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A remarkable property of cycloidal curves

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ABSTRACT

The purpose of this theoretical work is to establish a connection between the most important properties of plane curves: cycloids and sinusoids. For this, a drawing mechanism is considered, which simultaneously draws a sinusoid and two cycloids. Based on the results obtained using this mechanical method of obtaining curves, the following important, previously unknown, theoretical facts are established. Firstly, new in theoretical terms is that the sinusoid is not represented as a graph of a trigonometric function, but as a locus of points equidistant from the current points of two cycloids: an ordinary and another cycloid congruent to the original one, inverted and shifted along the axis by half a period. Secondly, the line passing through the current points of these cycloids is nothing like a normal to the resulting sinusoid. This property greatly simplifies the graphical construction of such a normal. And, finally, a simple trigonometric relationship was established between the angle of rotation of the generating circle and the angle of deviation of the normal from the vertical.

KEY WORDS: angle of rotation of the generating circle; cycloid; cycloidal curves; generating circle; mechanisms for drawing curves; normal to sinusoid; shortened cycloid; sinusoid; tangent to a sinusoid.

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Una propiedad destacable de las curvas cicloidales

RESUMEN

El propósito de este trabajo teórico es establecer una conexión entre las propiedades más importantes de las curvas planas: cicloides y sinusoides. Para esto, se considera un mecanismo de dibujo, que presenta simultáneamente una sinusoide y dos cicloides. Con base en los resultados obtenidos mediante este método mecánico de obtención de curvas, se establecen los siguientes hechos teóricos importantes, previamente desconocidos. En primer lugar, lo nuevo en términos teóricos es que la sinusoide no se representa como un gráfico de una función trigonométrica, sino como un lugar geométrico de puntos equidistantes de los puntos actuales de dos cicloides: una ordinaria y otra cicloide congruente con la original, invertida y desplazada a lo largo del eje por medio punto. En segundo lugar, la línea que pasa por los puntos actuales de estas cicloides no se parece en nada a una normal a la sinusoide resultante. Esta propiedad simplifica enormemente la construcción gráfica de una normal de este tipo. Y, finalmente, se estableció una relación trigonométrica simple entre el ángulo de rotación del círculo generador y el ángulo de desviación de la normal respecto a la vertical.

PALABRAS CLAVE: ángulo de rotación del círculo generador; cicloide; curvas cicloidales; círculo generador; mecanismos para dibujar curvas; normal a sinusoide; cicloide acortado; sinusoide; tangente a una sinusoide.

Introduction

The purpose of this article is to develop new theoretical knowledge regarding cycloidal and sinusoidal curves. On the basis of this theoretical knowledge, it is possible to construct various devices that associate rotational motion with harmonic motion, such as sinus mechanisms, propellers in water with a fish-like working organ, peristaltic pumps with a delicate effect on the pumped liquid, for example, blood, and other similar devices.

The cycloid was first considered by the French mathematician Roberval in 1634. When calculating the area under the cycloid graph, he considered an auxiliary curve formed by the projection of a point on a circle rolling in a straight line onto the vertical diameter of this circle. He carefully and vaguely called this curve the companion of the cycloid. This curve turned out to be an ordinary sinusoid (Gindikin, 2001).

Roberval, as it were, "encrypted" another approach to defining a sinusoid, not as a graph of a harmonic function, but as a kind of cycloidal curves with a general kinematic approach, in the same mathematical terms as for an ordinary cycloid. "The true spirit of geometry means something more: it requires an approach to the study of geometric images not from one, but from different points of view, because only this path leads to complete knowledge" (Litzman, 1960). In the case of a sinusoid, the first approach is a sinusoid as a graph of a sine, and an alternative approach, a sinusoid, is a quasi-cycloidal curve (the main parameters are the radius of the generating circle, its angle of rotation, the distance from a point to the center of the circle, in the case of considering a shortened cycloid). The term "wavelength" (cycloid period) is not used, it is used for reference only. The provisions of Roberval's treatise concerning the cycloid and the auxiliary satellite line (sinusoid) almost unchanged "migrated" into the modern Handbook of Higher Mathematics: "Cycloid and Sinusoid. The locus of the bases of the perpendiculars dropped from the point M of the cycloid to the diameter of the generating circle passing through the fulcrum is a sinusoid with a wavelength $2\pi R$ and amplitude *d*. The axis of this sinusoid coincides with the line of the centers of the cycloid" (Vygodsky, 2006).

> "By the end of the seventeenth century, mathematicians had discovered all the secrets of the cycloid and paid attention to other curves. It has often happened in the history of mathematics that a certain idea or problem will appear at exactly the right time. This was the case with the cycloid. The discoveries of its beautiful geometric and mechanical properties are closely related to the history of analytic geometry and differential calculus. The missions and battles that were fought over them led to significant achievements. No other curve could serve the same purpose" (Martin, 2010).

> "The heroic history of the cycloid ended at the end of the 17th century. It arose so mysteriously in solving a variety of problems that no one doubted that it played a completely exclusive role. The piety before the cycloid held out for a long time, but time passed, and it became clear that it was not connected with the fundamental laws of nature, like, say, conical sections. The problems that led to the cycloid played a huge role in the formation of mechanics and mathematical analysis, but when the magnificent buildings of these sciences were built, it turned out that these problems are private, far from the most important. An instructive historical illusion took place. However, getting acquainted with the instructive history of the cycloid, it is possible to see many fundamental facts from the history of science" (Gindikin, 2001).

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These two quotes kind of summarize the study of the properties of cycloidal curves at the end of the 17th century. However, even in our time, cycloids often become the solution to scientific problems, for example, associated with tsunamis. "The optimal trajectory for two arbitrary points in the ocean, as in the case of one point on the coast, will also be a cycloid passing through these two points" (Shokin et al., 1989).

It should also be noted that all of the above refers to one isolated cycloid, while this article discusses the properties of the mechanism of two cycloids.

Cycloid properties are currently actively used for educational purposes. "Cycloids are a great example of not only the need for parametric equations, but an example of how to integrate and differentiate them; they also require many of the necessary skills and abilities to use these skills to solve problems" (Roidt, 2011). "The fascinatingly presented biographies of great scientists will interest the widest circles of readers, from high school students to adults; those interested in mathematics will enjoy and benefit from getting to know the scientific achievements of the heroes of the book" (Gindikin, 2001).

1. Methodology

Let's set the task: to invent a mechanism for plotting curves that could draw simultaneously such "mechanical curves" as cycloids and sinusoids. Is it possible to build such a mechanism? Mathematics answers in the affirmative. "You can build other hinge mechanisms, at least theoretically, which will draw ellipses, hyperbolas and even any predetermined curve, whatever its degree" (Courant and Robbins, 2001). Such a mechanism for plotting cycloidal curves was found, and it turned out to be extremely simple and informative.

There is no up-to-date information on this area, since the topic has long been considered well-established and quite classical, and the expectation of new works is considered unlikely. A publication similar to this one could have appeared three hundred years ago, and it is surprising that this did not happen earlier. This article provides new knowledge for the uderstanding of cycloidal and sinusoidal curves; the provisions outlined in this article can be extended to other, more complex cycloidal curves (for example, epi- and hypocycloids). Existing new publications on cycloids are often teaching material for students and even for high school students. It should be noted that articles on cycloids are descriptive

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or educational in nature. This work sheds light on the unknown properties of cycloidal curves.

2. Results

Let us refer to Figure 1. A point P_1 located on the circumference of the upper generating disc at a distance R from the center of the disc describes, when rolling without sliding along a straight line (axis X), a well-known ordinary cycloid, which is a series of arches with "points" downward. The curve is periodic, located in the upper positive half-plane; Period (basis of the cycloid) $2\pi R$. Similarly, a point P_2 located on the circumference of the lower generating disk at the same distance R from the center of the disk describes, when rolling without sliding along a straight line (axis X), a congruent, mirrored, inverted cycloid ("points" up), shifted along the axis X by half a period relative to the upper cycloid. This shift is set initially. This cycloid is located already in the lower, negative half-plane. The angle t in radians is a generalized coordinate for both cycloids, since the producing discs roll synchronously without sliding.

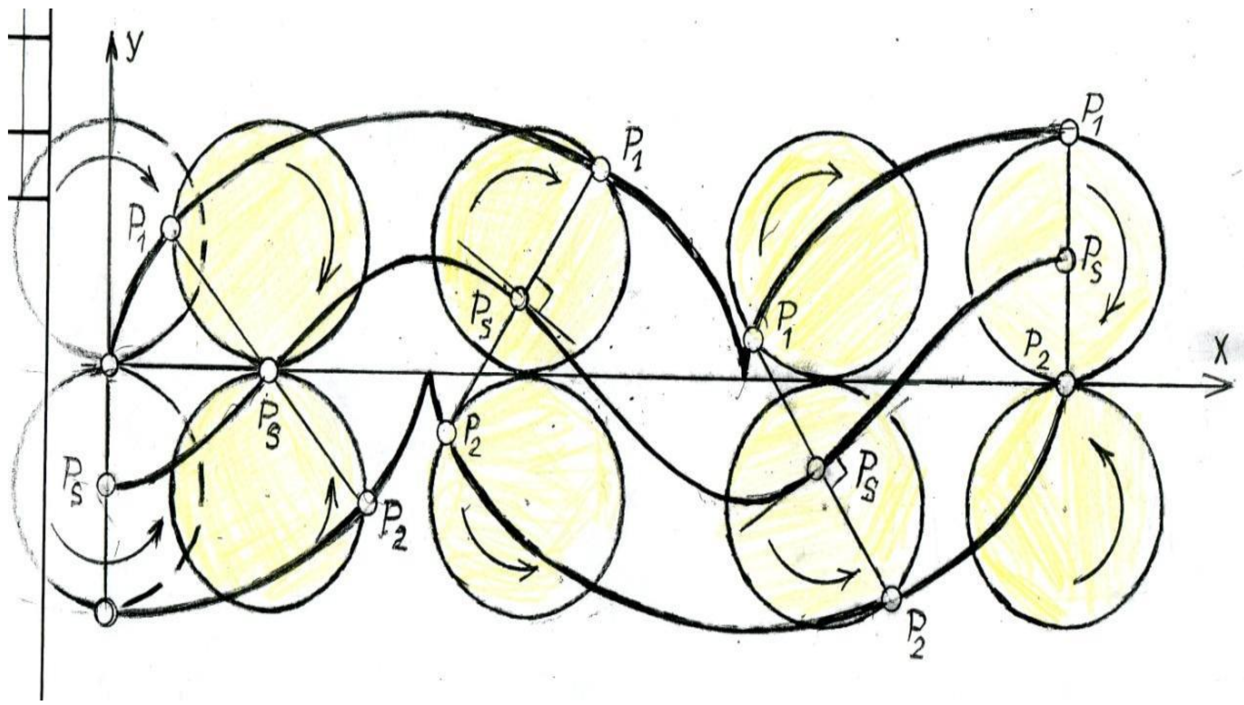

Figure 1. Scheme for plotting cycloidal curves and an ordinary sinusoid

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Let's connect the points P_1 and P_2, find the midpoint P_3, equidistant from the points P_1 and P_2; that is, P_3 is the midpoint of a segment P_1 P_2 of variable length. It will be shown below that the point P_3 draws nothing more than the old familiar Roberval's auxiliary line, the companion of the cycloid – a sinusoid of the same period as the cycloid and with an amplitude equal to the radius of the generating disk.

However, what is most surprising and striking is the fact that the normal to the sinusoid at an arbitrary angle t passes through the same current points P_1 and P_2 the cycloid corresponding to the same value of the generalized coordinate (angle t). A complete and generalized proof for the case of an arbitrary value of the ratio r/R is given below.

Thus, a cycloid, as a cycloidal curve and a sinusoid, as a kind of quasi - cycloidal curve (the point P_3 is not located on the extension of the radius of the generating circle, but is constructed in some way) are "equalized in rights" so that they can be considered not only "fellow travelers" but also "relatives", but rather even "Siamese twins" of geometry. The properties of these curves can be considered from a general point of view and are determined by the radius of the generating circle, its angle of rotation and the distance from a point to its center (for shortened cycloids), without involving the concepts of a cycloid basis and wavelength for a sinusoid.

Let two generating circles of radius R (Figure 1) roll synchronously along a direct straight line $y = 0$ (axis X) without sliding in the positive direction of the axis X. The condition of synchronicity means the presence of a common point of contact of the circles with the guide at any moment of rolling, i.e. the angles of rotation of the circles are always equal. One circle rolls over the "positive" side of the base (top) ($y \geq 0$), the other under the "bottom" ($y \leq 0$).

The starting points of the upper and lower cycloid are chosen in such a way that the lower inverted arch of the cycloid is offset along the guide by half the base. Then the following theorem holds, which consists of two points.

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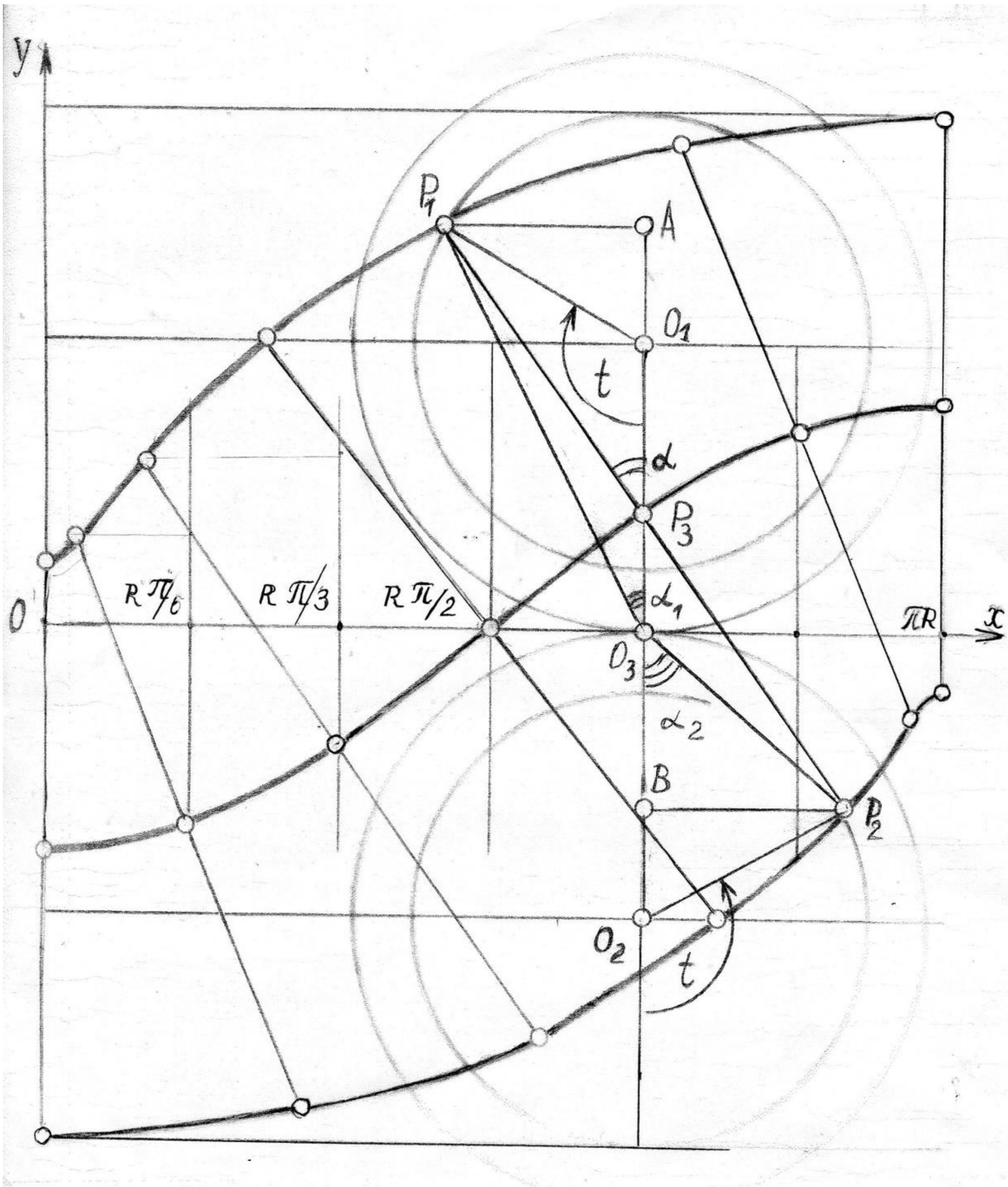

Figure 2. Scheme for obtaining harmonic oscillations (sinusoids) using two shortened cycloids

Taldykin's theorem:

1. The locus of the points of the midpoints of the segments connecting the current points of two congruent cycloids, mirrored relative to the directing line and shifted (displaced) along the half-base (half-period) guide is a sinusoid of the same period as the cycloids, with an amplitude equal to the distance from the point describing the cycloid to the center of the generating circle.

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2. The normal to such a sinusoid passes through the current points of the upper and lower cycloid.

Evidence. Let us consider a more general case when the points P_1 and P_2 during synchronous rolling of the generating circles along the guide (x-axis) without sliding describe shortened cycloids. Parametric equations of these cycloids

for P_1: $x_1 = Rt - \varepsilon R\sin t$, $y_1 = R - \varepsilon R\cos t$; (1)

for P_2: $x_2 = Rt + \varepsilon R\sin t$, $y_2 = -R - \varepsilon R\cos t$;

where x_1, y_1, x_2, y_2 are current coordinates of points *P1* and P_2; t is the angle of rotation of each circle in radians (generalized coordinate); R is the radius of each producing circle; r is the distance from the point P to the center of the generating circle; Truncated cycloids are characterized by the ratio $\varepsilon = r/R$; At $\varepsilon = 1$ we get an ordinary cycloid, at $\varepsilon = 0$ is a straight line along which the center of the generating circle moves.

The following properties are valid for such a mechanism of two synchronously rolling producing circles with a common point of contact with the directing line.

1. The point $P_3(t)$ is the middle of the segment P_1P_2 when rolling the generating circles describes a sinusoid with an amplitude εR and period equal to the periods of the cycloid $2\pi R$.

The coordinates of the midpoint P_3 of the segment P_1P_2 are equal to the half-sum of the coordinates of the points P_1 and P_2

$$x_s = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(Rt - \varepsilon R\sin t + Rt + \varepsilon R\sin t) = Rt, \ (2)$$

$$y_s = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}(R - \varepsilon R\cos t - R - \varepsilon R\cos t) = -\varepsilon R\cos t$$

Equations for a point P_3 are parametric equations of a sinusoid:

$x_s = Rt$, $y_s = -\varepsilon R\cos t$ (3)

2. Tangent and normal to a sinusoid. It is known from the course in differential geometry that for a plane smooth curve given in parametric form $x = x(t)$, $y = y(t)$ the equation of the normal to this curve is as follows:

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$y'(Y-y)+x'(X-x)=0$, *(4)*

where X,Y are current coordinates of normal points; x,y are coordinates of the curve point M. Substituting here the parametric equations of the sinusoid *(3)* and the equations of the first derivative of the sinusoid:

$x' = R; y' = \varepsilon R\sin t$; *(5)*

we get

$$Y\varepsilon\sin t+\varepsilon^2 R\cos t\cdot\sin t+X-Rt=0. \quad (6)$$

Thus, on the one hand, the equations of the normal to a sinusoid, given in a parametric form $x=Rt$, $y=-\varepsilon R\cos t$ are as follows

$$Y=-X\frac{1}{\varepsilon\sin t}-R\varepsilon\cos t+R\frac{t}{\varepsilon\sin t}; \quad (7)$$

On the other hand, the equation of a straight line passing through two given points $P_1(x_1;y_1)$, $P_2(x_2;y_2)$ is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}; \quad (8)$$

Substituting the current values for the points here $P_1(x_1;y_1)$, $P_2(x_2;y_2)$ (1) get

$$\frac{y-(R-\varepsilon R\cos t)}{(-R-\varepsilon R\cos t)-(R-\varepsilon R\cos t)}=\frac{x-(Rt-\varepsilon R\sin t)}{(Rt-\varepsilon R\sin t)-(Rt-\varepsilon R\sin t)}. \quad (9)$$

After a number of transformations, equation (9) can be written in the form

$$y=-x\frac{1}{\varepsilon\sin t}-R\varepsilon\cos t+R\frac{t}{\varepsilon\sin t} \quad (10)$$

Comparing the equation of a straight line passing through two given points $P_1(x_1;y_1)$, $P_2(x_2;y_2)$ (10) with the equation of the normal to the sinusoid (1), we see that they are identical. Therefore, we can conclude that the normal to the sinusoid at an arbitrary point $M(x=at;\ y=-\varepsilon R\cos\varphi)$ and the line passing through the points $P_1(t)$ and $P_2(t)$ that draw the cycloids coincide, that is, the normal to the sinusoid is a line passing through the current points of the upper and lower truncated cycloids.

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Basic relations between the angle of rotation of the generating circles (generalized coordinate) and the angle between the normal to the sinusoid and the vertical. It follows directly from Figure 2 that

$$P_1A = P_2B = \varepsilon R \sin\alpha \text{; } (11)$$

$$AO_1 = O_3P = O_2B = \varepsilon R \cos\alpha \text{; } (12)$$

then for a sinusoid:

$$tg\alpha = \frac{P_1A}{AP_3} = \frac{\varepsilon R \sin t}{R} = \varepsilon \sin t \text{; } (13)$$

It is well known that the normal to the cycloid passes through the fulcrum of the generating circle (Gindikin, 2001). Let's draw normals to cycloids, connecting points P_1 and P_2 with a common point O_3 of support of two generating circles. Tangent of the angle between the normal to the cycloid and the vertical for the upper cycloid

$$tg\alpha_1 = \frac{\varepsilon \sin t}{1 + \varepsilon \cos t} \text{; } (14)$$

$$tg\alpha_2 = \frac{\varepsilon \sin t}{1 - \varepsilon \cos t}$$ for the lower cycloid. *(15)*

Conclusion

The theorem on the construction of a flat curve was first formulated in the work: sinusoid using two truncated cycloids, one of which is of the usual form with its points downward, the other is inverted and shifted by half a period, relative to the original one. In this case, a point equidistant from the current points of the cycloid during the rolling of two identical generating circles draws a sinusoid of the same period as the cycloid, and the line passing through the current points of the cycloid is the normal to the constructed sinusoid. An elementary relationship has been established between the angle of rotation of the generating circle and the angle of deviation of the normal to the sinusoid from the vertical.

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