DYNAMICS OF SOFIC SHIFTS

ALI AKBAR K.

Department of Mathematics, Central University of Kerala, Kasaragod (India).

E-mail: aliakbar.pkd@gmail.com, aliakbar@cukerala.ac.in

ORCID: https://orcid.org/0000-0003-3542-3727

Reception: 13/08/2022 **Acceptance:** 28/08/2022 **Publication:** 29/12/2022

Suggested citation:

Ali A. K. (2022). Dynamics of Sofic Shifts. *3C Tecnología. Glosas de innovación aplicada a la pyme*, *11*(2), 13-23. https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

ABSTRACT

In this paper, we provide a characterization for the subshifts of finite type (SFT) in terms of Cellular automata (CA). In addition, we prove that

1. *The following are equivalent for a non-singleton subshift of finite type* $X_{\mathcal{F}}$.
   a) $X_{\mathcal{F}}$ *is transitive and* $Per(X_{\mathcal{F}})$, *the set of periodic points of* $X_{\mathcal{F}}$, *is cofinite*
   b) $X_{\mathcal{F}}$ *is weak mixing*
   c) $X_{\mathcal{F}}$ *is mixing.*
2. *For non-singleton sofic shifts, only the statements* (a) *and* (b) *are equivalent.*

KEYWORDS

subshift of finite type, sofic shifts, mixing, strongly connected digraph, labeled digraph, cellular automata

 https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

1 INTRODUCTION AND PRELIMINARIES

A dynamical system is a pair (X, f), where X is a metric space and f is a continuous self map. For each dynamical system (X, f), the period set $\text{Per}(f) = \{n \in \mathbb{N} : \exists\ x \in X \text{ such that } f^n(x) = x \neq f^m(x)\ \forall m < n\}$ consisting of the lengths of the cycles there, is a subset of the set $\mathbb{N}$ of positive integers. We say that a point $x \in X$ is periodic if $f^n x = x$ for some $n \in \mathbb{N}$, where f^n is the composition of f with itself n times. The smallest such positive integer n is called the period of x. Two dynamical systems (X, f), (Y, g) are said to be topological conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. If there is a continuous surjection $h : X \to Y$ such that $h \circ f = g \circ h$ then we say that (Y, g) is a factor of (X, f). A dynamical system (X, f) is said to be transitive if for any non-empty open sets U, V in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ and is said to be mixing if for any non-empty open sets U, V in X there exists $n \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq n$. A dynamical system (X, f) is said to be weak mixing, if given any four nonempty open sets U_1, V_1, U_2, V_2 in X there exists $m \in \mathbb{N}$ such that $f^m(U_1) \cap V_1$ and $f^m(U_2) \cap V_2$ are non-empty.

The subshifts form an important class of dynamical systems, because almost all dynamical systems are factor of some subshifts. There are plenty of books that explain how their study would throw light on still larger classes of dynamical systems (see [7], [10] and [14]). There are some dynamical systems such that the notions transitivity together with cofiniteness, weak mixing, and mixing which are equivalent (See [5], [8] for interval maps, See [2] for topological graph maps). It is natural to ask on which spaces a similar result will be true. We find that the same result is true in the class of non-singleton SFTs, and in the class of continuous 2-dimensional toral automorphisms. Note that SFTs, continuous 2-dimensional toral automorphisms and topological graph maps are different kinds of dynamical systems, and we cannot hope to have a similarity of proofs. In this paper, we concentrate on two-sided shifts. The case of one-sided shifts is similar.

Let $\mathcal{A}$ be a non-empty finite set (called alphabet) with discrete topology and consider the set $\mathcal{A}^{\mathbb{Z}}$, which denotes the set of doubly-infinite sequences $(x_i)_{i\in\mathbb{Z}}$ where each $x_i \in \mathcal{A}$, with product topology. It is compact and metrizable. The shift is the homeomorphism $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. A subset $A \subset \mathcal{A}^{\mathbb{Z}}$ is called σ-invariant if $\sigma(A) \subset A$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ forms a dynamical system called a full shift. A subshift is a σ- invariant non-empty closed subset X of a full shift, together with the restriction of σ to X. We denote the set of periods of all periodic points of σ in X by $Per(X)$. We call $Per(X)$ the period set of X. A word w on $\mathcal{A}$ is a concatenation $w_1 w_2 ... w_k$, where each $w_i \in \mathcal{A}$ and $(\overline{w})_n = w_r$ whenever $n \equiv r(\mathbf{mod}\ k)$. A subshift X is said to be a subshift of finite type (SFT) if $X = X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{no word in } \mathcal{F} \text{ occurs in } x\}$ for some finite set of words $\mathcal{F}$. A subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is called sofic if it is a factor of an SFT.

The notion of strongly connected digraphs is well known. For every SFT, there is an associated digraph and an associated matrix as described in [7]. This may or may not be strongly connected. Let $G = (V, E)$ be any directed graph (digraph) with vertex set V and edge set E. A subgraph $G' = (V', E')$ of G is said to be a full subgraph if $E' = E \cap \{(v_1, v_2) : v_1, v_2 \in V'\}$. A digraph is said to be simple if from every vertex v to a vertex w there is atmost one edge and it is said to be strongly connected if for every pair of vertices there exists a directed path. It is to be noted that a connected digraph may not be strongly connected. The SFT associated for a digraph G is denoted as X_G and the SFT associated for an $m \times m$ matrix A with entries 0 or 1 is denoted as X_A. Let $V' = \{v \in V : \text{there exists a cycle through this } v\}$. Consider a full subgraph of a simple digraph G with vertex set V', say $G' = (V', E')$. Define a relation R on V' in such a way that xRy if there is a directed path from x to y and from y to x. Then R is an equivalence relation on the set of vertices V'. Let $G_x^{'}$ be the full subgraph of G' with $[x]$, equivalence class of x, as the vertex set. This $G_x^{'}$ is strongly connected simple digraph and G' can be written as a finite union of such strongly connected simple digraphs, say $\bigcup_{i=1}^{n} G_i$. The SFT associated for a simple digraph G is denoted as X_G. Note that $Per(X_G) = Per(X_{G'})$.

Let $\mathcal{A}$ be an alphabet having atleast two elements. Let $r \in \mathbb{N}_0$. A function $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ is called a local rule. It induces a function $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by the rule

$(F(x))_n = f(x_{n-r}, x_{n-r-1}, ..., x_{n-1}, x_n, x_{n+1}, ..., x_{n+r-1}, x_{n+r})$ for all $n \in \mathbb{Z}$. The pair $(A^{\mathbb{Z}}, F)$ (simply the map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$) is called a cellular automaton (abbreviated as CA). A map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is

https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

a cellular automaton if and only if it is continuous and commutes with the shift (see [13]).

Our main results prove that:

1. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be any CA. Then $Fix(F)$ is an SFT or empty set. Conversely given any SFT X there exists a CA F such that $Fix(F) = X$.
2. The following are equivalent for a subset S of $\mathbb{N}$.
   a) $S = \text{Per}(X)$ for some mixing subshift of finite type X.
   b) Either $S = \{1\}$ or $\mathbb{N} \setminus F$ for some finite subset F of $\mathbb{N}$.
3. a) The following are equivalent for a non-singleton SFT X.
      i. X is transitive and $Per(X)$ is cofinite
      ii. X is weak mixing
      iii. X is mixing.
   b) In general, in the case of non-singleton sofic shifts, only the statements 3(a)ii and 3(a)iii are equivalent.

The results are intuitive in nature and provide a basic understanding of the dynamics of shift spaces. This review article will be useful to any reader interested in understanding basics of dynamics of shift spaces.

2 SUBSHIFTS OF FINITE TYPE AND SOFIC SHIFTS: TRANSITIVITY, WEAK MIXING, MIXING

One of the result in this paper is motivated by the following two known theorems.

Theorem 1. *The following are equivalent for a topological graph map $f : G \to G$ (See [5], [8] for interval maps, See [2] for topological graph maps).*

1. *f is transitive and $Per(f)$ is cofinite (ie., $\mathbb{N} \setminus Per(f)$ is finite)*
2. *f is weak mixing*
3. *f is mixing.*

Theorem 2. *The following are equivalent for a continuous toral automorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$.*

1. *T is transitive and $Per(T)$ is cofinite (ie., $\mathbb{N} \setminus Per(T)$ is finite)*
2. *T is weak mixing*
3. *T is mixing.*

Proof. Proof follows from the main results of [16] and [17].

2.1 SUBSHIFTS OF FINITE TYPE

Let A be a $k \times k$ adjacency matrix (i.e., the matrix with entries 0 or 1). We call the matrix primitive if there exists $N \in \mathbb{N}$ such that $A^N > 0$. Now we consider the following known proposition.

Proposition 1. *[7] An SFT induced by a matrix A with non-zero rows and columns is mixing if and only if A is primitive.*

Next we state the following known theorem. We denote any finite subset A of $\mathbb{N}$ as $A \subset\subset \mathbb{N}$, and *gcd* for the greatest common divisor.

Theorem 3. *[1] The following are equivalent for a subset S of $\mathbb{N}$.*

1. *$S = Per(X_G)$ for some strongly connected simple digraph G containing cycles of lengths $m_1, ..., m_k$ such that $gcd(m_1, m_2, ..., m_k) = 1$.*
2. *Either $S = \{1\}$ or $S = \mathbb{N} \setminus F$ for some $F \subset\subset \mathbb{N}$.*

Next we have:

Theorem 4. *A strongly connected simple digraph G induced by an adjacency matrix $A_{k\times k}$ with non zero rows and columns contains cycles of lengths $m_1, m_2, ..., m_k$ such that $gcd(m_1, m_2, ..., m_k) = 1$ if and only if A is primitive.*

Proof. Suppose that G is a strongly connected simple digraph, and contains cycles of lengths $m_1, m_2, ..., m_k$ such that $\gcd(m_1, m_2, ..., m_k) = 1$. Without loss of generality we can assume that these are simple cycles of G that contain all vertices. By a basic number theory result, there exists $n_0 \in \mathbb{N}$, $F \subset\subset \mathbb{N}$ such that for all $n \geq n_0$, $n \in \mathbb{N} \setminus F$ there exist $a_1, a_2, ..., a_k \in \mathbb{N}$ such that $n = a_1 m_1 + a_2 m_2 + ... + a_k m_k$. Therefore, given any vertex x there exists a cycle of length n for all $n \geq n_0$. Let $m = diam(G) = Max\{l(x, y) : x, y \in V(G)\}$ where $l(x, y)$ denotes the length of a directed path from x to y. Let $N = n_0 + m$. Hence $A^N > 0$ since for every $x, y \in V(G)$ there exists a path of length p for all $p \leq m$ and for every $x \in V(G)$ there exists a cycle of length n for all $n \geq n_0$. Write $N = n_0 + m - p + p$. Hence A is primitive.

Conversely, suppose that A is primitive and $\gcd(m_1, m_2, ..., m_l) = p > 1$ for all cycles of length m_i, $1 \leq i \leq p$, $p \in \mathbb{N}$. Let k = gcd of lengths of all cycles of G. Then there exist cycles of length $m_1, m_2, ..., m_l$ such that $\gcd(m_1, m_2, ..., m_l) = k$. Then k divides the lengths of all cycles. Also, there exists $s \in \mathbb{N}$ such that $A^s > 0$, and for every $x, y \in V(G)$ there exists a path of length s from x to y since A is primitive and G is strongly connected. Therefore k divides s and $s + 1$, which implies $k = 1$. A contradiction. Hence the proof.

Corollary 1. *A strongly connected simple digraph G contains cycles of length $m_1, ..., m_k$ such that $gcd(m_1, m_2, ..., m_n) = 1$ if and only if X_G is mixing.*

Proof. This follows from Proposition 1, Theorems 3 and 4.

Corollary 2. *The following are equivalent for a subset S of $\mathbb{N}$.*

1. *$S = Per(X)$ for some mixing subshift of finite type X.*
2. *Either $S = \{1\}$ or $\mathbb{N} \setminus F$ for some finite subset F of $\mathbb{N}$.*

Proof. Proof follows from Theorems 3, 4, and Corollary 1.

The proof of the following theorem relies mostly on the proof of Proposition 1, as given in [7].

Lemma 1. *An SFT X_A is weak mixing if and only if for every $1 \leq i_1, j_1, i_2, j_2 \leq k$ there exists $n \in \mathbb{N}$ such that $A^n(i_1, j_1) > 0$ and $A^n(i_2, j_2) > 0$ where A denotes a $k \times k$ adjacency matrix.*

Proof. Assume that X_A is weak mixing. Let $U_1 = \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = i_1\}$, $V_1 = \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = j_1\}$, $U_2 = \{(y_n)_{n\in\mathbb{Z}} \in X_A : y_0 = i_2\}$ and $V_2 = \{(y_n)_{n\in\mathbb{Z}} \in X_A : y_0 = j_2\}$ where $1 \leq i_1, j_1, i_2, j_2 \leq k$. These sets are open. Then there exists $n \in \mathbb{N}$ such that $\sigma^n(U_i) \cap V_i \neq \emptyset$, $i = 1, 2$. Hence $x_0 = i_1$, $y_0 = i_2$, $x_n = j_1$ and $y_n = j_2$. Note that $A^N(i, j) = \sum_{r_1=1}^{k} ... \sum_{r_{N-1}=1}^{k} A(i, r_1)A(r_1, r_2)...A(r_{N-2}, r_{N-1})A(r_{N-1}, j)$ for all $N \in \mathbb{N}$. But $A(i_1, x_1) = A(x_1, x_2) = A(x_2, x_3) = ... = A(x_{n-1}, j_1) = 1$ and $A(i_2, y_1) = A(y_1, y_2) = A(y_2, y_3) = ... = A(y_{n-1}, j_2) = 1$. Therefore $A^n(i_1, j_1) > 0$ and $A^n(i_2, j_2) > 0$.

 https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

Conversely, assume that for every $1 \leq i_1, j_1, i_2, j_2 \leq k$ there exists $N \in \mathbb{N}$ such that $A^N(i_1, j_1) > 0$ and $A^N(i_2, j_2) > 0$. Given non-empty open sets U_1, V_1, U_2, V_2 we can choose $(i_n^{(l)})_{n\in\mathbb{Z}} \in U_l$ and $(j_n^{(l)})_{n\in\mathbb{Z}} \in V_l$ such that for $M > 0$ sufficiently large; and $U_l \supset \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = i_k^{(l)}, -M \leq k \leq M\}$, $V_l \supset \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = j_k^{(l)}, -M \leq k \leq M\}$ for $l = 1, 2$ (It is possible since the set of symmetric cylinders form a base for the topology on $\mathcal{A}^{\mathbb{Z}}$).

By hypothesis, there exists $N > 0$ such that $A^N(i_M^{(l)}, j_{-M}^{(l)}) > 0$ for $l = 1, 2$. This means we can find a word $x_1^l ... x_{N-1}^l$ such that $A(i_M^{(l)}, x_1^l) = A(x_1^l, x_2^l) = ... = A(x_{N-1}^l, j_{-M}^{(l)}) = 1$.

Define $x_n^{(l)} = \begin{cases} i_n^{(l)} & \text{if } n \leq M \\ x_{n-M}^l & \text{if } M+1 \leq n \leq M+N-1 \\ j_{n-(2M+N)}^{(l)} & \text{if } M+N \leq n \end{cases}$

Then $\sigma^{2M+N}(U_l) \cap V_l \neq \emptyset$ for $l = 1, 2$. Hence X_A is weak mixing.

Next we have:

Theorem 5. *An SFT is weak mixing if and only if it is mixing.*

Proof. Let A be an adjacency matrix of order k. Assume that X_A is weak mixing. We have to prove that there exist cycles of lengths $m_1, m_2, ..., m_p$ such that $\gcd(m_1, m_2, ..., m_p) = 1$. Suppose not. Then there exists $s \in \mathbb{N} \setminus \{1\}$ such that s divides the lengths of all cycles (let $s = $ gcd of lengths all cycles). Let $1 \leq v_1, w_1, v_2 \leq k$ be such that $v_1 w_1$ is a block in x for some $x \in X_A$. Then there exist a cycle of length n through v_2 and a path of length n from w_1 to v_1, which implies s divides n and $n+1$. Hence $s = 1$. A contradiction. Hence X_A is mixing. Converse part is easy.

Hence we have:

Theorem 6. *The following are equivalent for a non-singleton SFT $X_{\mathcal{F}}$.*

(i) $X_{\mathcal{F}}$ is transitive and $Per(X_{\mathcal{F}})$ is cofinite.

1. *$X_{\mathcal{F}}$ is weak mixing.*

2. *$X_{\mathcal{F}}$ is mixing.*

Proof. We first observe that the period set $\mathrm{Per}(X_{\mathcal{F}})$ of a finite SFT $X_{\mathcal{F}}$ is finite. So $\mathrm{Per}(X_{\mathcal{F}})$ is not cofinite. Except singleton SFTs all other finite SFTs are not weak mixing and hence not mixing. Hence the theorem follows from Theorems 3, 4 and 5, and Proposition 1.

Remark 1. *Suppose some rows of A or columns of A is of full of zeros, say i th row and j th column. Then remove ith row and j th column. Doing this for all such i and j, we obtain another matrix $\tilde{A}$ of smaller size. Then X_A and $X_{\tilde{A}}$ are in a sense one and the same. Therefore the equivalence of (2) and (3) is true for all subshifts induced by adjacency matrices.*

Next we consider:

Theorem 7. *(see [15]) (Blokh, Barge-Martin) Let $f : I \to I$ be an interval map such that the periodic points are dense in I. Then the interval I decomposes into transitive components C_n in the following way.*

1. *C_n is a closed non-degenerate interval or C_n is the union of two disjoint closed non degenerate intervals,*

2. *$f|_{C_n}$ is transitive,*

3. *the complement set of $\bigcup C_n$ is included in $\{x \in X : f^2(x) = x\}$.*

In addition, the number of transitive components C_n is finite or countable and their interiors are pairwise disjoint.

https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

As similar to Theorem 7, now we have:

Theorem 8. *Let $X_{\mathcal{F}}$ be an SFT for some finite set of words $\mathcal{F}$ over an alphabet $\mathcal{A}$ with dense set of periodic points. Then there exists some finite set of words $\mathcal{G} \supset \mathcal{F}$, and an SFT $X_{\mathcal{G}}$ with dense set of periodic points and it is a finite union of transitive SFTs, and $Per(X_{\mathcal{F}}) = Per(X_{\mathcal{G}})$.*

Proof. Let X^G denotes the subshift of finite type associated for a simple digraph G such that $X^G = X_{\mathcal{F}}$. Let V be the set of all vertices of G. For $v_1, v_2 \in V$, we say that $v_1 \sim v_2$ whenever there is a path from v_1 to v_2 and vice-versa. Then $\sim$ forms an equivalence relation on V. Each equivalence class corresponds to a strongly connected simple digraphs. Let G' be the union of all such strongly connected simple digraphs. Observe that $Per(X^G) = Per(X^{G'})$. Now consider $\mathcal{G} \supset \mathcal{F}$ such that $X_{\mathcal{G}} = X^{G'}$. Hence the proof.

Consider a countable set $\{1, 2, ...\}$. With the discrete topology it is a non-compact metrizable space. Let $\sum = \{1, 2, ...\}^{\mathbb{Z}}$. With product topology $\sum$ is a totally disconnected, perfect and non-compact metric space. As in the finite case, the cylinder sets form a countable basis of clopen sets. The shift, σ, is a homeomorphism of the space to itself. The dynamical system $(\sum, \sigma)$ is the full shift on the symbols. If A is a countable, zero-one matrix, then as in the finite case, we use transition rules to define a shift-invariant subset of the full shift on countably many symbols, denoted by $\sum_A$. Then the subspace $\sum_A$ of $\sum$ is non-compact, metrizable and $\sigma : \sum_A \to \sum_A$ is the countable state Markov shift defined by A.

Next consider the following two known propositions.

Proposition 2. *[12] A countable state Markov shift $\sum_A$ is topologically transitive if and only if A is irreducible.*

Proposition 3. *[12] A countable state Markov shift $\sum_A$ is topologically mixing if and only if A is primitive.*

Now we have:

Theorem 9. *[12] The following are equivalent for countable Markov shift $\sigma : \sum_A \to \sum_A$.*

(i) $\sigma : \sum_A \to \sum_A$ is transitive and $Per(\sigma)$ is cofinite.

(ii) $\sigma : \sum_A \to \sum_A$ is weak mixing.

(iii) $\sigma : \sum_A \to \sum_A$ is mixing.

Proof. The proof follows from Propositions 2 and 3.

2.2 SOFIC SHIFTS

The notion of labeled digraph is well known (see [14], [7]). For every sofic shift there is a labeled digraph and vice versa (see [7]). As similar to digraphs, we can define simple labeled digraph and strongly connected labeled digraph. First we have to define it for corresponding digraphs. Then consider the corresponding labeled digraphs. As similar to digraphs, for every labeled digraph Γ there exists another labeled digraph Γ' such that $\mathrm{Per}(X_\Gamma) = \mathrm{Per}(X_{\Gamma'})$ and Γ' is a finite union of strongly connected simple labeled digraphs where X_Γ denotes the subshift induced by Γ. Let Γ be a finite labeled digraph, the edges of Γ are labeled by an alphabet $\mathcal{A}_m = \{1, 2, ..., m\}$. Note that we do not assume that the different edges of Γ are labeled differently. Let $E(\Gamma)$ denotes the set of all edges of Γ. The subset $X_\Gamma \subset \mathcal{A}_m^{\mathbb{Z}}$ consisting of all infinite directed paths in Γ is closed and shift invariant. If a subshift X is topologically conjugate to X_Γ for some labeled digraph Γ, then we say that Γ is a presentation of X.

Proposition 4. *[7] A subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it admits a presentation by a finite labeled digraph.*

https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

Theorem 10. *[1] The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $S = Per(X_\Gamma)$ *for some strongly connected labeled digraph* Γ *containing cycles of length* $m_1, m_2, ..., m_k$ *such that* $gcd(m_1, m_2, ..., m_k) = 1$.

(2) *Either* $S = \{1\}$ *or* $S = \mathbb{N} \setminus F$ *for some* $F \subset\subset \mathbb{N}$.

From the definition of transitivity, mixing, weak mixing and by using some ideas from Lemma 1, we can prove the following lemma for any directed labeled graph Γ. Recall that for every non-empty open set U in X_Γ we can choose $(i_n)_{n\in\mathbb{Z}} \in U$ such that for $M > 0$ sufficiently large; $U \supset \{(x_n)_{n\in\mathbb{Z}} : x_k = i_k, -M \leq k \leq M\}$.

Lemma 2. *Let* Γ *be a labeled digraph. Then the following are true.*

1. X_Γ *is transitive if and only if for every* $i, j \in E(\Gamma)$ *there exists a directed path of length* n *from* i *to* j *for some* $n \in \mathbb{N}$.
2. X_Γ *is weak mixing if and only if for every* $i_1, j_1, i_2, j_2 \in E(\Gamma)$ *there exist directed paths of length* n *from* i_1 *to* j_1 *and from* i_2 *to* j_2 *for some* $n \in \mathbb{N}$.
3. X_Γ *is mixing if and only if for every* $i, j \in E(\Gamma)$ *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *there is a directed path of length* n *from* i *to* j.

Theorem 11. *Let* Γ *be a strongly connected labeled digraph. Then* Γ *contains cycles of lengths* $m_1, m_2, ..., m_k$ *such that* $gcd(m_1, m_2, ..., m_k) = 1$ *if and only if* X_Γ *is mixing.*

Proof. We can provide a proof similar to that of Theorem 4. Here while proceeding the proof without loss of generality it is not possible to assume the cycles are simple. Still the conclusion is true.

Corollary 3. *The period set of a mixing SFT is*

either $\{1\}$ *or* $\mathbb{N} \setminus F$ *for some* $F \subset\subset \mathbb{N}$.

Proof. Proof follows from Theorems 10 and 11.

Theorem 12. *A sofic shift is weak mixing if and only if it is mixing.*

Proof. Because of Lemma 2 and Theorem 11, we can provide a proof similar to that of Theorem 5.

Remark 2. *There exists a sofic shift* X_Γ *which is transitive and its period set is cofinite, but it is not Mixing.*

Proof. Let X_Γ be the sofic shift based on the directed graph Γ with vertices 0 and 1, arcs labeled a, b, c from 0 to 1, and arcs labeled a, b, d from 1 to 0. Then X_Γ is the image of the topologically transitive subshift of finite type, based on Γ but with distinctly labeled edges. The period set of X_Γ is $\mathbb{N}$. But X_Γ is not topologically mixing by Theorems 11. Hence the remark follows.

Remark 3. *If* X_Γ *is a transitive non-singleton sofic shift, then the set of periodic points of* σ *in* X_Γ *is dense in* X_Γ. *But a compact dynamical system which is totally transitive and has a dense set of periodic points is weak mixing (See [4]). Therefore* X_Γ *is totally transitive if and only if* X_Γ *is weak mixing. In general, for a subshift, the conclusion of Theorem 12 need not be true. There is a subshift which is weak mixing but not mixing (Chacon shift, See [11]).*

https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

3 A CHARACTERIZATION OF AN SFT

The cellular automata play an important role in various contexts such as computer graphics, parallel computing and cell biology. It is natural to ask for a neat description of the sets of periodic points of cellular automata, unfortunately we do not have a complete answer. There have been some papers that discussed about the sets of periodic points for continuous self maps (See [3], [7], [9]). It is natural to ask: Which sets will arise as the set of all periodic points of continuous self maps? This question is too abstract. If we ask the same question in the class of some nice class of maps then we can expect a nice answer. In this section, we consider in the case of CA. Characterization of the sets of periodic points for a continuous self map of an interval is incomplete. J.-P. Delahaye gave partial results in this context (see Propositions 5, 6). This is our first motivation for considering CA. We completely solved in the case of a continuous 2-dimensional toral automorphism in [16] (see Theorem 13). This is our second motivation for considering CA. In this section we give a partial answer in the case CA. Our result is similar to the following propositions 5 and 6, and Theorem 13. It characterizes an SFT in terms of a CA.

Proposition 5. *[9] (i) The set of fixed points of a continuous function from $[0,1]$ to $[0,1]$ is a closed subset of $[0,1]$.*

(ii) For every closed subset F of $[0,1]$ there exists a continuous function f whose fixed point set is F .

Definition 1. *A subset F of $[0,1]$ is symmetric if for $x \in [0,1]$, $\frac{1}{2}+x \in F \Leftrightarrow \frac{1}{2}-x \in F$.*

Proposition 6. *[9] (i) The set of periodic points of period 1 or 2 of a continuous function from $[0,1]$ to $[0,1]$ is a closed subset of $[0,1]$.*

(ii) For every symmetric closed subset of $[0,1]$ there exists a continuous function from $[0,1] \to [0,1]$ whose set of periodic points period 1 or 2 is $F \cup \{\frac{1}{2}\}$.

Theorem 13. *[16]*

For any continuous toral automorphism T, the set $P(T)$ of periodic points of T is one of the following:

1. *$\mathbb{Q}_1 \times \mathbb{Q}_1$, where $\mathbb{Q}_1$ denotes the set of all rational points in $[0,1)$.*
2. *S_r for some $r \in \mathbb{Q} \cup \{\infty\}$; where $S_r = \{(x,y) \in \mathbb{T}^2 \ : \ rx+y \text{ is rational }\}$.*
3. *$\mathbb{T}^2$.*

Definition 2. *A dynamical system (X,f) has the shadowing property, if for any $\epsilon > 0$ there exists $\delta > 0$ such that any finite δ-chain is ϵ-shadowed by some point. A point $x \in X$ ϵ-shadows a finite sequence $x_0, x_1, ..., x_n$, if for all $i \leq n$, $d(F^i(x), x_i) < \epsilon$. A (finite or infinite) sequence $(x_n)_{n\geq 0}$ is a δ-chain, if $d(F(x_n), x_{n+1}) < \epsilon$ for all n.*

ie., $\forall \epsilon > 0,\ \exists \delta > 0, \forall x_0, ..., x_n,\ (\forall i, d(F(x_i), x_{i+1}) < \delta \implies \exists x, \forall i, d(F^i(x), x_i) < \epsilon)$.

Definition 3. *A dynamical system (X,f) is open, if $f(U)$ is open for any open $U \subset X$.*

There are two distinct topological characterization of SFT known in literature as follows.

Theorem 14. *[13] A subset $X \subset \mathcal{A}^{\mathbb{N}}$ is an SFT if and only if X has the shadowing property.*

Theorem 15. *[13] A subset $X \subset \mathcal{A}^{\mathbb{N}}$ is an SFT if and only if X is an open subset of $\mathcal{A}^{\mathbb{N}}$.*

Next we have:

Lemma 3. *For every SFT X, there exists a finite set of words $\mathcal{G}$ having odd length such that $X = X_{\mathcal{G}}$.*

Proof. Let X be a k-step SFT. Then there exists a finite set of words $\mathcal{F}$ having length atmost k such that $X = X_{\mathcal{F}}$. If k is odd then consider $\mathcal{G} = \{x \in W_k(\mathcal{A}^{\mathbb{Z}}) : y \text{ is a subword of } x \text{ for some } y \in \mathcal{F}\}$. If k is even then consider $\mathcal{G} = \{x \in W_{k+1}(\mathcal{A}^{\mathbb{Z}}) : y \text{ is a subword of } x \text{ for some } y \in \mathcal{F}\}$.

 https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

Claim: $X_{\mathcal{F}} = X_{\mathcal{G}}$.

Let $x \in X_{\mathcal{F}}$. Suppose $x \notin X_{\mathcal{G}}$. Then for some $y \in \mathcal{F}$, y is a subword of x. A contradiction. Therefore $x \in X_{\mathcal{G}}$. Next, let $x \in X_{\mathcal{G}}$. Which implies y is not a subword of x for all $y \in \mathcal{F}$. Then $x \in X_{\mathcal{F}}$. Hence the claim.

Next we have:

Theorem 16. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be any CA. Then $Fix(F)$ is an SFT or an empty set. Conversely given any SFT X there exists a CA F such that $Fix(F) = X$.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA defined by the local rule $f : \mathcal{A}^{2r+1} \to \mathcal{A}$. Let $\mathcal{F} = \{w \in \mathcal{A}^{2r+1} : f(w) \neq \text{the middle term of } w\}$. Note that $\mathcal{F}$ is finite (it may be empty or $\mathcal{A}^{2r+1}$). First, let $x \in X_{\mathcal{F}}$. Which implies $(F(x))_i = x_i$ for all i. ie., $F(x) = x$. Next, let $x \in \mathcal{A}^{\mathbb{Z}}$ such that $F(x) = x$. Then $f(x_{i-r}x_{i-r+1}...x_0...x_{i+r-1}x_{i+r}) = x_i$ for all i. ie., $x \in X_{\mathcal{F}}$. Hence $Fix(F) = X_{\mathcal{F}}$.

Conversely, given any SFT $X_{\mathcal{F}}$ without loss of generality assume that $\mathcal{F}$ contains words of same length (odd length) because of Lemma 3.

Define $f : A^{2r+1} \to A$ such that

$$f(w) = \begin{cases} \textit{the middle term of } w & \text{if } w \textit{ is forbidden} \\ \textit{some other alphabet} & \textit{otherwise} \end{cases}$$

Then $Fix(F) = X_{\mathcal{F}}$.

Remark 4. *In the statement of Theorem 16, we can replace $Fix(F)$ by $Fix(F^n)$.*

Proof. Let $f : A^{2r+1} \to A$ be the local rule of a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. The local rule $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ induces a function $\tilde{f} : \mathcal{A}^{2s+2r+1} \to \mathcal{A}^{2s+1}$ for all s. Then by inductively, define $f_n : \mathcal{A}^{2nr+1} \to \mathcal{A}$ such that $f_n(w) = f(\tilde{f}_{n-1}(w))$ where $\tilde{f}_m : \mathcal{A}^{2r+2(m-1)r+1} \to \mathcal{A}^{2r+1}$ denotes the induced function of f_m for $s = r$. Note that the length of $\tilde{f}(w)$ is equal to the difference between the length of w and $2r$. Let $\mathcal{F}_n = \{w : f_n(w) \neq \text{the middle term of } w\}$. Then $Fix(F^n) = X_{\mathcal{F}_n}$.

Converse part follows easily.

Remark 5. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be any CA. Then $Fix(F^n)$ is an SFT for each $n \in \mathbb{N}$. Conversely, given any SFT X there exists CA $F_n : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that $Fix(F_n^n) = X$.*

4 CONCLUSIONS

For each self-map f on a set X, we associate a subset of $\mathbb{N}$ namely, $Per(f)$. If f belongs to a certain nice class of functions, then not all subsets of $\mathbb{N}$ may arise as the set of periods. Which subsets of $\mathbb{N}$ will come in this class? We answer this question for mixing SFT?s, and for mixing sofic shifts.
It is natural to ask: On which class of dynamical systems the following statements are equivalent?

1. f is transitive and $Per(f)$ is cofinite.
2. f is weak mixing
3. f is mixing.

We have proved that the above statements are equivalent in the case of non-singleton SFT's but not true in the case of sofic shifts. Also we have obtained a characterization for SFT's in terms of Cellular automata.

ACKNOWLEDGMENT

The author is very thankful to the referee for giving valuable suggestions.

 https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23

REFERENCES

[1] **K. Ali Akbar** and **V. Kannan** (2018), Set of periods of subshifts, *Pro. Indian Acad. Sci. (Math. Sci.)*, 128:63.

[2] **LL. Alseda**, **M.A. Del Rio** and **J. A. Rodrigues** (2003), A survey on the Relation Between Transitivity and Dense periodicity for graph maps, *Journal of Difference equations and Applications, Vol.9(3/4)*, 281-288.

[3] **I.N. Baker** (1964), Fixpoints of polynomials and rational functions, *J. London Math. Soc., 39*, 615-622.

[4] **J. Banks** (1997), Regular Periodic decomposition for topologically transitive maps, *Ergodic Theory and Dynamical Systems, 17*, 505-529.

[5] **L.S. Block** and **W.A. Coppel** (1992), Dynamics in One Dimension, *Springer-Verlag, Berline, Volume 1513 of Lecture Notes in Mathematics.*

[6] **J.A. Bondy** and **U.S.R. Murty** (1982), Graph theory with application, *Elsevier science publishing Co., Inc, New York.*

[7] **M. Brin** and **G. Stuck** (2002), Introduction to Dynamical Systems, *Cambridge University Press.*

[8] **E.M. Coven** and **M.C. Hidalgo** (1991), On the topological entropy of transitive maps of the interval, *Bull. Aust. Math. Soc., 44*, 207-213.

[9] **J.-P. Delahaye** (1981), The set of periodic points, *Amer. Math. Monthly., 88*, 646-651.

[10] **R.L. Devaney**(1989), An Introduction to Chaotic Dynamical Systems, *Addison-wesley Publishing Company Advanced Book Program., Redwood City, CA, second edition.*

[11] **Eli Glasner** (2003), Ergodic Theory Via Joinigs, *American Mathematical Society, Mathematical surveys and monographs, Vol 101.*

[12] **B.P. Kitchens** (1997), Symbolic Dynamics, *Springer.*

[13] **P. Kurka** (2003), Topological and symbolic dynamics, *Societe Mathematique de France, Paris.*

[14] **D.A. Lind** and **B. Marcus** (1995), An introduction to symbolic dynamics and coding, *Cambridge University Press., New York, NY, USA.*

[15] **Sylvie Ruette** (2003), Chaos for continuous interval maps, *A survey of relationship between the various forms of chaos.*

[16] **I. Subramania Pillai**, **K. Ali Akbar**, **V. Kannan** and **B. Sankararao** (2010), The set of periodic points of a toral automorphism, *Journal of Mathematical Analysis and Applications., 366*, 367-371.

[17] **I. Subramania Pillai**, **K. Ali Akbar**, **V. Kannan** and **B. Sankararao** (2011), The set of periods of periodic points of a toral automorphism, *Topology Procedings, Volume 37*, 1-14.

https://doi.org/10.17993/3ctecno.2022.v11n2e42.13-23