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -40.5 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.15 & 0 & 4.85 \\ 40.3 & 0 & 0.2 \end{bmatrix}$$

MAP with zero correlation (MZC)

$$D_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5.25 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.95 & 0 & 0.05 \\ 0.15 & 0 & 5.1 \end{bmatrix}$$

The arrival process labeled MPC has correlated arrivals with the correlation between two successive interarrival times given by 0.5315, the arrival process corresponding to the one labeled MNC has correlated arrivals with the correlation between two successive interarrival times given by -0.4470 and the arrival process labeled MZC has zero correlation between two successive interarrival times.
Service time follows continuous phase-type distribution $(\boldsymbol{\beta}, S)$ of order 6 in Model I.
Here we take $\boldsymbol{\beta} = (0.2, 0.1, 0.2, 0.1, 0.2, 0.2)$

$$S = \begin{bmatrix} -6.7 & 0.5 & 0 & 1.2 & 0 & 0.5 \\ 0 & -5.5 & 0.5 & 0.2 & 1 & 0.8 \\ 0 & 0.1 & -5.5 & 0.9 & 0.3 & 1.2 \\ 0.1 & 0 & 0.8 & -6.5 & 0.3 & 0 \\ 0.3 & 0 & 0.5 & 0 & -4.5 & 1.3 \\ 0.2 & 0.3 & 0 & 0.4 & 0 & -5.5 \end{bmatrix}.$$

In Model II, $\boldsymbol{\alpha'} = (0.2, 0.5, 0.3), \boldsymbol{\beta'} = (0.1, 0.4, 0.5), \boldsymbol{\gamma'} = (\boldsymbol{\alpha'}, \mathbf{0}) = (0.2, 0.5, 0.3, 0, 0, 0)$.

$$S' = \begin{bmatrix} -28.19 & 0.5 & 0 \\ 0 & -28.21 & 0.1 \\ 0 & 0.2 & -28.46 \end{bmatrix}, \quad T' = \begin{bmatrix} -4.5583 & 0.3 & 0 \\ 0 & -4.582 & 0.3 \\ 0 & 0.1 & -4.6 \end{bmatrix},$$

$$L = \begin{bmatrix} T' & T'^0\boldsymbol{\beta'} \\ \mathbf{0} & S' \end{bmatrix} = \begin{bmatrix} -4.5583 & 0.3 & 0 & 0.4258 & 1.7033 & 2.1292 \\ 0 & -4.582 & 0.3 & 0.4282 & 1.7128 & 2.1410 \\ 0 & 0.1 & -4.6 & 0.45 & 1.8 & 2.25 \\ 0 & 0 & 0 & -28.1900 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -28.2100 & 0.1 \\ 0 & 0 & 0 & 0 & 0.2 & -28..46 \end{bmatrix}.$$

Service rate in Model I = Service rate in Model II= 3.7649. Fix $n = 6, n_1 = 3, n_2 = 3, m = 3, k_1 = 2, k_2 = 6, k = 7, CK1 = 12, CK2 = 20, RS = 40$.
Let $ER_1$and ER_2 denote the expected revenue in Model I and Model II respectively.

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10.1 MAP with positive correlation (MPC) and the clock follows generalized Erlang distribution

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	51.2838	50.6032	8.0993	0.6468	7.4011	185.9164
13-13.5	48.9982	48.3254	7.8161	0.7322	7.5104	191.9783
14-14.5	46.7988	46.1337	7.5605	0.8178	7.5929	196.6326
15-15.5	44.6870	44.0296	7.3284	0.9032	7.6535	200.1340
16-16.5	42.6620	42.0121	7.1163	0.9881	7.6960	202.6849
17-17.5	40.7213	40.0788	6.9215	1.0722	7.7237	204.4475
18-18.5	38.8617	38.2264	6.7416	1.1555	7.7390	205.5523
19-19.5	37.0791	36.4510	6.5748	1.2378	7.7439	206.1053
20-20.5	35.3697	34.7486	6.4194	1.3190	7.7401	**206.1931**
21-21.5	33.7293	33.1151	6.2741	1.3990	7.7289	205.8866
22-22.5	32.1541	31.5466	6.1377	1.4778	7.7113	205.2444
23-23.5	30.6401	30.0393	6.0093	1.5554	7.6883	204.3152
24-24.5	29.1838	28.5895	5.8879	1.6316	7.6604	203.1393
25-25.5	27.7819	27.1941	5.7727	1.7065	7.6288	201.7505
26-26.5	26.4132	25.8498	5.6633	1.7800	7.5934	200.1771

Table 1 – Effect of θ_i's on performance measures in Model I when the arrival process is MPC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	19.3578	18.8077	6.0942	0.0214	7.0839	209.7992
14-14.5	15.2272	14.7401	5.6427	0.0350	7.7778	242.6997
16-16.5	11.8777	11.3801	5.2328	0.0524	8.3303	269.3707
18-18.5	9.168-	8.6946	4.8598	0.0374	8.7588	290.5665
20-20.5	7.0021	6.5515	4.5207	0.0977	9.0821	307.0833
22-22.5	5.3114	4.8821	4.2149	0.1248	9.3219	319.7984
24-24.5	4.0354	3.6254	3.9425	0.1546	9.5012	329.6455
26-26.5	3.1050	2.7124	3.7025	0.1868	9.6401	337.4388
28-28.5	2.4421	2.0649	3.4912	0.2211	9.7500	343.6853
30-30.5	1.9721	1.6087	3.3036	0.2573	9.8350	348.6118
32-32.5	1.6352	1.2844	3.1352	0.2951	9.8966	352.3382
34-34.5	1.3886	1.0493	2.9829	0.3342	9.9367	354.9907
36-36.5	1.2035	0.8749	2.8443	0.3742	9.9582	356.7130
37-37.5	1.1279	0.8044	2.7796	0.3946	9.9629	357.2698
38-38.5	1.0611	0.7425	2.7177	0.4151	9.9640	357.6454
39-39.5	1.0019	0.6879	2.6585	0.4357	9.9618	357.8543
40-40.5	0.9490	0.6395	2.6018	0.4565	9.9565	**357.9098**
41-41.5	0.9015	0.5964	2.5474	0.4774	9.9485	357.8242
42-42.5	0.8587	0.5578	2.4952	0.4983	9.9379	357.6082
43-43.5	0.8200	0.5231	2.4451	0.5194	9.9250	357.2720
44-44.5	0.7847	0.4918	2.3970	0.5405	9.9100	356.8247
45-45.5	0.7525	0.4634	2.3507	0.5616	9.8929	356.2746

Table 2 – Effect of θ_i's on performance measures in Model II when the arrival process is MPC

Table 1 and Table 2 show the effect of $\theta_i's$ on various performance measures and the revenue function in Model I and II respecively when the arrival process is MPC. In Model I, when $\theta_i's$ values increases, the values of ER_1 increase and reach the maximum at θ_i's=19-19.5 and then decreases. The maximum revenue in this case is 206.1931. In Model II, when $\theta_i's$ values increases, the values of ER_2 increase and reach the maximum at $\theta_i's = 40 - 40.5$, and then decreases. The maximum value of ER_2 is 357.9098. When we compare Model I and II, the values expected revenue in Model II is greater than that of the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I. In both models values of RK_1 decreases when $\theta_i's$ values increases. Also in both models, RK_2 increases when $\theta_i's$ values increases. This is because when $\theta_i's$ values increase, the expected service time of the customer in each stage decreases.

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10.2 MAP with positive correlation (MPC) and the clock follows Erlang distribution

Now we consider the case when all the values of $\theta_i's$ are equal.

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	51.8634	51.1808	8.0856	0.6256	7.4584	188.7979
13	49.5574	48.8826	7.8021	0.7110	7.5679	194.8694
14	47.3366	46.6696	7.5466	0.7960	7.6500	199.5104
15	45.2033	44.5439	7.3147	0.8820	7.7100	202.9824
16	43.1571	42.5053	7.1028	0.9670	7.7517	205.4928
17	41.1959	40.5515	6.9084	1.0513	7.7784	207.2073
18	39.3166	38.6795	6.7289	1.1348	7.7926	208.2591
19	37.5154	36.8855	6.5626	1.2174	7.7964	208.7561
20	35.7882	35.1654	6.4076	1.2988	7.7914	**208.7864**
21	34.1312	33.5152	6.2628	1.3792	7.7790	208.4219
22	32.5401	31.9310	6.1270	1.4583	7.7603	207.7220
23	31.0113	30.4089	5.9990	1.5362	7.7362	206.7358
24	29.5411	28.9452	5.8781	1.6127	7.7074	205.5043
25	28.1260	27.5366	5.7635	1.6879	7.6746	204.0613
26	26.7629	26.1799	5.6545	1.7618	7.6381	202.4354

Table 3 – Effect of θ_i's on performance measures in Model I when the arrival process is MPC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	19.4795	18.9287	6.1059	0.0197	7.0686	209.0797
14	17.2946	16.7574	5.8739	0.0257	7.4369	226.4730
16	15.3286	14.8048	5.6532	0.0327	7.7676	242.2119
18	9.2393	8.7652	4.8685	0.0700	8.7578	290.4893
20	7.0610	6.6097	4.5287	0.0937	9.0849	307.1775
22	5.3588	4.9288	4.2221	0.1204	9.3277	320.0353
24	4.0723	3.6617	3.9489	0.1498	9.5094	329.9940
26	3.1329	2.7397	3.7081	0.1816	9.6501	337.8737
28	2.4629	2.0852	3.4961	0.2156	9.7614	344.1917
30	1.9877	1.6238	3.3080	0.2515	9.8477	349.1815
32	1.6470	1.2957	3.1392	0.2891	9.9104	352.9645
34	1.3977	1.0580	2.9865	0.3280	9.9516	355.6660
36	1.2108	0.8818	2.8476	0.3678	9.9739	357.4294
37	1.1345	0.8105	2.7827	0.3881	9.9790	358.0040
38	1.0671	0.7479	2.7207	0.4085	9.9804	358.3955
39	1.0073	0.6928	2.6614	0.4291	9.9785	358.6188
40	0.9539	0.6440	2.6045	0.4499	9.9735	**358.6872**
41	0.9060	0.6005	2.5500	0.4707	9.9657	358.6131
42	0.8629	0.5615	2.4978	0.4917	9.9553	358.4074
43	0.8238	0.5265	2.4476	0.5127	9.9426	358.0803
44	0.7882	0.4949	2.3993	0.5338	9.9277	357.6409
45	0.7558	0.4663	2.3530	0.5549	9.9108	357.0978

Table 4 – Effect of θ_i's on performance measures in Model II when the arrival process is MPC

Tables 3 and 4 show the effect of θ_i on performance measures and expected revenue (ER) when the arrival process is MPC. In Model I ER is maximum at $\theta = 20$ and the maximum revenue is 208.7864. In Model II ER is maximum at $\theta = 40$ and the maximum revenue is 358.6872. When θ_i's values increases, the values of RK_1 decrease at the same time the values of RK_2 increase. This is because the expected service time of the customer in each stage decreases. When we compare Models I and II, the values of expected revenue in Model II are greater than that of the corresponding values of expected revenue in Model I. Also the values of the rate of perfect service (RP) in Model II are greater than the corresponding values of RP in Model I.

10.3 MAP with negative correlation (MNC) and the clock follows generalized Erlang distribution

Tables 5 and 6 show the effect of $\theta_i's$ on various performance measures and the revenue function when the arrival process is MNC and the clock is a generalized Erlang clock. ER is maximum when θ_i's $= 15 - 15.5$ in Model I and the maximum revenue is 273.7589. In Model II ER is maximum when θ_i's $= 40 - 40.5$ and the maximum revenue is 357.9432. When we compare Model I and II, the values

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	18.1197	17.1597	11.4736	0.9058	10.4146	260.7828
13-13.5	12.6645	11.7209	11.0056	1.0202	10.5100	267.9301
14-14.5	9.2711	8.3467	10.5469	1.1295	10.5303	272.0594
15-15.5	7.0945	6.1913	10.1019	1.2330	10.4910	**273.7589**
16-16.5	5.6478	3.7669	9.6761	1.3309	10.4083	273.6003
17-17.5	4.6497	3.7914	9.2740	1.4239	10.2959	272.0712
18-18.5	3.9353	3.0993	8.8974	1.5120	10.1639	269.5493
19-19.5	3.4061	2.5918	.5466	1.5959	10.0197	266.3116
20-20.5	3.0018	2.2084	8.2205	1.6760	9.8681	262.5575
21-21.5	2.6844	1.9112	7.9172	1.7525	9.7122	258.4313
22-22.5	2.4294	1.6755	7.6350	1.8256	9.5543	254.0392
23-23.5	2.2204	1.4850	7.3718	1.8955	9.3958	249.4616
24-24.5	2.0461	1.3283	7.1260	1.9623	9.2380	244.7604
25-25.5	1.8985	1.1978	6.8959	2.0263	9.0815	239.9836

Table 5 – Effect of θ_i's on performance measures in Model I when the arrival process is MNC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	1.2636	0.6835	6.4269	0.0218	7.4496	220.4267
14-14.5	1.0887	0.5487	5.8233	0.0353	8.0099	249.8111
16-16.5	0.9632	0.3591	5.3212	0.0526	8.4581	273.4177
18-18.5	0.8684	0.2997	4.8976	0.0734	8.8175	292.4617
20-20.5	0.7938	0.2558	4.5356	0.0976	9.1056	307.8438
22-22.5	0.7335	0.2221	4.2230	0.1248	9.3354	320.2465
24-24.5	0.6834	0.1957	3.9503	0.1547	9.5174	330.1971
26-26.5	0.6410	0.1744	3.7105	0.1870	9.6594	338.1110
28-28.5	0.6046	0.1569	3.4980	0.2215	9.7681	344.3199
30-30.5	0.5728	0.1424	3.3083	0.2577	9.8486	349.0926
32-32.5	0.5449	0.1301	3.1381	0.2954	9.9054	352.6495
34-34.5	0.5200	0.1196	2.9845	0.3344	9.9418	355.1727
36-36.5	0.4977	0.1105	2.8451	0.3743	9.9611	356.8142
37-37.5	0.4874	0.1064	2.7802	0.3946	9.9650	357.3450
38-38.5	0.4775	0.1026	2.7182	0.4151	9.9655	357.7017
39-39.5	0.4681	0.0990	2.6588	0.4358	9.9630	357.8971
40-40.5	0.4592	0.0957	2.6020	0.4565	9.9574	**357.9432**
41-41.5	0.4506	0.0925	2.5476	0.4774	9.9492	357.8508
42-42.5	0.4424	0.0895	2.4954	0.4984	9.9386	357.6303
43-43.5	0.4346	0.0867	2.4452	0.5194	9.9256	357.2909
44-44.5	0.4270	0.0840	2.3971	0.5405	8.9104	356.8414
45-45.5	0.4197	0.0815	2.3508	0.5617	8.7875	356.2899

Table 6 – Effect of θ_i's on performance measures in Model II when the arrival process is MNC

of expected revenue in Model II is greater than that of the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I. In both models values of RK_1 decreases when $\theta_i's$ values increases. Also in both models, RK_2 increases when $\theta_i's$ values increases. This is because when $\theta_i's$ values increase, the expected service time of the customer in each stage decreases.

10.4 MAP with negative correlation (MNC) and the clock follows Erlang distribution

Tables 7 and 8 show the effect of θ on various performance measures and expected revenue, when the arrival process is MNC and the clock is an Erlang clock. ER is maximum at $\theta = 15$ and the maximum revenue is 278.5231 in Model I and ER is maximum at $\theta = 40$ and the maximum revenue is 358.7241 in Model II.

10.5 MAP with zero correlation (MZC) and the clock follows generalized Erlang distribution

Tables 9 and 10 show the effect of θ_i's on various performance measures and the revenue function when the arrival process is MZC and the clock is generalized Erlang clock. ER is maximum when θ_i's $= 16-16.5$ and the maximum revenue is 266.1353 in Model I and ER is maximum when θ_i's $= 40-40.5$ and the maximum revenue is 350.9024 in Model II . When we compare Model I and II, the values expected revenue in Model II is greater than the corresponding values of expected revenue in Model I. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values

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$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	19.9412	18.9776	11.4650	0.8768	10.5037	265.0314
13	13.7816	12.8336	11.0061	0.9923	10.6089	272.4372
14	9.9745	9.0450	10.5547	1.1029	10.6364	276.7431
15	7.5522	6.6436	10.1148	1.2078	10.6014	**278.5231**
16	5.9568	5.0703	9.6919	1.3071	10.5201	278.3574
17	4.8662	4.0022	9.2908	1.4012	10.4067	276.7548
18	4.0925	3.2510	8.9143	1.4905	10.2724	274.1164
19	3.5241	2.7044	8.5629	1.5754	10.1250	270.7381
20	3.0930	2.2945	8.2359	1.6565	9.9698	266.8317
21	2.7567	1.9786	7.9317	1.7338	9.8102	262.5493
22	2.4880	1.7294	7.6485	1.8077	9.6485	258.0019
23	2.2688	1.5288	7.3845	1.8784	9.4864	253.2722
24	2.0867	1.3646	7.1379	1.9460	9.3249	248.4235
25	1.9331	1.2282	6.9071	2.0106	9.1651	243.5047

Table 7 – Effect of θ_i's on performance measures in Model I when the arrival process is MNC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	1.2692	0.6879	6.4435	0.0200	7.4383	219.8071
14	1.0928	0.5517	5.8370	0.0330	8.0032	249.4221
16	0.9663	0.3609	5.3327	0.0497	8.4553	273.2238
18	0.8709	0.3010	4.9073	0.0700	8.8181	292.4344
20	0.7960	0.2567	4.5440	0.0936	9.1090	307.9595
22	0.7353	0.2229	4.2303	0.1204	9.3414	320.4850
24	0.6850	0.1963	3.9567	0.1499	9.5255	330.5411
26	0.6424	0.1749	3.7161	0.1818	9.6694	338.5454
28	0.6059	0.1574	3.5030	0.2159	9.7796	344.8316
30	0.5740	0.1428	3.3128	0.2519	9.8615	349.6701
32	0.5460	0.1305	3.1421	0.2893	9.9194	353.2827
34	0.5210	0.1199	2.9881	0.3281	9.9568	355.8528
36	0.4986	0.1108	2.8484	0.3679	9.9768	357.5335
37	0.4874	0.1067	2.7834	0.3882	9.9811	358.0814
38	0.4784	0.1029	2.7212	0.4086	9.9820	358.4535
39	0.4690	0.0993	2.6617	0.4292	9.9797	358.6628
40	0.4600	0.0959	2.6048	0.4499	9.9744	**358.7241**
41	0.4514	0.0927	2.5502	0.4707	9.9655	358.6404
42	0.4432	0.0897	2.4979	0.4917	9.9560	358.4299
43	0.4353	0.0869	2.4477	0.5127	9.9431	358.0995
44	0.4278	0.0842	2.3995	0.5338	9.9282	357.6579
45	0.4205	0.0817	2.3531	0.5549	9.9112	357.1132

Table 8 – Effect of θ_i's on performance measures in Model II when the arrival process is MNC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12-12.5	16.8978	15.9692	11.963	0.8768	10.0764	252.3641
13-13.5	12.9656	12.0530	10.6422	0.9873	10.1672	259.2338
14-14.5	10.1494	9.2542	10.2102	1.0943	10.1985	263.5313
15-15.5	8.1129	7.2366	9.7986	1.1970	10.1805	265.6982
16-16.5	6.6205	5.7639	9.4070	1.2949	10.1229	**266.1353**
17-17.5	5.5094	4.6729	9.0355	1.3881	10.0346	265.1977
18-18.5	4.6677	3.8516	8.6844	1.4765	9.9233	263.1907
19-19.5	4.0190	3.2230	8.3536	1.5605	9.7956	260.3689
20-20.5	3.5102	2.7339	8.0430	1.6403	9.6565	256.9832
21-21.5	3.1044	2.3473	7.7517	1.7162	9.5102	253.0616
22-22.5	2.7757	2.0372	7.4789	1.7885	9.3595	248.8655
23-23.5	2.5057	1.7850	7.2233	1.8574	9.2069	244.4474
24-24.5	2.2807	1.5773	6.9837	1.9232	9.0538	239.8815
25-25.5	2.0912	1.4043	6.7591	1.9861	8.9014	235.2248

Table 9 – Effect of θ_i's on performance measures in Model I when the arrival process is MZC

in Model I. In both models values of RK_1 decreases when $\theta_i's$ values increases. Also in both models, RK_2 increases when $\theta_i's$ values increases. This is because when $\theta_i's$ values increase, the expected service time of the customer in each stage decreases.

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12-12.5	1.2979	0.7292	6,3005	0.0213	7.3031	216.0909
14-14.5	1.0927	0.5633	5.7087	0.0346	7.8524	244.8974
16-16.5	0.9490	0.3537	5.2165	0.0516	8.2917	268.0396
18-18.5	0.8426	0.2848	4.8012	0.0720	8.6441	286.7090
20-20.5	0.7607	0.2350	4.4464	0.0956	8.9265	301.7886
22-22.5	0.6861	0.1942	4.0935	0.1115	9.0690	311.4098
24-24.5	0.6421	0.1691	3.8726	0.1517	9.3302	323.7022
26-26.5	0.5977	0.1466	3.6375	0.1834	9.4694	331.4603
28-28.5	0.5601	0.1285	3.4292	0.2171	9.5760	337.5471
30-30.5	0.5277	0.1138	3.2432	0.2526	9.6549	342.2260
32-32.5	0.4995	0.1016	3.0764	0.2896	9.7105	345.7129
34-34.5	0.4746	0.0913	2.9258	0.3278	9.7463	348.1865
36-36.5	0.4526	0.0827	2.7892	0.3670	9.7651	349.7957
37-37.5	0.4425	0.0788	2.7255	0.3869	9.7690	350.3161
38-38.5	0.4328	0.0752	2.6647	0.4070	9.7695	350.6657
39-39.5	0.4237	0.0719	2.6065	0.4272	9.7670	350.8573
40-40.5	0.4150	0.0688	2.5508	0.4475	9.7616	**350.9024**
41-41.5	0.4068	0.0659	2.4975	0.4680	9.7535	350.8119
42-42.5	0.3989	0.0632	2.4463	0.4886	9.7431	350.5957
43-43.5	0.3914	0.0607	2.3971	0.5092	9.7303	350.2630
44-44.5	0.3842	0.05583	2.3499	0.5299	9.7155	349.8224
45-45.5	0.3773	0.0561	2.3046	0.5506	9.6987	3492817.

Table 10 – Effect of θ_i's on performance measures in Model II when the arrival process is MZC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_1
12	18.0958	17.1635	11.0909	0.8490	10.1654	256.5463
13	13.8212	12.9044	10.6417	0.9602	10.2618	263.5676
14	10.7643	9.8647	10.2140	1.0681	10.2973	267.9635
15	8.5599	7.6789	9.8059	1.1719	10.2821	270.1769
16	6.9502	6.0887	9.4169	1.2710	10.2259	**270.6121**
17	5.7566	4.9151	9.0473	1.3654	10.1376	269.6305
18	4.8564	4.0352	8.6972	1.4550	10.0253	267.5456
19	4.1654	3.3645	8.3669	1.5400	9.8956	264.6201
20	3.6258	2.8447	8.0563	1.6208	9.7540	261.0680
21	3.1973	2.4354	7.7647	1.6977	9.6047	257.0590
22	2.8514	2.1063	7.4914	1.7708	9.4510	252.7250
23	2.5682	1.8432	7.2352	1.8406	9.2950	248.1673
24	2.3331	1.6255	6.9951	1.9071	9.1387	243.4630
25	2.1355	1.4446	6.7698	1.9708	8.9831	238.6709

Table 11 – Effect of θ_i's on performance measures in Model I when the arrival process is MZC

$\theta_i's$	ECS	ECQ	RK_1	RK_2	RP	ER_2
12	1.3046	0.7347	6.3168	0.0197	7.2919	215.4835
14	1.0974	0.5670	5.7222	0.0324	7.8457	244.5159
16	0.9525	0.3558	5.2278	0.0487	8.2889	267.8495
18	0.8454	0.2863	4.8108	0.0686	8.6446	286.6823
20	0.7630	0.2361	4.4546	0.0918	8.9298	301.9020
22	0.6974	0.1987	4.1471	0.1180	9.1576	314.1811
24	0.6438	0.1699	3.8789	0.1469	9.3381	324.0394
26	0.5992	0.1472	3.6430	0.1782	9.4792	331.8862
28	0.5614	0.1290	3.4341	0.2117	9.5873	338.0488
30	0.5289	0.1142	3.2476	0.2469	9.6675	342.7921
32	0.5006	0.1020	3.0803	0.2837	9.7243	346.3337
34	0.4757	0.0917	2.9293	0.3217	9.7610	348.8537
36	0.4535	0.0829	2.7924	0.3607	9.7806	350.5009
37	0.4434	0.0791	2.7286	0.3805	9.7848	351.0379
38	0.4337	0.0755	2.6677	0.4006	9.7856	351.4027
39	0.4246	0.0721	2.6094	0.4207	9.7843	351.6079
40	0.4159	0.0690	2.5536	0.4411	9.7782	**351.6654**
41	0.4076	0.0661	2.5001	0.4615	9.7704	351.5859
42	0.3997	0.0643	2.4488	0.4820	9.7601	351.3796
43	0.3921	0.0609	2.3996	0.5026	9.7476	351.0556
44	0.3849	0.0585	2.3523	0.5233	9.7329	350.6227
45	0.3780	0.0563	2.3068	0.5440	9.7163	350.0888

Table 12 – Effect of θ_i's on performance measures in Model II when the arrival process is MZC

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10.6 MAP with zero correlation (MZC)and the clock follows Erlang distribution

Tables 11 and 12 show the effect of θ on various performance measures and the revenue function, when the arrival process is MZC and the clock, is an Erlang clock. ER is maximum at $\theta = 16$ and the maximum revenue is 270.6121 in Model I and ER is maximum when $\theta = 40$ and the maximum revenue is 351.6654 in Model II. Also, the values of the rate of perfect service (RP) in Model II are greater than the corresponding values in Model I.

From Tables 1-12, we can conclude that in all cases, the values of ER and RP in Model II is greater than the corresponding values of ER and RP in Model I. Moreover the values of RK_1 and RK_2 in Model II are less than the corresponding values of RK_1 and RK_2 in Model I.

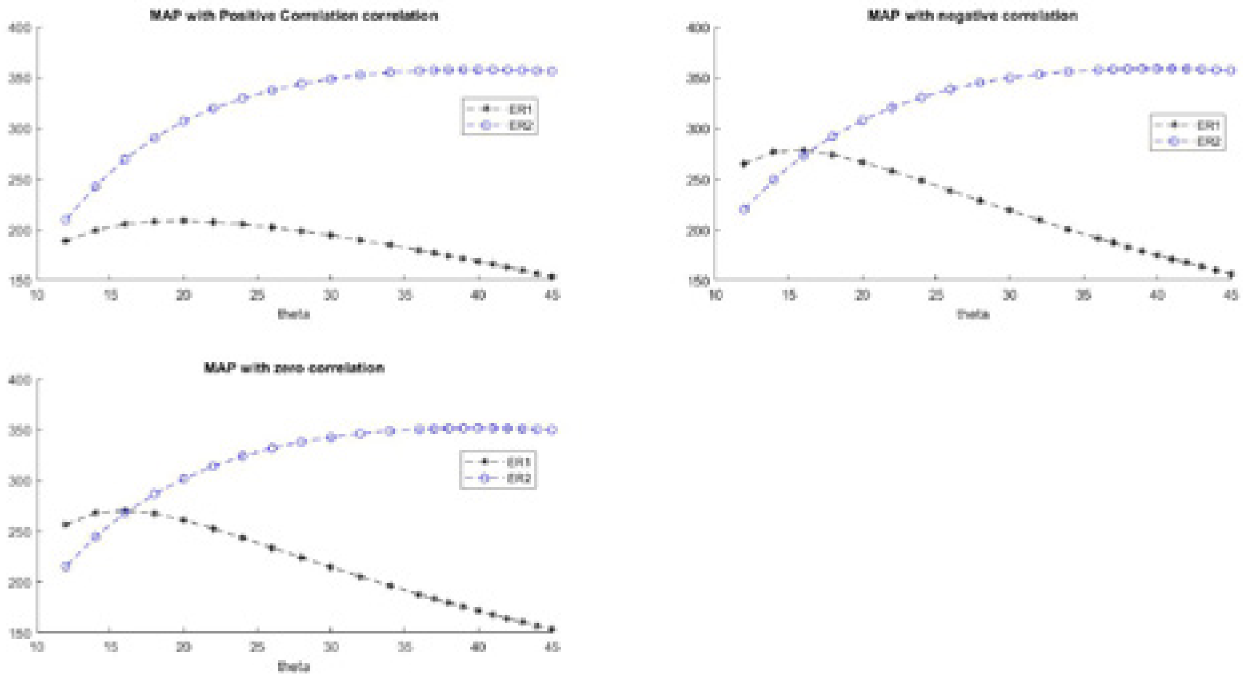

Figure 1 – Graph of Revenue Function

11 CONCLUSIONS

In this paper, we considered a $MAP/PH/1$ queue. We analysed this model by using the matrix-analytic method. We obtained the expected service time of a customer and also found the waiting time of a tagged customer. Also, we constructed a revenue function and other performance measures. To increase revenue, in Model II we consider the service time as the phase-type distribution (γ', L) of order $n = n_1 + n_2$, which is the convolution of the two phase-type distributions (α', T') of order n_1 and (β', S') of order n_2. We also performed some numerical experiments to evaluate some performance measures and also found that the revenue is maximum in Model II.

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