REIDEMEISTER NUMBER IN LEFSCHETZ FIXED POINT THEORY

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ABSTRACT

Several interesting numbers such as the homotopy invariant numbers the Lefschets number $L(f)$, the Nielsen number $N(f)$, fixed point index $i(X, f, U)$ and the Reidemeister number $R(f)$ play important roles in the study of fixed point theorems. The Nielsen number gives more geometric information about fixed points than other numbers. However the Nielsen number is hard to compute in general. To compute the Nielsen number, Jiang related it to the Reidemeister number $R(f_\pi)$ of the induced homomorphism $f_\pi : \pi_1(X) \to \pi_1(X)$ when X is a lens space or an H-space (Jian type space). For such spaces, either $N(f) = 0$ or $N(f) = R(f)$ the Reidemeister number of f_π and if $R(f) = \infty$ then $N(f) = 0$ which implies that f is homotopic to a fixed point free map. This is a review article to discuss how these numbers are related in fixed point theory.

KEYWORDS

Twisted conjugacy, Reidemiester number, Lefschetz number, Nielsen number, Jiang space

1 INTRODUCTION

Let $\phi : G \to G$ be an endomorphism of an infinite group G. One has an equivalence relation $\sim_\phi$ on G defined as $x \sim_\phi y$ if there exists a $g \in G$ such that $y = gx\phi(g)^{-1}$. The equivalence classes are called the Reidemeister classes of ϕ or ϕ-conjugacy classes. When ϕ is the identity, the Reidemeister classes of ϕ are the usual conjugacy classes. The Reidemeister classes of ϕ are the orbits of the action of G on itself defined as $g.x = gx\phi(g^{-1})$. The Reidemeister classes of ϕ containing $x \in G$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. The set of all Reidemeister classes of ϕ is denoted by $\mathcal{R}(\phi)$. We denote by $R(\phi)$ the cardinality of $\mathcal{R}(\phi)$ if it is finite and if it is infinite we set $R(\phi) := \infty$ and $R(\phi)$ is called the Reidemeister number of ϕ on G. We say that G has the R_∞-property if the Reidemeister number of ϕ is infinite for every automorphism ϕ of G. If G has the R_∞-property, we call G an R_∞-group.

The notion of Reidemeister number originated in the Nielsen–Reidemeister fixed point theory. See [?] and the references therein. The problem of determining which classes of groups have R_∞-property is an area of active research. Many mathematicians have been trying to determine which class of groups have the R_∞-property using the internal structure of the class, such as Lie group structure, C^*-algebra structure or purely algebraic properties of the class. There is no particular way to solve this problem, which makes it more difficult and interesting. If X is an H-space or a lens space, their fundamental groups are abelian. The Reidemeister number of an endomorphism of an abelian group is easily computable in many cases. In fact, if G is an abelian group, $\mathcal{R}(\phi)$ is an abelian group under the well defined operation $[x][y] := [xy], x, y \in G$.

The R_∞-property does not behave well with respect to finite index subgroups and quotients as the D_∞ and any free group of rank $n > 1$ has the R_∞-property although the infinite cyclic group and finitely generated free abelian groups, which are quotients of free groups, do not (ref. [5], [4]). Thus the R_∞-property is not invariant under quasi isometry, that is it is not *geometric* among the class of all finitely generated groups. The works of Levitt and Lustig [5] and Felshtyn [2] show that this property is geometric in the class of non-elementary hyperbolic groups. It is been proved in [7] that the R_∞-property is geometric for the class of all finitely generated groups that are quasi-isometric to irreducible lattices in real semisimple Lie groups with finite centre and finitely many connected components. The R_∞-property for irreducible lattices was proved in [6].

We have stated some results without proofs. For proofs and further readings, we refer the reader [1].

2 THE LEFSCHETZ NUMBER

Let X be a connected compact ANR and $f : X \to X$ a continuous map. We have seen fixed point theorems like Brouwer fixed point theorem that states "Any map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point"where $\mathbb{D}^n$ is the closed disk in $\mathbb{R}^n$ and the traditional Lefschetz fixed point theorem that states "If $L(f) \neq 0$ then f has a fixed point", where $L(f)$ is the Lefschetz number with respect to the rational homology. Our statement of the Lefschetz fixed point theorem differs from the traditional one. We will prove the theorem for $L(f, \mathbb{F})$, where $\mathbb{F}$ is any field, because it is often easier to compute $L(f, \mathbb{F})$ if the field is chosen properly than it is to compute $L(f)$, and the conclusion is for all maps homotopic to f rather than just for the map f. An important reason, however, was that the converse of the traditional statement is -"If $L(f) = 0$, then f is fixed point free"- and this is trivially false (we will see an example). On the other hand, the converse of our statement is -"If $L(f, \mathbb{F}) = 0$ for all fields $\mathbb{F}$, then there is a fixed point free map g homotopic to f". This is true. A proof can be seen in [1].

To define the Lefschetz number we need the following definitions. A subset A of a space X is called a *neighbourhood retract* of X if there exists an open subset U of X containing A and a retraction of U onto A, i.e., a map $r : U \to A$ such that the restriction of r to A is the identity map. A space X is an *absolute neighbourhood retract* (**ANR**) if it has the following property: If X imbeds into a separable metric space Y, then X is a neighbourhood retract of Y. The ANR property is a topological invariant. A compact space X is a *compact* $\mathbf{A}NR$ if and only if there exists an imbedding $i : X \to I^\infty$ such that

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$i(X)$ is a neighbourhood retract of I^∞, where

$$I^\infty = \sqcup_{n\in\mathbb{N}} [\frac{-1}{n}, \frac{1}{n}]$$

is the infinite Hilbert cube with the metric $d(x_n, y_n) = (\sum |x_n{}^2 - y_n{}^2|)^{1/2}$. I^∞ itself is an ANR by definition. Since $i : \mathbb{D}^n \to I^\infty$ defined by $(x_1, ...x_n) \mapsto (x_1, x_2/2, ...x_n/n, 0, 0....)$ is an imbedding such that $i(\mathbb{D}^n)$ is a retraction of I^∞, the n-cell $\mathbb{D}^n$ is an ANR. Any locally finite polyhedron is an ANR. Open subsets of an ANR and a neighbourhood retract of an ANR are also an ANR.

Throughout this note we will assume X is a connected compact ANR. Note that any compact ANR has only countably many connected components with each is open and an ANR. For a compact ANR, the homology $H_*(X, \mathbb{F})$ is a finitary graded $\mathbb{F}$-vector space and any map $f : X \to X$ induces a morphism $f_* : H_*(X, \mathbb{F}) \to H_*(X, \mathbb{F})$ between the homology groups which is a morphism of finitary graded vector space over $\mathbb{F}$.

Definition 1. *Let $f : X \to X$ be a map on X, $\mathbb{F}$ be a field. The Lefschetz number of f over $\mathbb{F}$ is defined to be the number:*

$$L(f, \mathbb{F}) := \sum (-1)^q Tr(f_q)$$

We denote $L(f) = L(f, \mathbb{Q})$.

We state the Lefschets fixed point theorem without proof.

Theorem 1 (Lefschetz Fixed Point Theorem ([1]))**.** *If X is a compact ANR and $f : X \to X$ is a map such that $L(f, \mathbb{F}) \neq 0$ for some field $\mathbb{F}$, then every map homotopic to f has a fixed point.*

For any field $\mathbb{F}$ the homology and cohomology are isomorphic and the induced morphism is the transpose of f_* so we can define the Lefschetz number using the cohomolgy too.

Observe that: (1) for the identity map 1_X of X, the Lefschetz number

$$L(1_X) = \sum_q (-1)^q Tr(1_q) = \sum (-1)^q dim(H_q(X, \mathbf{q})) = \sum (-1)^q b_q = \chi(X)$$

where $\chi(X)$ is the Euler characteristic and b_q is the q^{th} Betti number of X. (2) Since homotopic maps induce the same homomorphism on the homology groups, if $f : X \to X$ any continuous map and if g is homotopic to f then $L(g, \mathbb{F}) = L(f, \mathbb{F})$ for all field $\mathbb{F}$.

A space X is said to have the *fixed point property* if every continuous self map on X has a fixed point. Thus a contractible compact ANR X has fixed point property since $H_q(X, \mathbb{Q}) = 0, \forall q \neq 0$ and $H_0 = \mathbb{Q}$, thus $L(f) = 1$ (since every map on a path connected space induces the identity morphism on H_0) implies f has a fixed point by the Lefschetz fixed point theorem. For $X = \mathbb{S}^n$, the $n-$sphere, $\chi(X) = 0$ whenever n is odd. Thus the converse of the traditional fixed point theorem is false. Brouwer Fixed point theorem is an immediate consequence of the Lefschetz fixed point theorme, for; let $f : \mathbb{D}^n \to \mathbb{D}^n$ is any map. Since I^∞ is a contractible compact ANR and since retract of a space with fixed point property has the fixed point property, f has a fixed point.

3 Index for ANRs

For X, a compact ANR, a map $f : X \to X$, and an open set U of X without fixed points of f on its boundary it is possible to associate a number $i(X, f, U)$, the index of f on U. We define the index for such triples.

3.1 The axioms for an index

Let $\mathcal{C}_A$ denote the collection of all connected compact ANR spaces X where the Lefschetz number is defined since $H_*(X, \mathbf{q})$ is finitary. We define index for triples $i(X, f, U)$ with the following propeties:

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1. $X \in \mathcal{C}_A$,
2. $f : X \to X$ is a map,
3. U is open in X,
4. there are no fixed points of f on the boundary of U.

The collection of such triples (X, f, U) is denoted by $\mathcal{C}$. Observe that $(X, f, X), (X, f, \phi)$ satisfy these properties.
A (*fixed point*) *index* on $\mathcal{C}_A$ is a function $i : \mathcal{C} \to \mathbb{Q}$ which satisfies the following axioms:

1. (Localization). If $(X, f, U) \in \mathcal{C}$ and $g : X \to X$ is a map such that $g(x) = f(x)$ for all $x \in \bar{U}$ (the closure of U), then
$$i(X, f, U) = i(X, g, U).$$
2. (Homotopy). For $X \in \mathcal{C}_A$ and $H : X \times I \to X$ a homotopy, define $f_t : X \to X$ by $f_t(x) = H(x, t)$. If $(X, f_t, U) \in \mathcal{C}$ for all $t \in I$, then
$$i(X, f_0, U) = i(X, f_1, U).$$
3. (Additivity). If $(X, f, U) \in \mathcal{C}$ and $U_1, ...U_n$ is a set of mutually disjoint open subsets of U such that $f(x) \neq x$ for all
$$x \in U - \bigcup_{j=1}^{n} U_j,$$
then
$$i(X, f, U) = \sum_{j=1}^{n} i(X, f, U_j).$$
4. (Normalization). If $X \in \mathcal{C}_A$ and $f : X \to X$ is a map, then
$$i(X, f, X) = L(f).$$
5. (Commutativity) If $X, Y \in \mathcal{C}_A$ and $f : X \to Y, g : Y \to X$ are maps such that $(X, gf, U) \in \mathcal{C}$, then
$$i(X, gf, U) = i(Y, fg, g^{-1}(U)).$$

The localization axiom 1 obviously makes the definition of the index "local" in the sense that $i(X, f, U)$ is not affected by the behaviuor of f outside of $\bar{U}$. The normalization axiom 4 connects the index to Lefschetz theory. The homotopy and commutativity axioms are generalizations of properties of the Lefschetz number.

Lemma 1. *If there is an index i on $\mathcal{C}_A$ and if $(X, f, U) \in \mathcal{C}$, such that $i(X, f, U) \neq 0$, then f has a fixed point in U.*

Proof. Note that $i(X, f, \phi) = i(X, f, \phi) + i(X, f, \phi)$ by additivity 3 ($U = U_1 = U_2 = \phi$). Thus $i(X, f, \phi) = 0$ since it is rational. Suppose $f(x) \neq x$ on U. Then we can apply additivity 3 for the given open set U and $U_1 = \phi$, and we get $i(X, f, U) = i(X, f, \phi) = 0$. Which is a contradiction. Hence f has a fixed point in U.

Lemma 2. *Assume there is an index on $\mathcal{C}$ and if $X \in \mathcal{C}_A, f : X \to X$ a map such that $L(f) \neq 0$ then every map homotopic to f has a fixed point.*

Proof. Let g be any map homotopic to f; then $L(g) = L(f) \neq 0$. By the normalization axiom, $i(X, g, X) = L(g) \neq 0$, so g has a fixed point in X by 1.

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The last Lemma 1 makes the important point that Index Theory is more powerful than Lefschetz Theory in the sense that the existance of a function on $\mathcal{C}_A$ satisfying just two of the axioms of an index, namely additivity 3 and normalization 4, is enough to imply that the Lefschetz Fixed Point Theorem 1 is true for all maps on spaces in a collection $\mathcal{C}'$ on which an index is defined.

Example 1. *Let X be a compact connected ANR and $(X, f, U) \in \mathcal{C}$ where f is a constant map say $f(x) = x_0,\ \forall x \in X$. Then*

$$i(X, f, U) = \begin{cases} 0 & \text{if } x_0 \notin U \\ 1 & \text{if } x_0 \in U \end{cases}$$

For, if $x_0 \notin U$. Then by additivity 3 for the given U and $U_1 = \phi$, $i(X, f, U) = i(X, f, \phi) = 0$. Now suppose $x_0 \in U$. Let $Y = \{x_0\}$ the singleton space and $g : X \to Y, h : Y \to X$ be the maps $x \mapsto x_0, h = 1_Y$ respectively. Then $i(X, f, U) = i(X, hg, U) = i(Y, gh, Y) = L(gh) = L(1_Y)$ by the commutativity 5 and normalization 4 axioms. Since any map f on a path connected space induces the identity on the homology group H_0 and since Y is path connected and higher homology groups are trivial, $L(1_Y) = 1$. Hence $i(X, f, U) = 1$ in this case. The following theorem tells us that such an index exists on $\mathcal{C}_A$. Details can be seen in chapters IV and V of [1].

Theorem 2. *For the collection $\mathcal{C}_A$ of all connected compact ANR, there is a unique index defined on it satisfying all the five axioms.*

Now we are ready to define an index on the Nielsen classes of a map $f : X \to X$.

4 The Nielsen Number

For X, a compact ANR, and a map $f : X \to X$, we shall define a non-negative integer $N(f)$, called the *Nielsen number* of f. The Nielsen number is a lower bound for the number of fixed points of f.

4.1 Nielsen Classes

Assume that the set $Fixf$ of all fixed points of f is non-empty. Two points $x_0, x_1 \in Fixf$ are f-equivalent if there is a path $c : I \to X$ from x_0 to x_1 such that c and $f \circ c$ are homotopic with respect to the end points. This relation defines an equivalence relation on $Fixf$. The equivalence classes are called *Nielsen classes* or *fixed point classes* of f. It is known that the set of Nielsen classes of a map f on a connected, compact, ANR X is finite.

Theorem 3. *A map $f : X \to X$ on a compact ANR has only finitely many Nielsen classes.*

Hence we will denote the set of fixed point classes by $\mathcal{N}(f) := \{F_1, F_2, ..., F_n\}$.

4.2 Nielsen Number

Let $f : X \to X$ be a map with fixed point classes $F_1, ..., F_n$. Then for each $j = 1, .., n$, there is an open set $U_j \subset X$ such that $F_j \subset U_j$ and $\bar{U}_j \cap Fixf = F_j$. Let i be the index on $\mathcal{C}_A$. Note that $(X, f, U_j) \in \mathcal{C}$. Define the *index* $i(F_j)$ of the fixed point class F_j by $i(F_j) = i(X, f, U_j)$. This definition is independent of the choice of the open set $U_j \subseteq X$ such that $F_j \subseteq U_j$ and $\bar{U}_j \cap Fixf = F_j$ because, suppose U, V are two such open sets. If $x \in U - U \cap V$ then since x belongs to $U, x \notin F_k$ for $k \neq j$, while $x \notin V$, so $x \notin F_j$. Thus $x \notin Fixf$. By the additivity axiom, $i(X, f, U) = i(X, f, U \cap V)$. The same reasoning implies $i(X, f, V) = i(X, f, U \cap V)$. A fixed point class F of f is said to be *essential* if $i(F) \neq 0$ and *inessential* otherwise. The *Nielsen number* $N(f)$ of the map f is defined to be the number of fixed point classes of f that are essential.
A fixed point theorem with this number is that:

Theorem 4. *Any continuous map f on a connected compact ANR has at least $N(f)$ fixed points.*

Theorem 5. *Let $f,\ g : X \to X$ be homotopic maps; then $N(f) = N(g)$.*

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Thus any continuous function g homotopic to f has at least $N(f)$ fixed points.
The example 1 is an example with only one fixed point class and it is essential, i.e, $N(f) = 1$.

Computing $N(f)$ is difficult in general, in some cases it can be computed via the Reidemeister number $R(f)$ by knowing $L(f)$ and f_π , the induced homomorphism on the fundamental group of X. The main tool to compute the Nielsen number is the Jiang subgroup of the fundamental group. Before going to the computation method of Nielsen number we will see how it is related to the Reidemeister number $R(f)$.

5 The Reidemeister Number

Let $f : X \to X$ be a map on a connected compact ANR and let $Fixf = \{x \in X : f(x) = x\}$. Let $p : \widetilde{X} \to X$ be the universal covering of X and $\tilde{f} : \widetilde{X} \to \widetilde{X}$ be a lifting of f, ie. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called conjugate if there is a $\gamma \in \Gamma = \pi_1(X)$ such that $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$. Note that if $\tilde{f}$ is a lift of f and $\gamma \in \Gamma$, then $p(Fix\tilde{f}) = p(Fix\gamma\tilde{f}\gamma^{-1})$ and $p(Fix\tilde{f}) = p(Fix\tilde{f}')$, then $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$ for some $\gamma \in \Gamma$. This is an equivalence relation on the set $Fixf = \bigsqcup_{[\tilde{f}]} p(Fix\tilde{f})$, ie. $Fixf$ is a disjoint union of projections of fixed points of lifts from distinct lifting classes. The subset $p(Fix\tilde{f}) \subseteq Fixf$ is called the *fixed point class* of f determined by the lifting class $[\tilde{f}]$. Note that for any point $y \in \widetilde{X}$, since $(f \circ p)_\pi(\pi_1(\widetilde{X})) \subset p_\pi(\pi_1(X))$, there is a unique map $\tilde{f}$ such that $p \circ \tilde{f} = f \circ p$ and $\tilde{f}(y) = y$. In particular, any fixed point of f is a projection of a fixed point of some lift $\tilde{f}$ of f.

Now we define the Reidemeister number of a group homomorphism $\varphi : G \to G$. Let $\varphi : G \to G$ be a group homomorphism on a group G. One has an action of G on itself given by $g.x = gx\varphi(g)^{-1}$. Two elements $x, y \in G$ are said to be $\varphi-$ twisted conjugate, $x \sim_\varphi y$, if they are in the same orbit of this action. The orbits are called the φ-twisted conjugacy classes or the *Reidemeister classes* and $R(\varphi)$, the number of Reidemeister classes is called the *Reidemeister number* of φ. If $\varphi = Id$ then $R(\varphi)$ coincides with the number of conjugacy classes of G.

By fixing a lift $\tilde{f}$ of f and $\alpha \in \Gamma \simeq \pi_1(X)$, $\tilde{x} \in \widetilde{X}$, we obtain a unique element $\tilde{\alpha} \in \Gamma$ such that $\tilde{\alpha}\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x}))$. Thus we obtain a homomorphism $\phi : \pi_1(X) \to \pi_1(X)$ such that $\tilde{f}\alpha = \phi(\alpha)\tilde{f}, \forall\alpha \in \Gamma$. Also once we fix a lift $\tilde{f}$ of f, then every lift is of the form $\alpha\tilde{f}$ for some $\alpha \in \Gamma$. Let $\alpha, \beta \in \Gamma$. Then

$$[\alpha\tilde{f}] = [\beta\tilde{f}] \iff \beta\tilde{f} = \gamma\alpha\tilde{f}\gamma^{-1}$$

for some $\gamma \in \Gamma$. ie.

$$\iff \beta\tilde{f} = \gamma\alpha\phi(\gamma^{-1})\tilde{f}$$

By the uniqueness of lifts, we have $[\alpha\tilde{f}] = [\beta\tilde{f}] \iff \beta = \gamma\alpha\phi(\gamma^{-1})$ for some $\gamma \in \Gamma$. Furthermore, by choosing appropriate base point, ϕ can be identified with the induced homomorphism f_π on Γ. From now on, we write $R(f)$ for $R(f_\pi)$. It follows from the above definition that there is a one-one correspondence between the lifting classes of f and the Reidemeister classes of f_π.

Remark 1. *If we choose a different lifting $\tilde{f}'$ and thus a different homomorphism ϕ', we get a bijection between the ϕ-Reidemeister classes and the ϕ'-Reidemeister classes so that the cardianality of such sets is a constant.*

5.1 Relationship with Nielsen number

Suppose $x_1, x_2 \in Fixf$ are in the same Nielsen class, ie. there exists a path $c : I \to X$ from x_1 to x_2 such that $f \circ c$ and c are homotopic relative to the end points. Let $\tilde{f}$ be a lift of f and $\tilde{x}_1 \in Fix\tilde{f}$ such that $p(\tilde{x}_1) = x_1$. Lift c to a path $\tilde{c}$ starting from $\tilde{x}_1$ and ending at some $\tilde{x}_2$ in $\tilde{X}$. Then $\tilde{f} \circ \tilde{c}$ projects onto $f \circ c$ which is homotopic to c. Thus $\tilde{f} \circ \tilde{c}$ also ends at $\tilde{x}_2$. Hence $\tilde{f}(\tilde{x}_2) = \tilde{x}_2$. In otherwords, they belong to the same lifting class. Conversely, let $\tilde{x}_1, \tilde{x}_2 \in Fix\tilde{f}$ such that $p(\tilde{x}_1) = x_1 \neq x_2 = p(\tilde{x}_2)$. Let $\tilde{c} : I \to \widetilde{X}$ be a path from $\tilde{x}_1$ to $\tilde{x}_2$. Then $c = p \circ \tilde{c}$ is a path from x_1 to x_2 in X and $p(\tilde{f} \circ \tilde{c}) = f \circ p \circ \tilde{c} = f \circ c$, ie. $\tilde{f} \circ \tilde{c}$ projects onto $f \circ c$. In fact, the loop $\tilde{c}(\tilde{f} \circ \tilde{c})^{-1}$ projects to the loop $c(f \circ c)^{-1}$. Since $\widetilde{X}$ is simply-connected, the former loop is trivial in $\pi_1(\widetilde{X})$ and thus the later loop is homotopic to the trivial loop, ie. $c \sim f \circ c$. That is x_1 and x_2 are in the same Nielsen class. This shows that there is a one-to-one

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map, say ψ, from the set of Nielsen classes to the set of Reidemeister classes and which implies that $N(f) \leq R(f)$. Note that a lifting class $p(Fix\tilde{f})$ might be empty, but Nielsen classes are non-empty. Also $R(f)$ need not be finite while $N(f)$ is always finite. For example, if $f = 1_X$ then any two points are Nielsen equivalent, thus $N(f) \leq 1$ while $R(f)$ is the number of conjugacy classes in $\pi_1(X)$. In particular, if $\pi_1(X)$ is abelian then $R(f) = |\pi_1(X)|$.

6 Computing Nielsen Number

First, let us consider a simple example: For a simply connected space X, there is only one Nielsen class for any self map f of X, so $N(f) \leq 1$. In this case $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = 1$. $N(f)$ does not give more information than $L(f)$.

The main tool to calculate $N(f)$ is the Jiang subgroup $T(f) \leq \pi_1(X)$ introduced by B. Jiang(1963).

6.1 The Jiang Subgroup

Fix a point x_0 in a compact connected ANR X and a self map f on X. We denote by $Map(X)$ the set of all maps from X to itself with the supremum metric $d(f,g) = sup\{d(f(x), g(x)) \mid x \in X\}$, then it is a complete metric space. Let $p : Map(X) \to X$ be the map given by $p(g) = g(x_0)$. Then p induces a homomorphism $p_\pi : \pi_1(Map(X), f) \to \pi_1(X, f(x_0))$. The Jiang subgroup $T(f, x_0)$ is the image of the homomorphism p_π. Equivalently, an element $\alpha \in \pi_1(X, f(x_0))$ is said to be in the Jiang subgroup $T(f, x_0)$ of f if there is a loop H in $Map(X)$ based at f such that the loop c in X defined by $c(t) = H(t)(x_0)$ is homotopic to α.

Lemma 3. *The Jiang subgroup is independent of the base point, ie.* $T(f, x_0) \simeq T(f, x_1)$ *for any* $x_0, x_1 \in X$.

Theorem 6. *If* f *is such that* $T(f, x_0) \simeq \pi_1(X, x_0)$*. Then all the fixed point classes have the same index. If* $f : X \to X$ *is such that* $T(f, x_0) = \pi_1(X, x_0)$*, then* $L(f) = 0$ *implies* $N(f) = 0$. Proof.*If* $Fixf = \phi$*, then certainly* $N(f) = 0$*. Otherwise, let* $\{F_1, F_2, ..., F_n\}$ *be the different fixed point classes of* f*, and assume* $x_0 \in F_1$ *(Lemma 3). By Theorem 6* $i(F_j) = i(F_1)$ *for every* j*; so, by additivity (3) and normalization (4) axioms,*

$$0 = L(f) = \sum_j i(F_j) = n\ i(F_1)$$

Thus $i(F_1) = 0 \Rightarrow i(F_j) = 0$ *for every* j*, which implies* $N(f) = 0$. □

Lemma 4. *If* f *and* g *are homotopic, then* $T(f, x_0) \simeq T(g, x_0)$.

Lemma 5. $f : X \to X, x_0, x_1 \in X$*. Then there exists a map* $g : X \to X$ *such that both* $f^{-1}(x_0), x_1 \in g^{-1}(x_0)$.

This lemma implies that, given f, x_0 *as above, there is a map* $g \sim f$ *such that* $g(x_0) = x_0$*. Hence we can choose* $x_0 \in Fixf$ *(Lemmas 3, 4, 5). We will drop the base point from the fundamental group and the Jiang subgroup. The Jiang subgroup of the identity map on* X *is denoted by* $T(X)$ *and* $T(f) = T(f, x_0)$.

Theorem 7. *For any map* $f : X \to X$, $T(X) \subseteq T(f)$. Proof.*Let* $\alpha \in T(X) \leq \pi_1(X)$*. Then there is a loop* $[H] \in \pi_1(Map(X), 1_X)$ *based at the identity map such that* $[pH] = \alpha$*. Define a loop* H' *in* $Map(X)$ *(based at* f*) by* $H'(t)(x) = H(t)(f(x))$*. Then, since* $f(x_0) = x_0$*, it follows that* $H'(t)(x_0) = H(t)(x_0)$*, which proves that* $\alpha = [pH] = [pH'] \in T(f)$. □*An ANR is an* H-space *if there is an element* $e \in X$ *and a map* $\mu : X \times X \to X$ *such that* $\mu(x, e) = \mu(e, x) = x, \forall x \in X$*. (The fundamental group of an H-spaces is abelian), (*$\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$ *are the only spheres which are H-spaces). An important property of an H-space is:*

Theorem 8. *If* X *is a H-space, then* $T(X) = \pi_1(X)$. Proof.*We use* e *as the base point. Let* c *be any loop in* X *at* e *and define* $H : [0,1]^a Map(X) by H(t)(x) = \mu(c(t), x)$*. Thus* $[c] \in T(X)$. □

Note that for any H-space, $L(f) = 0$ *implies* $N(f) = 0$.

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Now on we will work with X a connected polyhedron and will fix a triangulation (K, τ) on X. A space X is aspherical *if $\pi_n(X) = 1$, for all $n \geq 2$.*

Theorem 9. *Let X be a connected aspherical polyhedron and $f : X^a X.ThenZ(f_\pi(\pi_1(X))) \subseteq T(f)$.*

Note that, if $f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))$, then $Z(f_\pi(\pi_1(X))) \subseteq \pi_1(X)$. If $f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))$, then $T(f) = \pi_1(X)$. Proof.*$f_\pi(\pi_1(X)) \subseteq Z(\pi_1(X))Z(f_\pi(\pi_1(X))) \subseteq T(f)T(f) = \pi_1(X)$.* □*Now on we assume $L(f) \neq 0$ (then there is at least one essential fixed point class, ie. $L(f) \neq 0N(f) \geq 1$ (by additivity 3), and X a compact ANR. If we apply the equivalence relation of f_π-equivalence (twisted action) to the Jiang subgroup $T(f) \subseteq \pi_1(X)$, then the set of equivalence classes is denoted by $T'(f)$. Let $J(f)$ be the cardinality of $T'(f)$. In other words, $J(f)$ is the number of f_π- twisted classes in $\pi_1(X)$ which contain elements of $T(f)$.*

Theorem 10. *If $\alpha \in T(f)$, then there is an essential fixed point class F of f such that $\psi(F) = [\alpha]$, the Reidemeister class containing α, where ψ is the map from the set of all Nielsen classes to the set of all Reidemeister classes of f discussed in section 5.1. It follows that $J(f) \leq N(f)$. If $T(f) = \pi_1(X)$, then $N(f) = R(f)$.* Proof.*$T(f) = \pi_1(X)$ implies that $J(f) = R(f)$ by definition. We know that $N(f) \leq R(f)$. Now the result follows from theorem 10, it states $J(f) \leq N(f)$.* □

Example 2. *Let $X = \mathbb{S}^1$, the circle, an aspherical H-space with $\pi_1(X) =$. Then $T(X) = T(1_X) = \pi_1(\mathbb{S}^1) =$. If $f : \mathbb{S}^{1a} S^1$ be any map, then $T(X) \subseteq T(f)T(f) = \pi_1(\mathbb{S}^1)$. Now $L(f) = 0 \iff f_\pi$ is the identity isomorphism (since $H_0(\mathbb{S}^1) == H_1$). Thus $L(f) = 0N(f) = 0$. If f_π is not the identity isomorphism, say $f_\pi(1) = q \neq 1$, then $T(f) = \pi_1(\mathbb{S}^1)N(f) = R(f) = \#Coker(1 - f_\pi) = |1 - q|$ since for an abelan group G and a given homomorphism $\phi : G^a G, theReidemeisternumberR(\phi) = \#Coker(1 - \phi)$.*

Example 3. *Let $f : \mathbb{S}^{2a} S^2$ be a rotation by an angle θ. Let $p, n \in \mathbb{S}^2$ be the south and north poles and are the only fixed points of f. Since $\mathbb{S}^2$ is simply connected, there is exactly one Nielsen class F and hence $N(f) \leq 1$. Note that f is homotopic to the identity map. Thus $L(f) = L(1) = \chi(\mathbb{S}^2) = 2 \neq 0N(f) \geq 1$. Hence $N(f) = 1$ and $i(F) = 1$.*

Example 4. *Let $f : \mathbb{S}^{na} S^n$ be the map $f(x) = -x$ for all $x \in \mathbb{S}^n$. Then $L(f) = 1 - deg(f)$, where degree of f is $deg(f) = (-1)^{n+1}$.*

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