EXTREME STATES, OPERATOR SPACES AND TERNARY RINGS OF OPERATORS

A. K. Vijayarajan
Kerala School of Mathematics, Kozhikode 673 571 (India).
E-mail:vijay@ksom.res.in
ORCID:0000-0001-5084-0567

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ABSTRACT

In this survey article on extreme states of operator spaces in $C^{¡}$*-algebras and related ternary ring of operators an extension result for rectangular operator extreme states on operator spaces in ternary rings of operators is discussed. We also observe that in the spacial case of operator spaces in rectangular matrix spaces, rectangular extreme states are conjugates of inclusion or identity maps implemented by isometries or unitaries. A characterization result for operator spaces of matrices for which the inclusion map is an extreme state is deduced using the above mentioned results.*

KEYWORDS

keyword 1, Keyword 2,...

1 INTRODUCTION

Arveson's extension theorem [3, Theorem1.2.3] for completely positive(CP) maps in the context of operator systems in C^*-algebras is a remarkable result in the study of boundary representations of C^*-algebras for operator systems. The theorem asserts that any CP map on an operator system $\mathcal{S}$ in a C^*-algebra $\mathcal{A}$ to $B(\mathcal{H})$ can be extended to a CP map from $\mathcal{A}$ to $B(\mathcal{H})$. A natural non-self-adjoint counterpart of the set up Arveson worked with can be considered to be consisting of operator spaces in ternary rings of operators(TROs) and completely contractive maps on them. An important recent work in this scenario is [10] where rectangular matrix convex sets and boundary representations of operator spaces were introduced. An operator space version of Arveson's conjecture, namely, every operator space is completely normed by it's boundary representations is also established in [10].

Apart from Arveson's fundamental work [3–5], we refer to the work of Douglas Farenick on extremal theory of matrix states on operator systems [8, 9] and Kleski's work on pure completely positive maps and boundary representations for operator systems [12].

In this article we study an extension result for rectangular extreme states on operator spaces in TROs. A characterization result for rectangular extreme state on operator spaces of matrices with trivial commutants is deduced.

2 Preliminaries

In this section we recall the fundamental notions that we require for the discussions later on in this article.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ be the space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{K} = \mathcal{H}$, we denote $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. A (concrete) *operator space* is a closed subspace of the (concrete) C^*-algebra $B(\mathcal{H})$. The space $B(\mathcal{H}, \mathcal{K})$ can be viewed as a subspace of $B(\mathcal{K} \oplus \mathcal{H})$, and hence it is always an operator space which we study more in later parts of this article.

An abstract characterisation of operator spaces was established by Ruan [16]. A subclass of operator spaces called ternary rings of operators referred to as TROs is of special interest to us here. These were shown to be the injective objects in the category of operator spaces and completely contractive linear maps by Ruan [17].

Definition 1. *A ternary ring of operators (TRO) T is a subspace of the C^*-algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ that is closed under the triple product $(x, y, z) \to xy^*z$ for all $x, y, z \in T$.*

A triple morphism between TROs is a linear map that preserves the triple product.

*A triple morphism between TROs can be seen as the top-right corner of a *-homomorphism between the corresponding linking algebras [11]; see also [6, Corollary 8.3.5].*

Clearly, an obvious, but important example of a TRO is the space $B(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces.

Definition 2. *Let X and Y be operator spaces. A linear map $\phi : X \to Y$ is called completely contractive (CC) if the linear map $\mathbb{I}_n \otimes \phi : \mathbb{M}_n \otimes X \to \mathbb{M}_n \otimes Y$ is contractive for all n.*

We denote the set of all CC maps from X to $B(\mathcal{H}, \mathcal{K})$ by $CC(X, B(\mathcal{H}, \mathcal{K}))$.

Definition 3. *A representation of a TRO T is a triple morphism $\phi : T \to B(\mathcal{H}, \mathcal{K})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.*

A representation $\phi : T \to B(\mathcal{H}, \mathcal{K})$ is *irreducible* if, whenever p, q are projections in $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively, such that $q\phi(x) = \phi(x)p$ for every $x \in T$, one has $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$.

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A linear map $\phi : \mathbf{T} \to B(\mathcal{H},\mathcal{K})$ is *nondegenerate* if, whenever p, q are projections in $B(\mathcal{H})$ and $B(\mathcal{K})$, respectively, such that $q\phi(x) = \phi(x)p = 0$ for every $x \in \mathbf{T}$, one has $p = 0$ and $q = 0$.

Definition 4. *A rectangular operator state on an operator space X is a non degenerate linear map $\phi : X \to B(\mathcal{H},\mathcal{K})$ such that $\|\phi\|_{cb} = 1$ where the norm here is the completely bounded norm.*

Several characterizations of nondegenerate and irreducible representations of TROs are obtained in [7, Lemma 3.1.4 and Lemma 3.1.5].

A concrete TRO $\mathbf{T} \subseteq B(\mathcal{H},\mathcal{K})$ is said to act irreducibly if the corresponding inclusion representation is irreducible.

3 systems and spaces

An operator system in a C^*-algebra is a unital selfadjoint closed linear subspace. Let $\mathcal{S}$ be an operator system in a C^*-algebra $\mathcal{A}$, and $\mathcal{B}$ be any other C^*-algebra.

A linear map $\phi : \mathcal{S} \to \mathcal{B}$ is called completely positive (CP) if the linear map $\mathbb{I}_n \otimes \phi : \mathbb{M}_n \otimes \mathcal{S} \to \mathbb{M}_n \otimes \mathcal{B}$ is positive for all natural numbers n.

We denote the set of all unital CP maps from $\mathcal{S}$ to $B(\mathcal{H})$ by $UCP(\mathcal{S}, B(\mathcal{H}))$.

One of the crucial theorems in this context is Arveson's extension theorem [3, Theorem1.2.3]. Arveson's extension theorem asserts that any CP map on an operator system $\mathcal{S}$ to $B(\mathcal{H})$ can be extended to a CP map from the C^*-algebra $\mathcal{A}$ to $B(\mathcal{H})$.

Given an operator space $X \subseteq B(\mathcal{H},\mathcal{K})$, we can assign an operator system $S(X) \subseteq B(\mathcal{K} \oplus \mathcal{H})$, called the *Paulsen system* [15, Chapter 8] which is defined to be the space of operators

$$\left\{ \begin{bmatrix} \lambda I_{\mathcal{K}} & x \\ y^* & \mu I_{\mathcal{H}} \end{bmatrix} : x, y \in X, \lambda, \mu \in \mathbb{C} \right\}$$

where $I_{\mathcal{H}}$, $I_{\mathcal{K}}$ denote the identity operators on $\mathcal{H}$, $\mathcal{K}$ respectively. It is well known that [15, Lemma 8.1] any completely contractive map $\phi : X \to B(\mathcal{H},\mathcal{K})$ on the operator space X extends canonically to a unital completely positive (UCP) map $S(\phi) : S(X) \to B(\mathcal{K} \oplus \mathcal{H})$ defined by

$$S(\phi)\left(\begin{bmatrix} \lambda I_{\mathcal{K}_0} & x \\ y^* & \mu I_{\mathcal{H}_0} \end{bmatrix} \right) = \begin{bmatrix} \lambda I_{\mathcal{K}} & \phi(x) \\ \phi(y)^* & \mu I_{\mathcal{H}} \end{bmatrix}.$$

4 Extreme states and boundary theorems

4.1 Commutant of a TRO and boundary theorem

We introduce the notion of commutant of a rectangular operator set $X \subseteq B(\mathcal{H},\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. In the context of Hilbert C^*-module by Arambašić [1] introduced a similar notion. We can now define commutants of operator spaces in general and TROs in particular. We will prove that in the case of TROs, the commutants satisfy the usual properties with respect to the relevant notions of invariant subspaces and irreducibility of representations and thereby justifying the term. We observe that commutant behaves well with respect to the Paulsen map which is crucial for us. Throughout the artilce unless mentioned otherwise T represents a TRO.

Definition 5. *For $X \subseteq B(H,K)$, the commutant of X is the set in $B(K \oplus H)$ denoted by $X^{\natural}$ and defined by*

$$X^{\natural} = \{A_1 \oplus A_2 \in B(\mathcal{K} \oplus \mathcal{H}) : A_1 \in B(\mathcal{K}) \text{ and } A_2 \in B(\mathcal{H}), A_1 x = x A_2 \text{ and } A_2 x^* = x^* A_1, \ \forall x \in X\}$$

where $(A_1 \oplus A_2)(\eta_1 \oplus \eta_2) = A_1\eta_1 \oplus A_2\eta_2$, $\eta_1 \in K$ and $\eta_2 \in H$.

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Remark 1. *For any non-empty set $X \subseteq B(\mathcal{H},\mathcal{K})$, the commutant $X^{\prime}$ is a von Neumann subalgebra of $B(\mathcal{K} \oplus \mathcal{H})$.*

Definition 6. *Let $\phi : T \to B(\mathcal{H},\mathcal{K})$ be a non-zero representation and $\mathcal{H}_1 \subseteq \mathcal{H}$ and $\mathcal{K}_1 \subseteq \mathcal{K}$ be closed subspaces. We say that the pair $(\mathcal{H}_1,\mathcal{K}_1)$ of subspaces is ϕ-invariant if $\phi(T)H_1 \subseteq K_1$ and $\phi(T)^*K_1 \subseteq H_1$.*

The following two results follow easily from the definitions above.

Lemma 1. *Let $\phi : T \to B(H,K)$ be a non-zero representation. Let p and q be the orthogonal projections on closed subspaces $H_1 \subseteq H$ and $K_1 \subseteq K$ respectively. Then (H_1,K_1) is a ϕ -invariant pair of subspaces if and only if $q \oplus p \in \phi(T)^{\prime}$.*

Corollary 1. *Let $\phi : T \to B(H,K)$ be a representation. Then ϕ is irreducible if and only if ϕ has no ϕ-invariant pair of subspaces other than $(0,0)$ and (H,K).*

The following result illustrate that the term *commutant* used above is an appropriate one.

Proposition 1. *Let $\phi : T \to B(H,K)$ be a non zero representation of a TRO T. Then ϕ is irreducible if and only if $\phi(T)^{\prime} = \mathbb{C}(I_K \oplus I_H)$.*

Proof. Assume ϕ is irreducible. Let $q \oplus p$ be a projection in $\phi(T)^{\prime}$. From the definition of commutant we have $\phi(x)p = q\phi(x)$. By irreducibility of ϕ we have either $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$. Hence $\phi(T)^{\prime} = \mathbb{C}(I_K \oplus I_H)$.

Conversely let $\phi(T)^{\prime} = \mathbb{C}(I_K \oplus I_H)$ and p, q be projections such that $\phi(x)p = q\phi(x)$. Applying adjoint we have $p\phi(x)^* = \phi(x)^*q$. Thus $q \oplus p \in \phi(T)^{\prime}$. By assumption, either $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$. Hence ϕ is irreducible.

The following result shows that commutant behaves well with respect to the Paulsen map.

Lemma 2. *For a rectangular matrix state $\phi : X \to \mathbb{M}_{n,m}(\mathbb{C})$ we have*

$$\phi(X)^{\prime} = [S(\phi)(S(X))]^{\prime}.$$

Proof. To show $\phi(X)^{\prime} \subseteq [S(\phi)(S(X))]^{\prime}$, consider $A_1 \oplus A_2 \in \phi(X)^{\prime}$, then we have

$$\begin{bmatrix} \lambda & \phi(x) \\ \phi(y)^* & \mu \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} \lambda A_1 & \phi(x)A_2 \\ \phi(y)^*A_1 & \mu A_2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda A_1 & A_1\phi(x) \\ A_2\phi(y)^* & \mu A_2 \end{bmatrix}$$
$$= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \lambda & \phi(x) \\ \phi(y)^* & \mu \end{bmatrix}$$

Hence $A_1 \oplus A_2 \in [S(\phi)(S(X))]^{\prime}$.

Conversely, since the Paulsen system contains a copy of scalar matrices, if $A \in [S(\phi)(S(X))]^{\prime}$, then $A = A_1 \oplus A_2$, where $A_1 \in B(K)$ and $A_2 \in B(H)$. Using the commutativity relation of the Paulsen map on the matrices

$$\begin{bmatrix} 0 & \phi(x) \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ \phi(x)^* & 0 \end{bmatrix}$$

we can conclude that $A_1\phi(x) = \phi(x)A_2$ and $A_2\phi(x)^* = \phi(x)^*A_1, \ \forall x \in X$. Hence $A = A_1 \oplus A_2 \in \phi(X)^{\prime}$. Thus $[S(\phi)(S(X))]^{\prime} \subseteq \phi(X)^{\prime}$.

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We prove a version of the rectangular boundary theorem proved in [10, Theorem 2.17] which is an analogue of boundary theorem of Arveson in the context of operator systems. Arveson's boundary theorem [4, Theorem 2.1.1] states that if $\mathcal{S} \subseteq B(\mathcal{H})$ is an operator system which acts irreducibly on $\mathcal{H}$ such that the C^*-algebra $C^*(\mathcal{S})$ contains the algebra of compact operators $\mathcal{K}(\mathcal{H})$, then the identity representation of $C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$ if and only if the quotient map $B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is not completely isometric on $\mathcal{S}$. Our context here is that of an operator space and the generated TRO.

Throughout we assume that X is an operator space, and T is a TRO containing X as a generating set. We say that a rectangular operator state $\phi : X \to B(\mathcal{H}, \mathcal{K})$ has *unique extension property* if any rectangular operator state $\tilde{\phi}$ on T whose restriction to X coincides with ϕ is automatically a triple morphism. Boundary representations for operator spaces were introduced in [10]. For our purposes we consider the following definition which is slightly different from the definition given in [10, Definition 2.7] but we remark that both are the same.

Definition 7. *An irreducible representation $\theta : T \to B(\mathcal{H}, \mathcal{K})$ is a boundary representation for X if θ_X is a rectangular operator state on X and θ_X has unique extension property.*

An exact sequence of TROs induces an exact sequence of the corresponding linking algebras, which is actually an exact sequence of C^*-algebras. The decomposition result for representations of C^*-algebras is well known [5, section 7] and the result for TROs follows from it.

If $0 \to T^0 \to T \to T^1 \to 0$ is an exact sequence of TRO's, then every non degenerate representation

$$\pi : T \to B(\mathcal{H}, \mathcal{K})$$

of T decomposes uniquely into a direct sum of representations $\pi = \pi_T^0 \oplus \pi_T^1$ where π_T^0 is the unique extension to T of a nondegenerate representation of the TRO-ideal [7, Definition 2.2.7] T^0, and where π_T^1 is a nondegenerate representation of T that annihilates T^0. When $\pi = \pi_T^0$ we say that π *lives on* T^0.

With X and T as above let $\mathcal{K}(\mathcal{H}, \mathcal{K})$ denotes the set of compact operators from $\mathcal{H}$ to $\mathcal{K}$. Then $\mathcal{K}_T = T \cap \mathcal{K}(\mathcal{H}, \mathcal{K})$ is a TRO-ideal and we have an obvious exact sequence of TRO's. Denote $\mathcal{K}_T$ to be the set of all irreducible triple morphisms $\theta : T \to B(\mathcal{H}, \mathcal{K})$ such that θ lives on $\mathcal{K}_T$ and θ_X is a rectangular operator state on X. The following is a boundary theorem in this context.

Theorem 1. *Let $X \subseteq B(\mathcal{H}, \mathcal{K})$ be an operator space such that TRO T generated by X acts irreducibly and $\mathcal{K}_T \neq \{0\}$. Then $\mathcal{K}_T$ contains a boundary representation for X if and only if the quotient map $q : T \to T/\mathcal{K}_T$ is not completely isometric on X.*

Proof. Assume that the quotient map q is not completely isometric on X. If $\mathcal{K}_T$ contains no boundary representations for X, then every boundary representation must annihilate $\mathcal{K}_T$ and consequently it factors through q. By [10, Theorem 2.9], there are sufficiently many boundary representations π_l, $l \in I$, for X so that

$$\|(q(x_{ij}))\| \leq \|(x_{ij})\| = \sup_{l \in I} \|\pi_l(x_{ij})\| \leq \|(q(x_{ij}))\|$$

for every $n \times n$ matrix (x_{ij}) over X and every $n \geq 1$. This is a contradiction.

Conversely, if the quotient map is completely isometric on X, then we show that no $\pi \in \mathcal{K}_T$ can be a boundary representation for X. Let $\pi : T \to B(\mathcal{H}, \mathcal{K})$ be an irreducible representation that lives on $\mathcal{K}_T$ and consider the map $Q : q(T) \to B(\mathcal{K} \oplus \mathcal{H})$ defined by $Q(q(a)) = \pi(a)$. Then Q is well defined and

$$\|Q(q(a))\| = \|\pi(a)\| \leq \|a\| = \|q(a)\|.$$

Similarly $\|Q_n(q(a_{ij}))\| \leq \|(q(a_{ij}))\|$ and hence Q is completely contractive. Then the Paulsen's map $S(Q) : S(q(T)) \to B(\mathcal{K} \oplus \mathcal{H})$ is unital and completely positive. By Arveson's extension theorem $S(Q)$

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extends to a completely positive map $S.Q/ : C^{¡}.S.q.T/// \rightarrow B.\mathcal{K}\ \mathcal{H}/$. Define $\psi : T \rightarrow B.\mathcal{H}, \mathcal{K}/$ via

$$S.Q/\left(\begin{bmatrix} 0 & q.a/ \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} ¡ & \psi.a/ \\ ¡ & ¡ \end{bmatrix}$$

Then clearly ψ is linear. Also for all a Ë T,

$$\psi.a/ = \begin{bmatrix} 0 & \psi.a/ \\ 0 & 0 \end{bmatrix} \leq S.Q/\left(\begin{bmatrix} 0 & q.a/ \\ 0 & 0 \end{bmatrix}\right) \leq S.Q/q.a/ \leq a.$$

Since π Ë $\boldsymbol{\mathcal{K}}_T$ and $\psi_X = \pi_X$ we have $\psi_{cb} = 1$. Also we have $\psi.\boldsymbol{\mathcal{K}}_T/ = 0$ and hence $\pi.\boldsymbol{\mathcal{K}}_T/ = 0$ which is a contradiction to the fact that π lives in $\boldsymbol{\mathcal{K}}_T$.

Remark 2. *The following result is a specific form of rectangular boundary theorem given above. Here we give an independent proof which directly uses the corresponding version of Arveson's boundary theorem and Lemma 2.*

Theorem 2. *Assume that X is an operator space in $\mathbb{M}_{n,m}.\mathbb{C}/$ such that* $\dim.X^{Í}/ = 1$. *If $\phi : X \rightarrow \mathbb{M}_{n,m}.\mathbb{C}/$ is a rectangular matrix state on $\mathbb{M}_{n,m}.\mathbb{C}/$ for which $\phi.x/ = x$, Åx Ë X, then $\phi.a/ = a$, Å a Ë $\mathbb{M}_{n,m}.\mathbb{C}/$.*

Proof. By Lemma 2, $\phi.X/^{Í} = [S.\phi/.S.X//]^{Í}$. Hence $\dim[S.\phi/.S.X//]^{Í} = 1$. Since $\phi.x/ = x$, Åx Ë X, we have $S.\phi/.y/ = y$, Å y Ë $S.X/$. Then by [9, Thoerem 4.2] , $S.\phi/.z/ = z$, Å z Ë $\mathbb{M}_{n+m}$. In particular,

$$\begin{bmatrix} 0 & \phi.a/ \\ 0 & 0 \end{bmatrix} = S.\phi/(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}/ = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix},$$

and hence $\phi.a/ = a$, Å a Ë $\mathbb{M}_{n,m}.\mathbb{C}/$.

4.2 Rectangular operator extreme states

We prove an important extension result in this section. In this section we prove that any rectangular operator extreme state on an operator space in a TRO can be extended to a rectangular operator extreme state on the TRO. Extension results in the same spirit concerning operator systems and UCP maps were proved by Kleski [12]. The following definition which appeared in [10, Definition 2.9] is important for our further discussion.
We begin by defining rectangular operator convex combination, and rectangular operator extreme states.

Definition 8. *Suppose that X is an operator space, and $\phi : X \rightarrow B.\mathcal{H}, \mathcal{K}/$ is a completely contractive linear map. A rectangular operator convex combination is an expression $\phi = \alpha_1^{¡}\phi_1\beta_1 + \bigtriangledown + \alpha_n^{¡}\phi_n\beta_n$, where $\beta_i : \mathcal{H} \rightarrow \mathcal{H}_{\rangle}$ and $\alpha_i : \mathcal{K} \rightarrow \mathcal{K}_{\rangle}$ are linear maps, and $\phi_i : X \rightarrow B.\mathcal{H}_{\rangle}, \mathcal{K}_{\rangle}/$ are completely contractive linear maps for $i = 1, 2, \bigtriangledown n$ such that $\alpha_1^{¡}\alpha_1 + \bigtriangledown + \alpha_n^{¡}\alpha_n = 1$ and $\beta_1^{¡}\beta_1 + \bigtriangledown + \beta_n^{¡}\beta_n = 1$. Such a rectangular convex combination is proper if α_i, β_i are surjective, and trivial if $\alpha_i^{¡}\alpha_i = \lambda_i 1$, $\beta_i^{¡}\beta_i = \lambda_i 1$, and $\alpha_i^{¡}\phi_i\beta_i = \lambda_i\phi_i$ for some λ_i Ë $[0, 1]$.*

A completely contractive map $\phi : X \rightarrow B.\mathcal{H}, \mathcal{K}/$ is a rectangular operator extreme state if any proper rectangular operator convex combination $\phi = \alpha_1^{¡}\phi_1\beta_1 + \bigtriangledown + \alpha_n^{¡}\phi_n\beta_n$ is trivial.

Rectangular operator states will be referred to as *rectangular matrix states* if the underlying Hilbert spaces are finite dimensional. The following theorem illustrates a relation between linear extreme states and and rectangular operator extreme states.

Theorem 3. *Let X_1 X_2 be operator spaces. If a completely contractive map $\phi : X_2 \rightarrow B.\mathcal{H}, \mathcal{K}/$ is linear extreme in the set $CC.X_2, B.\mathcal{H}, \mathcal{K}//$ of all completely contractive maps from X_2 to $B.\mathcal{H}, \mathcal{K}/$, and ϕ_{X_1} is rectangular operator extreme, then ϕ is a rectangular operator extreme state.*

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Proof. Let $\phi : X_2 \to B.\mathcal{H}, \mathcal{K}/$ be linear extreme and ϕ_{X_1} be rectangular operator extreme. Then $S.\phi/ : S.X_2/ \to B.\mathcal{K}\ \mathcal{H}/$ is linear extreme in $UCP.S.X_2/, B.\mathcal{K}\ \mathcal{H}//$. For, let ψ_1, ψ_2 Ë $UCP.S.X_2/, B.\mathcal{K}\ \mathcal{H}//$ and $0 < t < 1$ be such that

$$S.\phi/ = t\Psi_1 + .1 * t/\Psi_2.$$

Then $S.\phi/ * t\Psi_1$ is UCP, so by ([10], Lemma 1.11) there exists a completely contractive map $\phi_1 : X_2 \to B.\mathcal{H}, \mathcal{K}/$ such that

$$t\Psi_1 \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right) = w^{1\dot{}2} \begin{bmatrix} \lambda I_{\mathcal{K}} & \phi_1.x/ \\ \phi_1.y/^{¡} & \mu I_{\mathcal{H}} \end{bmatrix} w^{1\dot{}2},$$

where

$$w = t\Psi_1 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = t \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & I_{\mathcal{H}} \end{bmatrix}.$$

So

$$t\Psi_1 \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right) = tS.\phi_1/ \left(\begin{bmatrix} \lambda & x \\ y^{¡} & \mu \end{bmatrix} \right).$$

Thus $\Psi_1 = S.\phi_1/$. Similarly $\Psi_2 = S.\phi_2/$ for some cc map $\phi_2 : X_2 \to B.\mathcal{H}, \mathcal{K}/$. Hence

$$\begin{aligned} S.\phi/ &= tS.\phi_1/ + .1 * t/S.\phi_2/ \text{ and therefore} \\ \phi &= t\phi_1 + .1 * t/\phi_2. \end{aligned}$$

Since ϕ is a linear extreme point, we have $\phi = \phi_1 = \phi_2$. Thus

$$\begin{aligned} S.\phi/ &= S.\phi_1/ = S.\phi_2/ \\ S.\phi/ &= \Psi_1 = \Psi_2. \end{aligned}$$

Thus $S.\phi/ : S.X_2/ \to B.\mathcal{K}\ \mathcal{H}/$ is a linear extreme point. Since $S.\phi/_{S.X_1/}$ is pure [10, Proposition 1.12], by [12, Proposition 2.2] $S.\phi/$ is a pure UCP map on $S.X_2/$. Hence ϕ is a rectangular operator extreme state on X_2.

Here we consider the set of bounded operators from X to $B.\mathcal{H}, \mathcal{K}/$ with the weak¡ topology called the bounded weak topology or BW-topology, identifying this set with a dual Banach space. In its relative BW-topology, $CC.X, B.\mathcal{H}, \mathcal{K}//$ is compact (see [3, Section 1.1] or [15, Chapter 7] for details).

Theorem 4. *Let $X_1\ \ X_2$ be operator spaces. Then every rectangular operator extreme state on X_1 has an extension to a rectangular operator extreme state on X_2.*

Proof. Let ϕ Ë $CC.X_1, B.\mathcal{H}, \mathcal{K}//$ be a rectangular operator extreme state and let

$$\mathcal{F} = \hat{}\psi \text{ Ë } CC.X_2, B.\mathcal{H}, \mathcal{K}// : \psi_{X_1} = \phi`.$$

Then clearly $\mathcal{F}$ is linear convex, and BW-compact. We claim that it is a face. For let ψ_1, ψ_2 Ë $CC.X_1, B.\mathcal{H}, \mathcal{K}//$ and $0 < t < 1$ be such that $t\psi_1 + .1 * t/\psi_2$ Ë $\mathcal{F}$.
Then

$$t\psi_{1_{X_1}} + .1 * t/\psi_{2_{X_1}} = \phi$$

$$tS.\psi_1/_{.S.X_1/} + .1 * t/S.\psi_2/_{S.X_1/} = S.\phi/.$$

Since $S.\phi/$ is pure,

$$S.\psi_1/_{S.X_1/} = S.\psi_2/_{S.X_2/} = S.\phi/.$$

$$\psi_{1_{X_1}} = \psi_{2_{X_2}} = \phi.$$

$\Rightarrow \psi_1, \psi_2$ Ë $\mathcal{F}$ and hence $\mathcal{F}$ is a face. Thus $\mathcal{F}$ has a linear extreme point say $\phi^{Í}$ which is a linear extreme point of $CC.X_2, B.\mathcal{H}, \mathcal{K}//$. By Theorem 3, it follows that $\phi^{Í} : X_2 \to B.\mathcal{H}, \mathcal{K}/$ is a rectangular operator extreme state.

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Now, in view of the above results, extension of rectangular operator extreme state from an operator space to the generated TRO is immediate.

Corollary 2. *If X is an operator space and T is the TRO generated by X and containing X, then any rectangular operator extreme state on the operator space X can be extended to a rectangular operator extreme state on the TRO T.*

Proof. Follows directly from the Theorem 4 by taking $X_1 = X$ and $X_2 = T$.

4.3 Rectangular matrix extreme states

Here we take up the special case of finite dimensional rectangular operator states and show that they are isometric or unitary 'conjugates' of the identity map. Here we investigate the relation between matrix extreme states on operator spaces and commutants of images of operator spaces under rectangular matrix extreme states. For operator spaces in rectangular matrix algebras with trivial commutants, we deduce that rectangular matrix extreme states are certain 'conjugates' of the identity state.

Proposition 2. *If $\phi : X \to \mathbb{M}_{n,m}.\mathbb{C}/$ is a rectangular matrix extreme state on the operator space X, then* $\dim .\phi.X/^{\acute{I}}/ = 1$.

Proof. The commutant $\phi.X/^{\acute{I}}$ is a unital ¡-subalgebra of $\mathbb{M}_{n+m}.\mathbb{C}/$ and is therefore the linear span of its projections. Choose any nonzero projection $p\ q$ Ë $\phi.X/^{\acute{I}}$. Then $\phi = q\phi p + .I * q/\phi.I * p/$. Since ϕ is a rectangular matrix extreme point, we have $p^{¡}p = \lambda_1 I$, $q^{¡}q = \lambda_1 I$ and $.I * p/^{¡}.I * p/ = \lambda_2 I$, $.I * q/^{¡}.I * q/ = \lambda_2 I$ and $q\phi p = \lambda_1 \phi$, $.I * q/\phi.I * p/ = \lambda_2 \phi$, for some λ_1, λ_2 Ë $[0, 1]$. Thus $\lambda_1^2 = \lambda_1$ and $\lambda_2^2 = \lambda_2$. This gives $p = I$ and $q = I$. Therefore $\phi.X/^{\acute{I}} = \mathbb{C}I$. Hence $\dim .\phi.X/^{\acute{I}}/ = 1$.

Theorem 5. *Assume that X is an operator space in $\mathbb{M}_{n,m}.\mathbb{C}/$ and that* $\dim .X^{\acute{I}}/ = 1$.

1. *If $\phi : X \to \mathbb{M}_{r,s}.\mathbb{C}/$ is a rectangular matrix extreme state on X, then $r \leq n$, $s \leq m$ and there are isometries $w : \mathbb{C}^r \to \mathbb{C}^n$ and $v : \mathbb{C}^s \to \mathbb{C}^m$ such that $\phi.x/ = w^{¡}xv$, Å x Ë X.*
2. *A rectangular matrix state ϕ on X with values in $\mathbb{M}_{n,m}.\mathbb{C}/$ is rectangular matrix extreme if and only if there exist unitaries v Ë $\mathbb{M}_n.\mathbb{C}/$ and u Ë $\mathbb{M}_m.\mathbb{C}/$ such that $\phi.x/ = v^{¡}xu$, Å x Ë X.*

Proof. (1): Let $X\ \ \mathbb{M}_{n,m}.\mathbb{C}/$, and $\dim .X^{\acute{I}}/ = 1$. Let $\phi : X \to \mathbb{M}_{r,s}.\mathbb{C}/$ be a rectangular matrix extreme state. By Corollary 2, ϕ can be extended to a rectangular extreme state $\Phi : \mathbb{M}_{n,m}.\mathbb{C}/ \to \mathbb{M}_{r,s}.\mathbb{C}/$. Then $S.\Phi/ : S.\mathbb{M}_{n,m}.\mathbb{C}// \to B.\mathbb{C}^r\ \ \mathbb{C}^s/$ is pure. So by [12, Theorem 3.3] there exists a boundary representation $w : \mathbb{M}_{n+m}.\mathbb{C}// \to B.\mathcal{L}/$ for $S.\mathbb{M}_{n,m}.\mathbb{C}//$ and an isometry $u : \mathbb{C}^r\ \ \mathbb{C}^s \to \mathcal{L}$ such that

$$S.\phi/.y/ = u^{¡}w.y/u, \text{ for all } y \text{ Ë } S.\mathbb{M}_{n,m}.\mathbb{C}//.$$

By [10, Proposition 2.8] we can decompose $\mathcal{L}$ as an orthogonal direct sum $\mathcal{L} = K_w\ \ H_w$ in such a way that $w = S.\pi/$ for some irreducible representation $\pi : \mathbb{M}_{n,m}.\mathbb{C}/ \to B.\mathcal{H}_w, \mathcal{K}_w/$.

From the construction of $\mathcal{H}_w$ and $\mathcal{K}_w$, it follows that u maps $\mathbb{C}^r\ 0$ to $\mathcal{K}_w$ and $0\ \mathbb{C}^s$ to $\mathcal{H}_w$. By defining the maps u_1 and u_2 as $u_1.x/ = u.x\ \ 0/$, x Ë $\mathbb{C}^r$ and $u_1.y/ = u.0\ \ y/$, y Ë $\mathbb{C}^s$ we see that $u_1 : \mathbb{C}^r \to \mathcal{K}_w$ and $u_2 : \mathbb{C}^s \to \mathcal{H}_w$ are isometries such that $u = u_1\ u_2$. Then $\Phi.x/ = u_1^{¡}\pi.x/u_2$, Å$x$ Ë $\mathbb{M}_{n,m}.\mathbb{C}/$ and thus Φ is a compression of an irreducible representation of $\mathbb{M}_{n,m}.\mathbb{C}/$. Since every irreducible representation of $\mathbb{M}_{n,m}.\mathbb{C}/$ is unitarily equivalent to the identity representation [7, Lemma 3.2.3] we have that ϕ is a compression of the identity representation. That is, there are isometries $w : \mathbb{C}^r \to \mathbb{C}^n$ and $v : \mathbb{C}^s \to \mathbb{C}^m$ such that

$$\phi.x/ = w^{¡}xv, \text{ Å } x \text{ Ë } X.$$

Since v and w are isometries, we conclude that $r \leq n$ and $s \leq m$.

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(2) Let $\phi : X \to \mathbb{M}_{n,m}(\mathbb{C})$ be a rectangular matrix extreme state. Then by part (a), there are isometries (unitaries in this case) $u \in \mathbb{M}_n(\mathbb{C})$, $v \in \mathbb{M}_m(\mathbb{C})$ such that

$$\phi(x) = v^* x u, \ \forall x \in X.$$

Conversely let $\phi(x) = v^* x u$, $\forall x \in X$ for unitaries u and v then

$$S(\phi)\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}\right) = \begin{bmatrix} u^* & 0 \\ 0 & v^* \end{bmatrix} \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \ \forall \ x, y \ \in \ X.$$

Then by [9, Theorem 4.3] $S(\phi)$ is pure. Hence ϕ is a rectangular matrix extreme state by ([10, Proposition2.12]).

The following result is now an immediate consequence of Proposition 2 and Theorem 5.

Theorem 6. *Let $X \subseteq \mathbb{M}_{n,m}(\mathbb{C})$ be an operator space. Then the inclusion map $i(x) = x, \forall x \in X$, is a rectangular matrix extreme state if and only if $dim(X^{\perp}) = 1$.*

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