Global stability of the Euler-Bernoulli beams excited by multiplicative white noises

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ABSTRACT

This paper considers the global stability of the Euler-Bernoulli beam excited by multiplicative white noise. Based on the theory of global random attractors, the Hausdorff dimensions of the global random attractors for the system is obtained. According to the relationship between Hausdorff dimensions and global Lyapunov exponents, the global stability of the stochastic beam is derived.

KEYWORDS

Global stochastic stability, Stochastic Euler-Bernoulli beam, Global Lyapunov exponents

1 INTRODUCTION

The dynamics of the beams are important problem in the elastokinetics [1]. Tajik [2] proposed the stability analysis of motion equations of unbalanced spinning pre-twisted beam. Dai [3] investigated the limit point bifurcations and jump of cantilevered microbeams according to Galerkin method and modal truncation. The bifurcation and chaos for transverse motion of axially accelerating viscoelastic beams was studied by Chen [4]. Pellicano [5] analyzed the linear subcritical behavior, bifurcation analysis and stability of a simply supported beam subjected to an axial transport of mass. Based on the direct method of multiple scales, Mao [6] dealt with problems of stability and saddle-node bifurcations for supercritically moving beam. Using Melnikov method [7], Zhang [8, 9] investigated the multi-pulse global bifurcations of a cantilever beam. Zhou [10] studied the chaos and subharmonic bifurcation of a composite laminated buckled beam with a lumped mass. Applying the phase plane and positive position feedback approach, Hamed [11] investigated the stability and bifurcation of the cantilever beam system which carrying an intermediate lumped mass, to name but a few. For more details, one can see refer to Refs [12–15] and the references therein.

Stochastic stability is one of the most important issue in the research areas of stochastic dynamics [16]. It is well known that, when the problem associated with stability are considered, there exists a very useful tool named Lyapunov exponents which can be distinguished as local type and global type [17]. Generally speaking, the solution of system is stability when biggest Lyapunov exponent associated is less than 0. With respect to the investigations on the stability and other dynamics behaviors for the beams by using local Lyapunov exponents, one can refer to the Refs [18–20] and the references therein. Similarly, the global Lyapunov exponents are also the powerful tools in studying the global dynamics for stochastic beams. Unfortunately, calculating the global Lyapunov exponents is not an easy thing, but if we only consider the global stability of the system, we can use the Hausdorff dimension of global attractors associated with system to describe the signs of the biggest global Lyapunov exponent. The method which can be used to get the Hausdorff dimension estimations associated with the global Lyapunov exponents was due to Debussche [21]. Employing this method, together with the support relationship between global random attractors and probability invariant measures proposed by Crauel [22, 23]. Chen et al [24] consider the global dynamics of the Euler-Bernoulli beams with additive white noises. With respect to investigation on the global dynamics of the nonautonomous the Euler-Bernoulli beams by global attractors theory, see Chen et al [25].

Let $D = (0, L)$, this paper consider the Euler-Bernoulli beam equation excited multiplicative white noise in the following form

$$u_{tt} + \overline{\alpha}(u_t) - \Delta^2 u + \left[\beta\|\nabla u\|^2 - p\right](-\Delta)u = \sigma u\dot{W}, \tag{1}$$

with the hinged boundary condition

$$x = 0 : u = u_{xx} = 0; x = L : u = u_{xx} = 0, \tag{2}$$

and the initial value

$$t = \tau : u = u_0, u_t = u_1, \tag{3}$$

where $u = t(t, x), x \in D$ is the lateral displacement of the beam, $\overline{\alpha}(u_t)$ denotes the damping, $\beta > 0, \sigma$ are constants, the negative and positive of $p \in \mathbb{R}$ can show the stretch and compress of the beam. $\|\nabla u\|^2$ denotes the geometry of the beam bending for its elongation. W is the one dimensional two-sided real-valued standard Wiener process, $\sigma u\dot{W}$ represents the multiplicative white noise.

Let $\|u\| \equiv \|u\|_{L^2(D)}$,$\|u\|_s \equiv \|u\|_{H_0^s(D)}$,$(u,v) \equiv (u,v)_{L^2(D)}$,$(u,v)_s = (u,v)_{H_0^s(D)}$, where $H^s(D)$, $H_0^s(D)$, $s \in \mathbb{R}$ are the usual Sobolev Spaces,for more detailed, see [26]. $A = \Delta^2$ with boundary condition (2), then $\mathcal{D}(A) = \{u | u \in H^4(D) \bigcap H_0^1(D), \Delta u = 0\}$, and then A is self-adjoint, positive, unbounded linear operators and $A^{-1} \in \mathscr{L}(L^2(D))$ is compact. then, their eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and the corresponding eigenvalues $\{e_i\}_{i=1}^{\infty}$ form an orthonormal basis in $L^2(D)$. Following the the mechanism in [27] p55, the power of $(-\Delta)^s, s \in \mathbb{R}$ can also be defined,particularly, $\mathcal{D}(A^{\frac{1}{2}}) = H_0^1(D) \bigcap H^2(D)$. Moreover, for any $s_1, s_2 \in \mathbb{R}, s_1 > s_2$, $\mathcal{D}(A^{s_1})$ can be compact imbedding in $\mathcal{D}(A^{s_2})$, and the following holds

$$\|u\|_{s_1} \geq \lambda_1^{\frac{s_1 - s_2}{2}} \|u\|_{s_2}, \quad \forall u \in \mathcal{D}(A^{s_1}). \tag{4}$$

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Suppose $\overline{\alpha}(u_t) = \alpha A^{\frac{1}{2}} u_t$, here $\alpha > 0$ is a constant, $A^{\frac{1}{2}} u_t$ is called the damping with strongly form, then following abstract form of Euler-Bernoulli beam equation excited multiplicative white noises

$$\begin{cases} u_{tt} + \alpha A^{\frac{1}{2}} u_t + Au + \left[\beta \|\nabla u\|^2 - p\right](-\Delta)u = \sigma u \dot{W}, \\ x = 0 : u = u_{xx} = 0; x = L : u = u_{xx} = 0, \\ u(x,\tau) = u_0(x), \partial_t u(x,\tau) = u_1(x). \end{cases} \tag{5}$$

The rest of paper is organized as follows. Preliminaries and main lemmas are listed in Section 2. Section 3 is devoted to derive main results and proof associated. The proofs for the main lemmas are given in Section 4. Finally, the conclusions are presented in section (4).

2 PRELIMINARIES AND MAIN LEMMAS

2.1 PRELIMINARIES

Let $E_1 = \mathcal{D}(A^{\frac{1}{2}}) \times L^2(D), E_2 = \mathcal{D}(A^{\frac{3}{4}}) \times H_0^1(D)$ equipped with Graph norms and the induced inner products, then they are all Hilbert spaces. Let $(X, \|\cdot\|_X)$ be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, P)$ be a probability space. We consider $\Omega = \{\omega \mid \omega(\cdot) \in \mathbb{C}(\mathbb{R}, \mathbb{R}), \omega(0) = 0\}, \mathcal{F}$ is the σ-algebra and P is the Wiener measure. Set a family of measure preserving and ergodic transformations $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(\cdot), \forall\ t \in \mathbb{R}$, Consider the following system

$$\begin{cases} dz + \mu z dt = dW \\ z(-\infty) = 0, \end{cases} \tag{6}$$

and the solution of system (6) is given by

$$z\left(\theta_t\omega\right) := -\mu \int_{-\infty}^{0} e^{\mu\tau} \left(\theta_t\omega\right)(\tau) d\tau. \tag{7}$$

$z(\theta_t\omega)$ is Ornstein-Uhlenbeck process (in Short O-U process). The following results on O-U process belong to Fan [28].

Lemma 1. *The Ornstein-Uhlenbeck process $z\left(\theta_t\omega\right)$ defined in system (7) satisfies*

$$\mathbb{E}\left[\left|z\left(\theta_t\omega\right)\right|\right] = \frac{1}{\sqrt{\pi\mu}},\ \mathbb{E}\left[\left|z\left(\theta_t\omega\right)\right|^2\right] = \frac{1}{2\mu}, \tag{8}$$

and there exists a constant $t_1(\omega) > 0$ satisfying

$$\int_{-t}^{0} \left|z\left(\theta_s\omega\right)\right| ds < \frac{1}{\sqrt{\pi\mu}} t, \int_{-t}^{0} \left|z\left(\theta_e\omega\right)\right|^2 ds < \frac{1}{2\mu} t, \forall\ t \geq t_1, \tag{9}$$

and the mapping $t \mapsto z\left(\theta_t\omega\right)$ grows sublinearly, i.e.

$$\lim_{t \to \pm\infty} \frac{z\left(\theta_t\omega\right)}{t} = 0.$$

Moreover If $\mu \geq 2\beta, \beta > 0$, then

$$\mathbb{E}\left(e^{\beta \int_s^{s+t} |z(\theta_\tau\omega)|^2 d\tau}\right) \leq e^{\frac{\beta t}{\mu}}, \forall\ s \in \mathbb{R}, t \geq 0, \tag{10}$$

when $\mu^3 \geq r^2, r \geq 0$, the following holds

$$\mathbb{E}\left(e^{r \int_s^{s+t} |z(\theta_\tau\omega)| d\tau}\right) \leq e^{\frac{rt}{\sqrt{\mu}}}, \forall\ s \in \mathbb{R}, t \geq 0. \tag{11}$$

The following is Random dynamical system which is due to Aronld [29].

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Definition 1. *The flow* $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ *is a family of measure preserving transformations in probability space, such that* $(t, \omega) \to \theta_t \omega$ *is measurable,* $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ *for all* $s, t \in \mathbb{R}$. *Then* $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}}\right)$ *is called a metric dynamical system.*

Definition 2. *A random dynamical system (RDS) on Polish space* (X, d) *with Borel* σ *algebra* $\mathcal{B}(X)$ *on* $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}}\right)$ *is a measurable mapping*

$$\begin{aligned} \phi :& \mathbb{R}^+ \times \Omega \times X \to X \\ & (t, \omega, x) \mapsto \phi(t, \omega)x \end{aligned}$$

such that $\mathbb{P} - a.s.$

1. $\phi(0, \omega) = id$ *on* X.
2. $\phi(t + s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega)$ *for all* $s, t \in \mathbb{R}^+$.

The theory of global random attractors is as following, one can refer to Crauel and Flandoli [22,30] and Schmalfuss [31].

Definition 3. *A random set* $K(\omega)$ *is said to absorb the set* $B \subset X$ *for a RDS* ϕ, *if* $P - a.s.$ *there exists* $t_B(\omega)$ *such that*

$$\phi(t, \theta_{-t}\omega) B \subset K(\omega), \quad \forall t \geq t_B(\omega).$$

Definition 4. *Let* $\mathcal{B} \subset 2^X$ *is a collection of subsets of* X, *then a closed random set* $\mathcal{A}(\omega)$ *is called random attractor associated with the* RDS ϕ, *if* $\mathbb{P} - a.s.$

1. $\mathcal{A}(\omega)$ *is a random compact set.*
2. $\mathcal{A}(\omega)$ *is invariant i.e.* $\phi(t, \omega), \mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$ *for all* $t \geq 0$.
3. *For every* $B \in \mathcal{B}$,

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \theta_{-t}\omega) B, \mathcal{A}(\omega)) = 0,$$

where dist $(\cdot, \cdot)$ *denotes the Hausdorff semidistance defined by*

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y), A, B \subset X.$$

The following random fixed point is important in investigating the global stability.

Definition 5. *[29, 32] Let* $\varphi(t, \omega)$ *be a RDS,* $a(\omega)$ *is a random set and consists of one point* $(\mathbb{P} - a.s)$. $a(\omega)$ *is called the random fixed point if the following holds*

$$\varphi(t, \omega)a(\omega) = a(\theta_t \omega), \quad \forall t \in \mathbb{R}^+.$$

The coming theorem is very useful to verify the existence of global random attractors in this paper.

Theorem 1. *[24] Suppose* $S_\varepsilon(t, \omega)$ *is a RDS on Polish space* (X, d), *and suppose that* ϕ *possesses an absorbing set in* X *and for any nonrandom bounded set* $B \subset X$, $\lim_{t\to+\infty} S_\varepsilon(t, \theta_{-t}\omega) B$ *is relative compact P-a.s. Then* ϕ *possesses uniqueness random attractors defined by the following*

$$\mathscr{A}(\omega) = \overline{\bigcup_{B\in X} \Lambda_B(\omega)},$$

where union is taken over all bounded $B \subset X$, *and* $\Lambda_B(\omega)$ *given by*

$$\Lambda_B(\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi(t, \theta_{-t}\omega) B}.$$

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Some powerful transformation on system (5) are derived as following. Set $v_1 = u_t$, we get

$$\begin{cases} u_t = v_1, \\ v_{1t} = -\alpha A^{\frac{1}{2}} v_1 - Au - \left[\beta \|\nabla u\|^2 - p\right](-\Delta)u + \sigma u \dot{W}, \\ x = 0 : u = u_x = 0; x = L : u = u_{xx} = 0, \\ u(x,\tau) = u_0(x), v_1(x,\tau) = u_1(x), \end{cases} \tag{12}$$

let $v_2 = v_1 + \varepsilon u$, here $\varepsilon > 0$, we have

$$\frac{d\boldsymbol{U}}{dt} = \boldsymbol{QU} + \boldsymbol{X}_1(\omega, \boldsymbol{U}),\ \boldsymbol{U}_\tau = (u_0, u_1 + \varepsilon u_0)^T, \tag{13}$$

where

$$\boldsymbol{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} u \\ v_2 \end{pmatrix},\ \boldsymbol{Q} = \begin{pmatrix} -\varepsilon I, I \\ -A + \varepsilon\left(\alpha A^{\frac{1}{2}} - \varepsilon\right), -\left(\alpha A^{\frac{1}{2}} - \varepsilon\right) I \end{pmatrix},$$

$$\boldsymbol{X}_1(\omega, \boldsymbol{U}) = \begin{pmatrix} X_{11}(\omega, \boldsymbol{U}) \\ X_{12}(\omega, \boldsymbol{U}) \end{pmatrix} = \begin{pmatrix} 0 \\ -\left[\beta\|\nabla u\|^2 - p\right](-\Delta)u + \sigma u \dot{W} \end{pmatrix}.$$

System (5) and system (13) are equivalent, thus, the dynamical behavior of system (5) can be reflected by system (13).

Set $v = u_t + \varepsilon u - \sigma u z(\theta_t \omega)$, where $z(\theta_t\omega)$ O-U process formulated by (6), it gives that

$$\frac{d\boldsymbol{V}}{dt} = \boldsymbol{QV} + \boldsymbol{X}_2(\theta_t\omega) + \boldsymbol{X}_3(V),\ \boldsymbol{V}_\tau = (u_0, u_1 + \varepsilon u_0 - \sigma u_0 z(\theta_t\omega))^T, \tag{14}$$

where

$$\boldsymbol{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},\ \boldsymbol{X}_3(V) = \begin{pmatrix} X_{31}(V) \\ X_{32}(V) \end{pmatrix} = \begin{pmatrix} 0 \\ -\left[\beta\|\nabla u\|^2 - p\right](-\Delta)u \end{pmatrix},$$

$$\boldsymbol{X}_2(\theta_t\omega) = \begin{pmatrix} X_{21}(\theta_t\omega) \\ X_{22}(\theta_t\omega) \end{pmatrix} = \begin{pmatrix} \sigma u z(\theta_t\omega) \\ \sigma\left(\mu - \alpha A^{\frac{1}{2}} + 2\varepsilon - \sigma z(\theta_t\omega)\right) u z(\theta_t\omega) - \sigma v z(\theta_t\omega) \end{pmatrix}.$$

System (14) is a system with random coefficient, which can be studied ω by ω.

2.2 MAIN LEMMAS

In order to obtain the global stochastic stability of the system (13) based on the global random attractors theory, the first step should be utilized to verify the system (13) can induce a RDS.

Lemma 2. *For any $\tau \in \mathbb{R}$ and initial value $\boldsymbol{V}_\tau \in E_1$, system* (14) *possesses a unique local mild solution $\boldsymbol{V}(t, \tau, \omega; \boldsymbol{V}_\tau) \in C([\tau, \tau + T], E_1), t \in [\tau, \tau + T], \forall\, T > 0$.*

Let $\varphi(t, \tau, \omega)$ be the solution mapping determined by system (14), which means $\boldsymbol{V}(t, \tau, \omega; \boldsymbol{V}_\tau) = \varphi(t, \tau, \omega)\boldsymbol{V}_\tau$, we have $\varphi(t, 0, \omega) = \varphi(0, -t, \theta_t\omega), \forall\, t \geq 0, \forall\, \omega \in \Omega$. Defining

$$S(t, \omega) := \varphi(t, 0, \omega), \quad \forall\, t \geq 0, \forall \omega \in \Omega,$$

gives the RDS associated with system (14), which together with the relationship between $\iota_\varepsilon : (u, v_1)^T \to [y, v_1 - \sigma u z(\theta_t\omega)]^T = (u, v)^T$ implies that system (13) can also generate a RDS $S_\varepsilon(t, \omega)$ with the following from

$$S_\varepsilon(t, \omega) = \iota^{-1}(\theta_t\omega), S(t, \omega)\iota(\omega) : E_1 \to E_1,$$

is the RDS induced by system (13). The following are very important in proof for the existence of the global random attractors for the system (14).

Let

$$p \leq \frac{2}{3}\lambda_1^{\frac{1}{4}},\ 0 < \varepsilon = \min\left\{1, \frac{\alpha\left(8\lambda_1^{\frac{1}{4}} - 12p\right)}{\lambda_1^{\frac{1}{4}}\left(10\lambda_1^{\frac{1}{4}} - 15p + 4\alpha^2\lambda_1^{\frac{1}{4}}\right)}\right\}, \tag{15}$$

then

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Lemma 3. *For any given* $\boldsymbol{U}=[U_1,U_2]^T\in E_1$*, the following holds*

$$(\boldsymbol{QU},\boldsymbol{U})_{E_1}\leq-\frac{\varepsilon}{2}\|\boldsymbol{U}\|_{E_1}^2-\frac{\varepsilon}{4}\|U_2\|^2-\frac{3p\varepsilon}{4}\|A^{\frac{1}{4}}U_1\|^2.$$

Based on the Lemma 3, we have

Lemma 4. *For any given bounded set* $B\in E_1$*, there exists a random variable* $r_1(\omega)>0$ *and* $T_B(\omega)\geq 0$*, for* $\forall\ t\geq T_B(\omega)$*, the following holds,*

$$\|S_\varepsilon(t,\theta_{-t}\omega)B\|_{E_1}\leq r_1(\omega).$$

Furthermore,

$$\mathbb{E}\left(r_1^2(\omega)\right)<\infty.$$

Lemma 4 shows that $S_\varepsilon(t,\omega)$ has a global absorbing set. In order to obtain the existence of the global random attractors for the system $S_\varepsilon(t,\omega)$, besides the existence of the global absorbing set, we need verify that the $S_\varepsilon(t,\omega)$ is asymptotically compact.

To begin with, we decompose the solution $\boldsymbol{V}$ generated by system (13) with the initial value $\boldsymbol{U}_\tau(\omega)=(u_0,u_1+\varepsilon u_0)^T$ into two parts $\boldsymbol{U}=\boldsymbol{U}^a+\boldsymbol{U}^b=(u^a,u_t^a+\varepsilon u^a)^T+\left(u^b,u_t^b+\varepsilon u^b\right)^T$, where $\boldsymbol{U}^a$ solves

$$\begin{cases}\frac{d\boldsymbol{U}^a}{dt}=\boldsymbol{QU}^a+\left(0,\sigma u^a\dot{W}\right)^T\\[2em]\boldsymbol{U}_\tau^a(\omega)=0,\end{cases}\tag{16}$$

and $\boldsymbol{U}^b$ solves

$$\begin{cases}\frac{d\boldsymbol{U}^b}{dt}=\boldsymbol{QU}^b+\left(0,-\left[\beta\|\nabla u\|^2-p\right](-\Delta)u+\sigma u^b\dot{W}\right)^T\\ \boldsymbol{U}_\tau^b=(u_0,u_1+\varepsilon u_0)^T.\end{cases}\tag{17}$$

Split the solution $\boldsymbol{V}$ of system (14) with the initial value $\boldsymbol{V}_\tau(\omega)=(u_0,u_1+\varepsilon u_0-\sigma u_0 z\left(\theta_{-\tau}\omega\right))^T$ into two parts $\boldsymbol{V}=\boldsymbol{V}^a+\boldsymbol{V}^b=(u^a,v^a)^T+\left(u^b,v^b\right)^T=(u^a,u_t^a+\varepsilon u^a)^T+\left(u^b,u_t^b+\varepsilon u^b-\sigma uz\left(\theta_t\omega\right)\right)^T$, where $\boldsymbol{V}^a$ solves

$$\begin{cases}\frac{d\boldsymbol{V}^a}{dt}=\boldsymbol{QV}^a+\left(\sigma u^a z\left(\theta_t\omega\right),\sigma\left(\mu-\alpha A^{\frac{1}{2}}+2\varepsilon-\sigma z\left(\theta_t\omega\right)\right)u^a z\left(\theta_t\omega\right)-\sigma v^a z\left(\theta_t\omega\right)\right)^T\\ \boldsymbol{V}_\tau^a(\omega)=0,\end{cases}\tag{18}$$

and $\boldsymbol{V}^b$ solves

$$\begin{cases}\frac{d\boldsymbol{V}^b}{dt}=\boldsymbol{QV}^b+\left(\sigma u^b z\left(\theta_t\omega\right),\sigma\left(\mu-\alpha A^{\frac{1}{2}}+2\varepsilon-\sigma z\left(\theta_t\omega\right)\right)u^b z\left(\theta_t\omega\right)-\sigma v^b z\left(\theta_t\omega\right)\right)^T\\ \qquad+\left(0,\left[\beta\|\nabla u\|^2-p\right]\Delta u\right)^T\\ \boldsymbol{V}_\tau^b=(u_0,u_1+\varepsilon u_0-\sigma u_0 z\left(\theta_{-\tau}\omega\right))^T.\end{cases}\tag{19}$$

For the solutions of system (18) and (19), we have the following priori estimates respectively.

Lemma 5. *The solution* $\boldsymbol{U}^a$ *of system* (16) *satisfies*

$$\lim_{t\to+\infty}\left\|A^{\frac{1}{4}}\boldsymbol{U}^a\right\|_{E_1}=0.$$

Lemma 6. *There exists* $r_2(\omega)>\infty,T_2(\omega)\geq 0$*, the solutions* $\boldsymbol{U}^b$ *satisfies*

$$\left\|\boldsymbol{U}^b\right\|_{E_1}\leq r_2(\omega),\quad\forall\ t\geq T_2(\omega).$$

Lemma 7. *There exists* $r_3(\omega)>0,T_3(\omega)\geq 0$*, the following holds*

$$\left\|A^{\frac{1}{4}}\boldsymbol{U}^b\right\|_{E_1}\leq r_3(\omega),\quad\forall\ t\geq T_3(\omega).$$

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The variation equations of system (13)

$$\frac{d\widehat{\boldsymbol{U}}}{dt}=\boldsymbol{Q}\widehat{\boldsymbol{U}}+\widehat{\boldsymbol{X}}(\boldsymbol{U})\widehat{\boldsymbol{U}}, \tag{20}$$

where

$$\begin{aligned}\widehat{\boldsymbol{X}}(\boldsymbol{U})&=\begin{pmatrix}\widehat{X}_1(\boldsymbol{U}),\widehat{X}_3(\boldsymbol{U})\\ \widehat{X}_2(\boldsymbol{U}),\widehat{X}_4(\boldsymbol{U})\end{pmatrix}\\ &=\begin{pmatrix}0,0\\ \left[\beta\|\nabla u\|^2-p\right](-\Delta)\cdot+2\beta(\nabla u,\nabla\cdot)(-\Delta)u+\sigma\dot{W},0\end{pmatrix}.\end{aligned}$$

On the other hand

$$\frac{d\widehat{\boldsymbol{V}}}{dt}=\boldsymbol{Q}\widehat{\boldsymbol{V}}+\widehat{\boldsymbol{X}}_2\left(V\right)\widehat{\boldsymbol{V}}+\widehat{\boldsymbol{X}}_3(V)\widehat{\boldsymbol{V}}, \tag{21}$$

here $\widehat{\boldsymbol{V}}=\{\widehat{V}_1,\widehat{V}_2\}$,

$$\begin{aligned}\widehat{\boldsymbol{X}}_2\left((\theta_t\omega)\right)&=\begin{pmatrix}\widehat{X}_{21}((\theta_t\omega)),\widehat{X}_{23}((\theta_t\omega))\\ \widehat{X}_{22}((\theta_t\omega)),\widehat{X}_{24}((\theta_t\omega))\end{pmatrix}\\ &=\begin{pmatrix}\sigma z\left(\theta_t\omega\right),0\\ \sigma\left(\mu-\alpha+2\varepsilon-\sigma z\left(\theta_t\omega\right)\right)z\left(\theta_t\omega\right),-\sigma z\left(\theta_t\omega\right)\end{pmatrix},\\ \widehat{\boldsymbol{X}}_3(V)&=\begin{pmatrix}\widehat{X}_{31}(\boldsymbol{V}),\widehat{X}_{33}(\boldsymbol{V})\\ \widehat{X}_{32}(\boldsymbol{V}),\widehat{X}_{34}(\boldsymbol{V})\end{pmatrix}\\ &=\begin{pmatrix}0,0\\ \left[\beta\|\nabla u\|^2-p\right](-\Delta)\cdot+2\beta(\nabla u,\nabla\cdot)(-\Delta)u,0\end{pmatrix}.\end{aligned}$$

Let $\widehat{\boldsymbol{U}}=\left[\widehat{U}_1,\widehat{U}_2\right]$ be the solution of system (20) with initial $t=0:\widehat{\boldsymbol{U}}^0\in E_1$ and $\widehat{\boldsymbol{V}}=\left[\widehat{V}_1,\widehat{V}_2\right]$ be the solution of system (21) with initial value $t=0:\widehat{\boldsymbol{V}}_0=\boldsymbol{I}\in E_1$.

Let $\boldsymbol{U}^{(1)}=\left[U_1^{(1)},U_2^{(1)}\right],\boldsymbol{U}^{(2)}=\left[U_1^{(2)},U_2^{(2)}\right]$ are two solutions of system (13), then

$$\frac{d\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}}{dt}=\boldsymbol{Q}\left(\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}\right)+\boldsymbol{X}_1\left(\boldsymbol{U}^{(1)}\right)-\boldsymbol{X}_1\left(\boldsymbol{U}^{(2)}\right), \tag{22}$$

Analogously, let $\boldsymbol{V}^{(1)}=\left[V_1^{(1)},V_2^{(1)}\right]=[u_1,v_1],\boldsymbol{V}^{(2)}=\left[V_1^{(2)},V_2^{(2)}\right]=[u_2,v_2]$ are two solutions of system (14) with initial values $\boldsymbol{V}_0^{(1)},\boldsymbol{V}_0^{(2)}$, where $\boldsymbol{V}_0^{(2)}=\boldsymbol{V}_0^{(1)}+\boldsymbol{I},\boldsymbol{I}=[I_1,I_2]\in E_1$, then

$$\begin{aligned}\frac{d\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}}{dt}&=\boldsymbol{Q}\left(\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right)+\boldsymbol{X}_2\left(\boldsymbol{V}^{(1)}\right)-\boldsymbol{X}_2\left(\boldsymbol{V}^{(2)}\right)\\ &+\boldsymbol{X}_3\left(\boldsymbol{V}^{(1)}\right)-\boldsymbol{X}_3\left(\boldsymbol{V}^{(2)}\right).\end{aligned} \tag{23}$$

In addition, set $\boldsymbol{\Gamma}=[\Gamma_1,\Gamma_2]^T=\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}-\widehat{\boldsymbol{U}}$, we have the following two Lemmas.

Lemma 8. *For* $\forall\ u_1,u_2\in\mathcal{D}(A^{\frac{1}{2}})$, *there exist constants* $c_1(\omega),c_2(\omega)$ *in* (51), *such that*

$$\begin{aligned}\left\|\widehat{X}_{32}\left(u_1\right)-\widehat{X}_{32}\left(u_2\right)\right\|_{\mathscr{L}\left(\mathcal{D}(A^{\frac{1}{2}}),L^2(D)\right)}&\le c_1(\omega)\left\|A^{\frac{1}{2}}u_1-A^{\frac{1}{2}}u_2\right\|,\\ \left\|\widehat{X}_{32}\left(u_1\right)\right\|_{\mathscr{L}\left(\mathcal{D}(A^{\frac{1}{2}}),L^2(D)\right)}&\le c_2(\omega).\end{aligned} \tag{24}$$

Furthemnore,

$$\mathbb{E}\left(c_1(\omega)\right)<\infty,\mathbb{E}\left(c_2(\omega)\right)<\infty.$$

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Lemma 9. *Let* $S_\varepsilon(\omega) := S_\varepsilon(t,\omega)$, *then* $S_\varepsilon(\omega)$ *is uniformly quasidifferentiable on* $\mathscr{A}(\omega)$.

The following relationship between Hausdorff dimension and global Lyapunov exponents which will be used to assert the global stochastic stability.

Theorem 2. *[21]Suppose that, for* $k = 1,\ldots,d$, $\sup_{u\in A(\omega)} \omega_k(DS_\varepsilon(\omega,u))$ *is integrable. Then we can generalize the notion of global Lyapunov exponents introduced in Ref. [17] by setting*

$$\Lambda_k = \mathbb{E}\left(\ln \sup_{u\in A(\omega)} \omega_k(DS_\varepsilon(\omega,u))\right) - \mathbb{E}\left(\ln \sup_{u\in A(\omega)} \omega_{k-1}(DS_\varepsilon(\omega,u))\right),$$

for $k \geq 2$ *and*

$$\Lambda_1 = \mathbb{E}\left(\ln \sup_{u\in A(\omega)} \omega_1(DS_\varepsilon(\omega,u))\right).$$

And it is easy to see that there exists $\overline{\omega}_d$ *satisfying* (26)(27) *if and only if*

$$\Lambda_1 + \ldots + \Lambda_d < 0.$$

3 MAIN RESULTS AND PROOFS

By Lemma 4, we get that $S_\varepsilon(t,\omega)$ has a global absorbing set, which along with Lemma 5, Lemma 6 and Lemma 7 gives that $S_\varepsilon(t,\omega)$ is asymptotically compact. Thus, employing the Theorem 1, it is asserted that there exist global random attractors $\mathscr{A}(\omega), \omega \in \Omega$ for system 13.

Based on the results and the estimations on global random attractors, the Hausdorff dimension of the global random attractors $\mathscr{A}(\omega)$ can be got by the method proposed by Debussche [21]. The outline of this method is as follows.

Firstly, it is verified that $S_\varepsilon(\omega)$ is almost surely *uniformly differentiable* on $A(w)$, i.e. for $\forall u \in A(w)$, there exist a linear operator $DS_\varepsilon(\omega,u)$ in $\mathscr{L}(H)$, the space of continuous linear operator from H to H, such that if u and $u+h$ are in $A(w)$:

$$|S_\varepsilon(\omega)(u+h) - S_\varepsilon(\omega)u - DS_\varepsilon(\omega,u)h| \leq K(\omega)|h|^{1+\alpha}, \tag{25}$$

where $K(w) \geq 1, w \in \Omega$ is a random variable, and $\alpha > 0$ is a constant.

Secondly, there exists an integrable random variable $\overline{\omega}_d$, such that

$$\omega_d(DS_\varepsilon(\omega,u)) \leq \overline{\omega}_d(\omega), \quad \forall u \in \mathscr{A}(\omega), \ \ \mathbb{P}-a.s, \tag{26}$$

and

$$\mathbb{E}\left(\ln\left(\overline{\omega}_d\right)\right) < 0. \tag{27}$$

Thirdly, there exists a random variable $\overline{\alpha}_1$ such $\mathbb{P}$ almost surely

$$\overline{\alpha}_1 \geq 1, \ \alpha_1(DS_\varepsilon(\omega,u)) \leq \overline{\alpha}_1(\omega), \quad \forall u \in \mathscr{A}(\omega), \ \ \mathbb{P}-a.s, \tag{28}$$

and

$$\mathbb{E}\left(\ln \overline{\alpha_1}\right) < \infty. \tag{29}$$

Finally,

$$\mathbb{E}(\ln K) < \infty. \tag{30}$$

If the (26),(27),(28)(29) and (30) hold, the Hausdorff dimension of $\mathscr{A}(w)$ is less than d.

The main results in this paper is given in the following Theorem.

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Theorem 3. *Let*

$$d=\min\left\{n\in\mathbb{Z}^{+}\mid \frac{1}{n}\sum_{i=1}^{n}\lambda_i^{-\frac{1}{2}}<\frac{\frac{\varepsilon}{2}-\frac{8}{\varepsilon}\left(\beta\overline{r}^{2}+|p|\right)^{2}-\frac{2M_7}{\sqrt{\pi\mu}}-\frac{2M_8}{2\mu}}{\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}}\right\},$$

when M_7, M_8 are constants described by (56) *stated in Section 4, then the Hausdorff dimension of $\mathscr{A}(\omega)$*

$$d_H(\mathscr{A}(\omega))<d.$$

Obviously, employing and Theorem 2, we have that when

$$\lambda_1^{-\frac{1}{2}}\le\frac{\frac{\varepsilon}{2}-\frac{8}{\varepsilon}\left(\beta\overline{r}^{2}+|p|\right)^{2}-\frac{2M_7}{\sqrt{\pi\mu}}-\frac{2M_8}{2\mu}}{\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}},$$

the Hausdorff dimension of for system (13) *is 0, which indicates that the global random attractors consists of only one random fixed point which is global stability.*

Proof. The solution mapping of system (20) denoted by $DS_\varepsilon(t,\omega,\boldsymbol{U})$, and $DS(t,\omega,\boldsymbol{V})$ signifies the solution mapping of system (21). Moreover, let $DS_\varepsilon(\omega,\boldsymbol{U}):=DS_\varepsilon(1,\omega,\boldsymbol{U})$. By Lemma 9, we attain $S_\varepsilon(\omega)$ is almost surely uniform differentiable, and the conditions (25) and (30) hold. In the light of (90) and (91) stated in the proof of Lemma 9 in Section 4, the (28) and (29) are satisfied. On the other hand, we have

$$\overline{\omega}_n(DS(\omega,\boldsymbol{V}))=\sup_{\substack{\left\|\overline{\boldsymbol{v}}_0^{(i)}\right\|<1\\ i=1,\dots,n}}\exp\left(\frac{1}{t}\int_0^t\operatorname{Tr}(\boldsymbol{Q}+\widehat{\boldsymbol{X}}_2(V)+\widehat{\boldsymbol{X}}_3(V))\circ Q_n(s)ds\right),$$

where $Q_n(s)=Q_n\left(s,\tau,\boldsymbol{V}_\tau;\widehat{\boldsymbol{V}}_1^0,\dots,\widehat{\boldsymbol{V}}_n^0\right)$ is the orthogonal projector from E_1 onto the space spanned by $\widehat{\boldsymbol{V}}_1(t),\dots,\widehat{\boldsymbol{V}}_n(t)$, here $\boldsymbol{V}=S(t,\tau)\boldsymbol{V}_\tau$ and $\widehat{\boldsymbol{V}}_1,\dots,\widehat{\boldsymbol{V}}_n$ are the solution of system (21) with initial values $\widehat{\boldsymbol{V}}^0=\widehat{\boldsymbol{V}}_1^0,\dots,\widehat{\boldsymbol{V}}_n^0$ respectively. For any given time s, $\boldsymbol{V}_i(s)=\{\mu_i(s),\nu_i(\tau)\},i=1,\dots,n$ is an orthonormal basis of $Q_n(s)E_1$, then

$$\operatorname{Tr}(\boldsymbol{Q}+\widehat{\boldsymbol{X}}_2(V)+\widehat{\boldsymbol{X}}_3(V))\circ Q_n(s)=\sum_{i=1}^{n}\left((\boldsymbol{Q}+\widehat{\boldsymbol{X}}_2(V)+\widehat{\boldsymbol{X}}_3(V))\boldsymbol{V}_i(s),\boldsymbol{V}_i(s)\right)_{E_1}.$$

In term of Lemma 3, we get

$$(\boldsymbol{Q}\boldsymbol{V}_i,\boldsymbol{V}_i)_{E_1}\le-\frac{\varepsilon}{2}\|\boldsymbol{V}_i\|_{E_1}^2-\frac{\varepsilon}{4}\|\nu_i\|^2,$$

and

$$\begin{aligned}\left(\widehat{\boldsymbol{X}}_3\boldsymbol{V}_i,\boldsymbol{V}_i\right)_{E_1}&=-\left(\left[\beta\|\nabla u\|^2-p\right](-\Delta)\mu_i,\nu_i\right)-\left(2\beta\left(\nabla u,\nabla\mu_i\right)(-\Delta)u,\nu_i\right)\\&\le\left\|\left[\beta\|\nabla u\|^2-p\right](-\Delta)\mu_i\right\|\|\nu_i\|+\|2\beta\left(\nabla u,\nabla\mu_i\right)(-\Delta)u\|\|\nu_i\|\\&\le|\beta\|\nabla u\|^2-p|\,\|\Delta\mu_i\|\,\|\nu_i\|+\frac{2\beta}{\lambda_1^{\frac{1}{8}}}\|\Delta u\|^2\|\nabla\mu_i\|\,\|\nu_i\|\\&\le\frac{8}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2\|\Delta\mu_i\|^2+\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\|\nabla\mu_i\|^2+\frac{\varepsilon}{4}\|\nu_i\|^2,\end{aligned}\tag{31}$$

moreover, we have

$$\left(\widehat{\boldsymbol{X}}_2\widehat{\boldsymbol{V}},\widehat{\boldsymbol{V}}\right)\le\left(M_7\left|z\left(\theta_t\omega\right)\right|+M_8\left|z\left(\theta_t\omega\right)\right|^2\right)\left(\left\|\widehat{V}_1\right\|_{H^2}^2+\left\|\widehat{V}_2\right\|_{H^2}^2\right),\tag{32}$$

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where M_7, M_8 defined by (57) stated in Section 4. Hence, we can obtain

$$\begin{aligned}
&\operatorname{Tr}(\boldsymbol{Q}+\widehat{\boldsymbol{X}}_2(\boldsymbol{V})+\widehat{\boldsymbol{X}}_3(\boldsymbol{V}))\circ Q_n(s)\\
&\leq -\frac{\varepsilon}{2}\|\boldsymbol{V}_i\|_{E_1}^2+\frac{8}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2\|\Delta\mu_i\|^2+\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\|\nabla\mu_i\|^2\\
&+\left(M_7\left|z\left(\theta_t\omega\right)\right|+M_8\left|z\left(\theta_t\omega\right)\right|^2\right)\left(\left\|\widehat{V}_1\right\|_{H^2}^2+\left\|\widehat{V}_2\right\|_{H^2}^2\right)\\
&\leq -\frac{n\varepsilon}{2}+\frac{8n}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2+\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n\lambda_i^{-\frac{1}{2}}\\
&+2n\left(M_7\left|z\left(\theta_t\omega\right)\right|+M_8\left|z\left(\theta_t\omega\right)\right|^2\right).
\end{aligned}$$

Thus,

$$\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{V}))=\sup_{\substack{\left\|\overline{\boldsymbol{U}}_0^{(i)}\right\|<1\\ i=1,\ldots,n}}\exp\left(\frac{1}{t}\int_0^t\operatorname{Tr}(\boldsymbol{Q}+\widehat{\boldsymbol{X}}_2(V)+\widehat{\boldsymbol{X}}_3(V))\circ Q_n(s)ds\right)$$

$$=\exp\left(\frac{1}{t}\int_0^t-\frac{n\varepsilon}{2}+\frac{8n}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2+\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n\lambda_i^{-\frac{1}{2}}+2n\left(M_7\left|z\left(\theta_t\omega\right)\right|+M_8\left|z\left(\theta_t\omega\right)\right|^2\right)ds\right),$$

which together with (9) gives that $\exists\ T_4(\omega)>0, \forall\ t\geq T_4$,

$$\begin{aligned}
&\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{V}))\\
&\leq \exp\left(-\frac{n\varepsilon}{2}+\frac{8n}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2+\frac{2nM_7}{\sqrt{\pi\mu}}+\frac{2nM_8}{2\mu}+\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n\lambda_i^{-\frac{1}{2}}\right).
\end{aligned}\tag{33}$$

Let

$$T(\theta_t\omega)=\begin{pmatrix}1 & 0\\ \sigma z\left(\theta_t\omega\right) & 1\end{pmatrix}.$$

Clearly, $T(\theta_t\omega)$ is a linear operator from E_1 to itself, then we have

$$DS_\varepsilon(t,\omega)=T(\theta_t\omega)DS(T,\omega).$$

Let O_n be the space spanned by $e_1,\ldots,e_n$ for any $n\in\mathbb{N}$, then the quadratic form $\chi\in O_n\mapsto\|T(\theta_t\omega)\chi\|_{E_1}^2$ is well defined, continuous, and nonnegative on O_n. Let $\alpha_1\geq\cdots\geq\alpha_n$ be the eigenvalues associated with $\chi_1,\chi_2,\ldots,\chi_n\in O_n$ satisfying

$$(T(\theta_t\omega)\chi_i,T(\theta_t\omega)\chi_j)_{E_1}=\alpha_i^2\delta_{ij}.$$

Set $\chi_j=(\xi_j,\eta_j)^T, j=1,2,\ldots,n$, it can be derived

$$\begin{aligned}
\alpha_i^2&=(T(\theta_t\omega)\chi_i,T(\theta_t\omega)\chi_i)_{E_1}\\
&\leq(\xi_j,\xi_j)_{H^2}+(\sigma z\left(\theta_t\omega\right)\xi_j+\eta_j,\sigma z\left(\theta_t\omega\right)\xi_j+\eta_j)\\
&\leq(\xi_j,\xi_j)_{H^2}+(\eta_j,\eta_j)+2\sigma|z\left(\theta_t\omega\right)|\,\|\xi_j\|\,\|\eta_j\|+\sigma^2|z\left(\theta_t\omega\right)|^2(\xi_j,\xi_j)\\
&\leq\|\xi_j\|_{H^2}^2+\|\eta_j\|^2+\frac{1}{\lambda_1^{\frac{1}{2}}}\left(\sigma|z\left(\theta_t\omega\right)|+\sigma^2|z\left(\theta_t\omega\right)|^2\right)\|\xi_j\|_{H^2}^2+\sigma|z\left(\theta_t\omega\right)|\,\|\eta_j\|^2\\
&\leq 1+\frac{\sigma}{\lambda_1^{\frac{1}{2}}}|z\left(\theta_t\omega\right)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z\left(\theta_t\omega\right)|^2+\lambda_1^{-1}+\lambda_1^{-1}\sigma|z\left(\theta_t\omega\right)|.
\end{aligned}$$

So

$$\ln\alpha_i\leq\ln\left(\frac{\sigma}{2\lambda_1^{\frac{1}{2}}}|z\left(\theta_t\omega\right)|+\frac{\sigma}{2\lambda_1^{\frac{1}{2}}}|z\left(\theta_t\omega\right)|^2+\frac{1}{2\lambda_1}+\frac{\sigma}{2\lambda_1}|z\left(\theta_t\omega\right)|\right),$$

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for any $1 \le i \le n$. Since

$$\omega_n(T(\theta_t\omega)) = \alpha_1\alpha_2\ldots\alpha_n,$$

we obtain

$$\ln\omega_n(T(\theta_t\omega)) \le \ln\left(\frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)| + \frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)|^2 + \frac{n}{2\lambda_1} + \frac{n\sigma}{2\lambda_1}|z(\theta_t\omega)|\right), \tag{34}$$

which combine with Lemma 1 gives that

$$\begin{aligned}\mathbb{E}\left(\omega_n(T(\theta_t\omega))\right) &\le \mathbb{E}\left(\frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)| + \frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)|^2 + \frac{n}{2\lambda_1} + \frac{n\sigma}{2\lambda_1}|z(\theta_t\omega)|\right)\\ &\le M_9,\end{aligned} \tag{35}$$

where M_9 given by (58) stated in Section 4. Since (33) and (34), we get that

$$\begin{aligned}\ln\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{U})) &= \ln\overline{\omega}_n(T(\theta_t\omega)) + \ln\overline{\omega}_n(DS(T,\omega))\\ &\le \ln\left(\frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)| + \frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)|^2 + \frac{n}{2\lambda_1} + \frac{n\sigma}{2\lambda_1}|z(\theta_t\omega)|\right)\\ &\quad - \frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}},\end{aligned}$$

then

$$\begin{aligned}\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{U})) \le &\left(\frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)| + \frac{n\sigma}{2\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)|^2 + \frac{n}{2\lambda_1} + \frac{n\sigma}{2\lambda_1}|z(\theta_t\omega)|\right)\\ &\times\exp\left(-\frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\right).\end{aligned}$$

Employing (35), we obtain

$$\begin{aligned}&\mathbb{E}\left(\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{U}))\right)\\ &\le M_9\mathbb{E}\left(\exp\left(-\frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\right)\right).\end{aligned}$$

Let

$$\overline{\omega}_n(\omega) = M_9\exp\left(-\frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\right),$$

then

$$\mathbb{E}\left(\overline{\omega}_n(DS_\varepsilon(\omega,\boldsymbol{U}))\right) \le \mathbb{E}\left(\overline{\omega}_n(\omega)\right), \tag{36}$$

and

$$\begin{aligned}&\mathbb{E}\left(\ln\overline{\omega}_n(\omega)\right) \le \ln M_9\\ &\times\mathbb{E}\left(-\frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\right),\end{aligned} \tag{37}$$

together with Lemma 4 we find

$$\ln M_9 \times \mathbb{E}\left(-\frac{n\varepsilon}{2} + \frac{8n}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2nM_7}{\sqrt{\pi\mu}} + \frac{2nM_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\right) \le +\infty.$$

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Therefore, we get $d_H(\mathscr{A}(\omega)) < d$. Especially, if

$$\lambda_1^{-\frac{1}{2}} \leq \frac{\frac{\varepsilon}{2} - \frac{8}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 - \frac{2M_7}{\sqrt{\pi\mu}} - \frac{2M_8}{2\mu}}{\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}},$$

$d_H(\mathscr{A}(\omega)) = 0$, which merges with the Theorem 2 shows that the largest global Lyapunov exponent of $S_\varepsilon(\omega)$ is

$$\begin{aligned}\lambda_1 =& \mathbb{E}\left(\ln \overline{\omega}_1(\omega)\right) \\ \leq& \mathbb{E}\left(-\frac{\varepsilon}{2} + \frac{8}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 + \frac{2M_7}{\sqrt{\pi\mu}} + \frac{2M_8}{2\mu} + \frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}\sum_{i=1}^{n}\lambda_i^{-\frac{1}{2}}\right) < 0,\end{aligned} \tag{38}$$

thus, we can conclude there exists a random fixed point of system (14) which is global stochastic stability.

4 PROOFS FOR LEMMAS

This section is intended to complete the proofs of Lemmas listed in subsection 2.2.

Proof for Lemma 2: Firstly, we display $\boldsymbol{X}_2(\boldsymbol{U}) + \boldsymbol{X}_3(\boldsymbol{U}) : E_1 \to E_1$ satisfies local Lipschitz condition. Since

$$\begin{aligned}&\left\|\boldsymbol{X}_3(U) - \boldsymbol{X}_3(V) + \boldsymbol{X}_2\left(U, \theta_t\omega\right) - \boldsymbol{X}_2\left(V, \theta_t\omega\right)\right\|_{E_1} \\ &\leq \left\|\boldsymbol{X}_3(U) - \boldsymbol{X}_3(V)\right\|_{E_1} + \left\|\boldsymbol{X}_2\left(U, \theta_t\omega\right) - \boldsymbol{X}_2\left(V, \theta_t\omega\right)\right\|_{E_1},\end{aligned}$$

let $c > 0, \tau \in \mathbb{R}$ are given constant, for $\forall\, \boldsymbol{U}, \boldsymbol{V} \in E_1, \|\boldsymbol{U}\|_{E_1} \leq c, \|\boldsymbol{V}\|_{E_1} \leq c$, combining Lemmas 4 and 7, we get there exists a positive constant $C_1(T, \tau, \omega, c)$ such that

$$\begin{aligned}&\left\|\boldsymbol{X}_3(U) - \boldsymbol{X}_3(V)\right\|_{E_1} \\ =&\left\|\left(\beta\left\|\nabla U_1\right\|^2 - p\right)\left((-\Delta)U_1 - (-\Delta)V_1\right) + \left(\beta\left\|\nabla U_1\right\|^2 - \beta\left\|\nabla V_1\right\|^2\right)(-\Delta)V_1\right\| \\ \leq&\left|\beta\left\|\nabla U_1\right\|^2 - p\right|\left\|\Delta U_1 - \Delta V_1\right\| + \beta\left(\left\|\nabla U_1\right\| + \beta\left\|\nabla V_1\right\|\right)\left\|\Delta V_1\right\|\left\|\Delta U_1 - \Delta V_1\right\| \\ \leq& C_1(T, \tau, \omega, c)\|\mathbf{U} - \mathbf{V}\|_{E_1}.\end{aligned}$$

On the other hand, by Lemma 1, there exists a positive constant $C_2(T, \tau, \omega, c)$ which satisfies

$$\begin{aligned}&\left\|\boldsymbol{X}_2\left(U, \theta_t\omega\right) - \boldsymbol{X}_2\left(V, \theta_t\omega\right)\right\|_{E_1} \\ &= \left|\sigma z\left(\theta_t\omega\right)\right|\left\|U_1 - V_1\right\|_{H^2} + \left|\sigma z\left(\theta_t\omega\right)\right|\left\|V_2 - U_2\right\| \\ &+ \left\|\sigma\left(\mu + 2\varepsilon - \sigma z\left(\theta_t\omega\right)\right) z\left(\theta_t\omega\right)\left(U_1 - V_1\right)\right\| + \left|\sigma\alpha z\left(\theta_t\omega\right)\right|\left\|A^{\frac{1}{2}}V_1 - A^{\frac{1}{2}}U_1\right\| \\ &\leq \left|\sigma z\left(\theta_t\omega\right)\right|\left\|A^{\frac{1}{2}}V_1 - A^{\frac{1}{2}}U_1\right\| + \left|\sigma\alpha z\left(\theta_t\omega\right)\right|\left\|A^{\frac{1}{2}}V_1 - A^{\frac{1}{2}}U_1\right\| \\ &+ \frac{\left|\sigma\left(\mu + 2\varepsilon - \sigma z\left(\theta_t\omega\right)\right) z\left(\theta_t\omega\right)\right|}{\lambda_1^{\frac{1}{2}}}\left\|A^{\frac{1}{2}}V_1 - A^{\frac{1}{2}}U_1\right\| + \left|\sigma z\left(\theta_t\omega\right)\right|\left\|V_2 - U_2\right\| \\ &\leq C_2(T, \tau, \omega, c)\|\boldsymbol{U} - \boldsymbol{V}\|_{E_1},\end{aligned}$$

The indicated above conclude $\boldsymbol{X}_2(\boldsymbol{U}) + \boldsymbol{X}_3(\boldsymbol{U}) : E_1 \to E_1$ satisfies local Lipschitz condition.

The semigroup method (Theorem 2.5.4 in [33]) is employed to achieve the existence and uniqueness of solution for system (14). Based on Lemma 3.5 in Ref. [24] and Lemma 2.2.3 in Ref. [33], we have $\boldsymbol{Q}$ is m-accretive in $\mathcal{E}_1$, then it can induce a linear semigroup of contractions formulated by $e^{\boldsymbol{Q}t}, t \in \mathbb{R}^+$, which together with the assertion that $\boldsymbol{X}_2\left(U, \theta_t\omega\right) + \boldsymbol{X}_3(\boldsymbol{U}) : E_1 \to E_1$ satisfies local Lipschitz condition can guarantee the system (14) possesses a unique local mild solution with the form

$$V\left(t, \tau, \omega; V_\tau\right) = e^{\boldsymbol{Q}(t-\tau)}V_\tau + \int_\tau^t e^{\boldsymbol{Q}(t-s)}\left(\boldsymbol{X}_2\left(\theta_s\omega\right) + \boldsymbol{X}_3(V)(s)\right) ds,$$

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where, $t \geq \tau, t, \tau \in \mathbb{R}$.

Proof for Lemma 3 : Since

$$
\begin{aligned}
&(\boldsymbol{QU}, \boldsymbol{U})_{E_1}\\
&\leq -\varepsilon \|U_1\|_{H^2}^2 + \varepsilon \|U_2\|^2 + \varepsilon\alpha \left\|A^{\frac{1}{2}}U_1\right\| \|U_2\| - \varepsilon^2 \|U_1\| \|U_2\| - \alpha \left\|A^{\frac{1}{4}}U_2\right\|^2\\
&\leq -\varepsilon \|U_1\|_{H^2}^2 + \varepsilon \|U_2\|^2 + \frac{\varepsilon}{2}\left(1 - \frac{3p}{2\lambda_1^{\frac{1}{4}}}\right) \left\|A^{\frac{1}{2}}U_1\right\|^2 + \frac{\varepsilon\alpha^2\lambda_1^{\frac{1}{4}}}{2\lambda_1^{\frac{1}{4}} - 3p}\|U_2\|^2 - \alpha \left\|A^{\frac{1}{4}}U_2\right\|^2\\
&\leq -\frac{\varepsilon}{2} \|U_1\|_{H^2}^2 + \varepsilon \|U_2\|^2 + \frac{\varepsilon\alpha^2\lambda_1^{\frac{1}{4}}}{2\lambda_1^{\frac{1}{4}} - 3p}\|U_2\|^2 - \alpha \left\|A^{\frac{1}{4}}U_2\right\|^2 - \frac{3p\varepsilon}{4}\|A^{\frac{1}{4}}U_1\|^2\\
&\leq -\frac{\varepsilon}{2} \|U_1\|_{H^2}^2 + \left(-\frac{\alpha}{\lambda_1^{\frac{1}{4}}} + \varepsilon + \frac{\varepsilon\alpha^2\lambda_1^{\frac{1}{4}}}{2\lambda_1^{\frac{1}{4}} - 3p}\right)\|U_2\|^2 - \frac{3p\varepsilon}{4}\|A^{\frac{1}{4}}U_1\|^2\\
&\leq -\frac{\varepsilon}{2}\|\boldsymbol{U}\|_{E_1}^2 - \frac{\varepsilon}{4}\|U_2\|^2 - \frac{3p\varepsilon}{4}\|A^{\frac{1}{4}}U_1\|^2,
\end{aligned}
$$

where, p and ε satisfies (15). Thus complete the proof.

Before deriving the other proofs for Lemmas, the following estimations and quantities are introduced

$$
\begin{aligned}
2\left(-\left[\beta\left\|A^{\frac{1}{4}}u\right\|^2 - p\right]A^{\frac{1}{2}}u, v\right) \leq& -\frac{1}{2\beta}\frac{d}{dt}\left[\beta\|A^{\frac{1}{4}}u\|^2 - p\right]^2\\
&+\frac{\varepsilon p^2}{4\beta} + \frac{3p\varepsilon}{2}\|A^{\frac{1}{4}}u\|^2 - \frac{\varepsilon}{4\beta}[\beta\|A^{\frac{1}{4}}u\|^2 - p]^2\\
&+\frac{4\sigma^2|z(\theta_t\omega)|^2}{7\varepsilon\beta}\left[\beta\|A^{\frac{1}{4}}u\|^2 - p\right]^2,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&2\left(\sigma A^{\frac{1}{2}}uz(\theta_t\omega), A^{\frac{1}{2}}u\right) \leq \frac{\varepsilon}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{2\sigma^2|z(\theta_t\omega)|^2}{\varepsilon}\|A^{\frac{1}{2}}u\|^2,\\
&2\left(\sigma\left(\mu + 2\varepsilon - \alpha A^{\frac{1}{2}} - \sigma z(\theta_t\omega)\right)uz(\theta_t\omega) - \sigma vz(\theta_t\omega), v\right)\\
&\leq \frac{2}{\varepsilon}\sigma^2|z(\theta_t\omega)|^2\|v\|^2 + \frac{\varepsilon}{2}\|v\|^2 + \sigma^2|z(\theta_t\omega)|^2\left(\frac{4}{\varepsilon}\|u\|^2 + \frac{\varepsilon}{4}\|v\|^2\right)\\
&+\frac{\varepsilon}{2}\|v\|^2 + 4\sigma^2|z(\theta_t\omega)|^2\left(\frac{(\mu+2\varepsilon)^2}{\varepsilon}\|u\|^2 + \frac{\alpha^2}{\varepsilon}\|A^{\frac{1}{2}}u\|^2\right)
\end{aligned}
\tag{40}
$$

$$
2\left(\sigma A^{\frac{1}{2}}u^a z(\theta_t\omega), A^{\frac{1}{2}}u^a\right) \leq \frac{\varepsilon}{2}\|A^{\frac{1}{2}}u^a\|^2 + \frac{2\sigma^2|z(\theta_t\omega)|^2}{\varepsilon}\|A^{\frac{1}{2}}u^a\|^2,
\tag{41}
$$

$$
\begin{aligned}
&2\left(\sigma(\mu + 2\varepsilon - \alpha A^{\frac{1}{2}} - \sigma z(\theta_t\omega))u^a z(\theta_t\omega) - \sigma v^a z(\theta_t\omega), A^{\frac{1}{2}}v^a\right)\\
&\leq \frac{2}{\varepsilon}\sigma^2|z(\theta_t\omega)|^2\|v^a\|^2 + \frac{\varepsilon}{2}\|v^a\|^2 + \sigma^2|z(\theta_t\omega)|^2\left(\frac{4}{\varepsilon}\|u^a\|^2 + \frac{\varepsilon}{4}\|v^a\|^2\right)\\
&+\frac{\varepsilon}{4}\|v^a\|^2 + 8\sigma^2|z(\theta_t\omega)|^2\left(\frac{(\mu+2\varepsilon)^2}{\varepsilon}\|u^a\|^2 + \frac{\alpha^2}{\varepsilon}\|A^{\frac{1}{2}}u^a\|^2\right),
\end{aligned}
\tag{42}
$$

https://doi.org/10.17993/3ctecno.2023.v12n1e43.386-412

$$
\begin{aligned}
&(-2\left[\beta\|A^{\frac{1}{4}}u\|^2-p\right]A^{\frac{1}{2}}u^a,A^{\frac{1}{2}}v^b)\\
&\leq \frac{3\beta^2}{\lambda_1^{\frac{1}{2}}}\|A^{\frac{1}{2}}u\|^4\left\|A^{\frac{1}{2}}u^a\right\|^2+3p^2\|\boldsymbol{V}^a\|_{E_1}^2\\
&+2\frac{d}{dt}\left\|A^{\frac{1}{2}}u^b\right\|^2+2\varepsilon^2\left\|A^{\frac{1}{2}}u^b\right\|^2+2\sigma^2|z(\theta_t\omega)|^2\left\|A^{\frac{1}{2}}u^b\right\|^2,\\
&-2\left[\beta\|A^{\frac{1}{4}}u\|^2-p\right](A^{\frac{1}{2}}u^b,A^{\frac{1}{2}}v^b)\\
&=-2\left[\beta\|A^{\frac{1}{4}}u\|^2-p\right](A^{\frac{1}{2}}u^b,A^{\frac{1}{2}}u_t+\varepsilon u^b-\sigma uz\left(\theta_t\omega\right))\\
&\leq -\frac{d}{dt}\left[\beta\|A^{\frac{1}{4}}u\|^2-p\right]\|A^{\frac{1}{2}}u^b\|^2+\frac{\beta}{2}\|A^{\frac{1}{2}}u\|^3\|u_t\|,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&2\left(\sigma\left(\mu+2\varepsilon-\alpha A^{\frac{1}{2}}-\sigma z\left(\theta_t\omega\right)\right)u^bz\left(\theta_t\omega\right)-\sigma vz\left(\theta_t\omega\right),A^{\frac{1}{2}}v^b\right)\\
&\leq\frac{\varepsilon}{2}\|A^{\frac{3}{4}}u^b\|^2+\frac{2\sigma^2|z(\theta_t\omega)|^2}{\varepsilon}\|A^{\frac{1}{4}}u^b\|^2+\frac{2}{\varepsilon}\sigma^2|z\left(\theta_t\omega\right)|^2\|A^{\frac{1}{4}}v^b\|^2+\frac{\varepsilon}{2}\|A^{\frac{1}{4}}v^b\|^2\\
&+\frac{\varepsilon}{2}\|A^{\frac{1}{4}}v^b\|^2+4\sigma^2|z(\theta_t\omega)|^2\left(\frac{(\mu+2\varepsilon)^2}{\varepsilon}\|A^{\frac{1}{4}}u^b\|^2+\frac{\alpha^2}{\varepsilon}\|A^{\frac{3}{4}}u^b\|^2\right)\\
&+\sigma^2\left|z\left(\theta_t\omega\right)\right|^2\left(\frac{4}{\varepsilon}\|A^{\frac{1}{4}}u^b\|^2+\frac{\varepsilon}{4}\|A^{\frac{1}{4}}v^b\|^2\right),
\end{aligned}
\tag{44}
$$

and

$$
\begin{aligned}
\frac{\beta}{2}\|A^{\frac{1}{2}}u\|^3\|u_t\| &\leq \frac{\beta}{2}\|\boldsymbol{V}\|_{E_1}^3(\|v\|+\|\varepsilon u\|+\|\sigma uz\left(\theta_t\omega\right)\|)\\
&\leq \frac{\beta}{2}\|\boldsymbol{V}\|_{E_1}^3\left(\|\boldsymbol{V}\|_{E_1}+\varepsilon\|\boldsymbol{V}\|_{E_1}+\sigma|z\left(\theta_t\omega\right)|\|\boldsymbol{V}\|_{E_1}\right)\\
&\leq \left(\frac{\beta+\beta\varepsilon}{2}+\frac{\beta}{2}|\sigma z\left(\theta_t\omega\right)|\right)\|\boldsymbol{V}\|_{E_1}^4,
\end{aligned}
$$

By Lemma 4, Lemma 5 and Lemma7, $\widehat{M}(t,\tau,\omega)$ defined as follow is bounded,

$$
\begin{aligned}
\widehat{M}(t,\tau,\omega)=&\frac{3\beta^2}{\lambda_1^{\frac{1}{2}}}\|A^{\frac{1}{2}}u\|^4\left\|A^{\frac{1}{2}}u^a\right\|^2+3p^2\|\boldsymbol{V}^a\|_{E_1}^2+\frac{\beta}{2}\|A^{\frac{1}{2}}u\|^3\|u_t\|\\
&+\left(2\varepsilon^2+\frac{3p\varepsilon}{2}+2-\varepsilon+2\left|p\right|\right)\|A^{\frac{1}{2}}u^b\|^2<+\infty,
\end{aligned}
\tag{45}
$$

$$
M_0=\max\left\{\frac{8\alpha^2+2}{\varepsilon}+\frac{8(\mu+2\varepsilon)^2+4}{\varepsilon\lambda_1^{\frac{1}{2}}},\frac{2}{\varepsilon}+\frac{\varepsilon}{4}\right\},\tag{46}
$$

$$
M=\max\left\{\frac{8(\mu+2\varepsilon)^2+6}{\varepsilon\lambda_1^{\frac{1}{2}}}+\frac{8\alpha^2}{\varepsilon},\frac{\varepsilon^2+8}{4\varepsilon},\frac{8}{7\varepsilon}\right\},\tag{47}
$$

$$
M_1=\frac{4\beta r_1^2(\omega)+3|p|\lambda_1^{\frac{1}{4}}}{2\lambda_1^{\frac{1}{4}}},\tag{48}
$$

$$
M_2=\frac{3\beta r_1^4(\omega)}{\lambda_1^{\frac{1}{4}}}+2|p|r_1^2(\omega)+\frac{\varepsilon p^2}{4\beta},\tag{49}
$$

$$
M_3=\max\left\{1,\frac{2}{\varepsilon}+\frac{\varepsilon}{4},\frac{4(\mu+2\varepsilon)^2+2}{\varepsilon\lambda_1^{\frac{1}{2}}}+\frac{4\alpha^2}{\varepsilon}+\frac{2\left|p\right|+4}{\lambda_1^{\frac{1}{4}}}+\frac{\varepsilon}{4\lambda_1^{\frac{1}{2}}}\right\},\tag{50}
$$

$$
c_1(\omega)=\max\left\{1,\frac{6\beta r_1(\omega)}{\lambda_1^{\frac{1}{4}}}\right\},\ \ c_2(\omega)=\max\left\{1,\frac{\vec{r}^2(\omega)}{2}+\frac{2\beta r_1(\omega)}{\lambda_1^{\frac{1}{8}}}\right\},\tag{51}
$$

https://doi.org/10.17993/3ctecno.2023.v12n1e43.386-412

$$C(p)=\begin{cases}1, & p\leqslant -2,\\ (1-\frac{p+2}{\lambda_1^{\frac{1}{4}}}), & -2<p<\frac{\lambda_1^{\frac{1}{4}}-2}{2},\end{cases} \tag{52}$$

$$q=\frac{\sigma\left(\mu+2\varepsilon+2\lambda_1^{\frac{1}{2}}+\alpha\lambda_1^{\frac{1}{2}}\right)}{2\lambda_1^{\frac{1}{2}}}, \tag{53}$$

$$-M_4=-\varepsilon+c_2(\omega), \tag{54}$$

$$M_5=\frac{\sigma\left(\mu+2\varepsilon+\alpha\lambda_1^{\frac{1}{4}}\right)}{2\lambda_1^{\frac{1}{4}}}+|\sigma|, \tag{55}$$

$$M_6=\varepsilon-\frac{16}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2-\frac{64\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{2}}}, \tag{56}$$

$$M_7=\sigma+\frac{\sigma}{\lambda_1^{\frac{1}{2}}}+\frac{\sigma\left(\mu+2\varepsilon+\alpha\lambda_1^{\frac{1}{2}}\right)}{2\lambda_1^{\frac{1}{2}}},\ M_8=\frac{\sigma^2}{2\lambda_1^{\frac{1}{2}}}, \tag{57}$$

$$M_9=\frac{n\sigma}{2\sqrt{\lambda_1\pi\mu}}+\frac{n\sigma}{2\lambda_1\sqrt{\pi\mu}}+\frac{n\sigma}{4\mu\lambda_1^{\frac{1}{2}}}+\frac{n}{2\lambda_1}. \tag{58}$$

Proof for Lemma 4 : Taking the inner product of V by $\boldsymbol{V}=[u,v]^T$ in E_1, we get that

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{V}\|_{E_1}^2 &\leq 2(\boldsymbol{QV},\boldsymbol{V})_{E_1}+2\left(\sigma A^{\frac{1}{2}}uz\left(\theta_t\omega\right),A^{\frac{1}{2}}u\right)\\ &+2\left(-\left[\beta\left\|A^{\frac{1}{2}}u\right\|^2-p\right]A^{\frac{1}{2}}u,v\right)\\ &+2\left(\sigma\left(\mu+2\varepsilon-\alpha A^{\frac{1}{2}}-\sigma z\left(\theta_t\omega\right)\right)uz\left(\theta_t\omega\right)-\sigma vz\left(\theta_t\omega\right),v\right).\end{aligned} \tag{59}$$

Set

$$\mathcal{H}(t,\tau,\omega)=\overline{\mathcal{H}}(u,v)=\frac{1}{2\beta}\left[\beta\left\|A^{\frac{1}{4}}u\right\|^2-p\right]^2+\left\|A^{\frac{1}{2}}u\right\|^2+\|v\|^2.$$

By (40) and Lemma 3, we find that

$$\begin{aligned}\frac{d}{dt}\overline{\mathcal{H}}(u,v) &\leq -\frac{\varepsilon}{2}\overline{\mathcal{H}}(u,v)+\frac{\varepsilon p^2}{4\beta}+\frac{4\sigma^2|z(\theta_t\omega)|^2}{7\varepsilon\beta}\left[\beta\|A^{\frac{1}{4}}u\|^2-p\right]^2\\ &+\left(\frac{\alpha^2+2}{\varepsilon}+\frac{(\mu+2\varepsilon)^2+4}{\varepsilon\lambda_1^{\frac{1}{2}}}\right)\sigma^2|z(\theta_t\omega)|^2\|A^{\frac{1}{2}}u\|^2\\ &+\left(\frac{2}{\varepsilon}+\frac{\varepsilon}{4}\right)\sigma^2|z\left(\theta_t\omega\right)|^2\|v\|^2\\ &\leq -\frac{\varepsilon}{2}\overline{\mathcal{H}}(u,v)+M\sigma^2|z(\theta_t\omega)|^2\overline{\mathcal{H}}(u,v)+\frac{\varepsilon p^2}{4\beta},\end{aligned} \tag{60}$$

where M is formulated by (47). Then, for any $t>0$, the following holds

$$\mathcal{H}\left(0,-t,\omega\right)\leq e^{-\frac{\varepsilon}{2}t+\int_{-t}^0 M\sigma^2|z(\theta_s\omega)|^2ds}\mathcal{H}(-t,-t,\omega)+\frac{\varepsilon p^2}{4\beta}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0 M\sigma^2|z(\theta_k\omega)|^2dk}ds.$$

https://doi.org/10.17993/3ctecno.2023.v12n1e43.386-412

Since $v = u_t + \varepsilon u - \sigma u z\left(\theta_t\omega\right)$, we get

$$\begin{aligned} \|\boldsymbol{U}\|_{E_1}^2 \leq & 2\left\|A^{\frac{1}{2}}u\right\|^2 + 2\left\|u_t + \varepsilon u - \sigma u z\left(\theta_t\omega\right)\right\|^2 + 2\left\|\sigma u z\left(\theta_t\omega\right)\right\|^2 \\ \leq & 2\left(1 + \frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\|\boldsymbol{V}\|_{E_1}^2. \end{aligned} \tag{61}$$

Thus, we have

$$\begin{aligned} \|\boldsymbol{U}\left(0,-t,\omega\right)\|_{E_1}^2 \leq & 2\left(1 + \frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\|\boldsymbol{V}\left(0,-t,\omega\right)\|_{E_1}^2 \\ \leq & 2\left(1 + \frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)e^{-\frac{\varepsilon}{2}t + \int_{-t}^0 M\sigma^2|z(\theta_s\omega)|^2 ds}\mathcal{H}(-t,-t,\omega) \\ & + \left(1 + \frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\frac{\varepsilon p^2}{2\beta}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s + \int_s^0 M\sigma^2|z(\theta_k\omega)|^2 dk} ds. \end{aligned} \tag{62}$$

Since the random variable $z(\theta_t\omega)$ is tempered, along with Lemma 1, we can infer that

$$\frac{2\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2 e^{-\frac{\varepsilon}{2}t + \int_{-t}^0 M\sigma^2|z(\theta_s\omega)|^2 ds}\mathcal{H}(-t,-t,\omega) \to 0, t \to +\infty,$$

and

$$\frac{2\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2 \frac{\varepsilon p^2}{4\beta}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s + \int_s^0 M\sigma^2|z(\theta_k\omega)|^2 dk} ds \to 0, t \to +\infty,$$

similarly

$$2e^{-\frac{\varepsilon}{2}t + \int_{-t}^0 M\sigma^2|z(\theta_s\omega)|^2 ds}\mathcal{H}(-t,-t,\omega) \to 0, t \to +\infty,$$

and

$$\frac{\varepsilon p^2}{2\beta}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s + \int_s^0 M\sigma^2|z(\theta_k\omega)|^2 dk} ds < \infty.$$

Let

$$\rho_1(\omega) = \frac{\varepsilon p^2}{2\beta}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s + \int_s^0 M\sigma^2|z(\theta_k\omega)|^2 dk} ds. \tag{63}$$

According to (59) - (63), we obtain that there exists $T_1(\omega) > 0$, such that

$$\|\boldsymbol{U}(t,\theta_{-t}\omega)\|_{E_1} = \|\boldsymbol{U}\left(0,-t,\omega\right)\|_{E_1} \leq r_1(\omega), \quad \forall t \geq T_1(\omega), \tag{64}$$

where $r_1(\omega) = \sqrt{\rho_1(\omega)}$.

On the other hand, utilizing (10), we obtain

$$\begin{aligned} \mathbb{E}\left(\rho_1(\omega)\right) = & \frac{\varepsilon p^2}{2\beta}\int_{-\infty}^0 e^{-\frac{\varepsilon}{2}s}\mathbb{E}\left(e^{M\sigma^2\int_s^0|z(\theta_k\omega)|^2 dk}\right) ds \\ \leq & \frac{\varepsilon p^2}{2\beta}\int_{-\infty}^0 e^{-\frac{\varepsilon}{2}s - \frac{sM\sigma^2}{\mu}} ds \\ = & \frac{\varepsilon p^2}{2\beta}\int_{\overline{T}}^0 e^{-\frac{\varepsilon}{2}s - \frac{sM\sigma^2}{\mu}} ds - \frac{\varepsilon p^2}{2\beta}\int_{-\infty}^{\overline{T}} e^{\left(\overline{\varepsilon} + \frac{M\sigma^2}{\mu}\right)s} ds \\ = & \frac{\varepsilon p^2}{2\beta}\int_{\overline{T}}^0 e^{-\frac{\varepsilon}{2}s - \frac{sM\sigma^2}{\mu}} ds - \frac{\varepsilon p^2 e^{\overline{\varepsilon} + \frac{M\sigma^2}{\mu}}\overline{T}}{2\beta\left(\overline{\varepsilon} + \frac{M\sigma^2}{\mu}\right)} \\ < & +\infty. \end{aligned} \tag{65}$$

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Proof for Lemma 5 : Taking the inner product $(\cdot,\cdot)_{E_1}$ of (18) with $A^{\frac{1}{2}}\boldsymbol{V}^a$ gives

$$\begin{aligned}\frac{d}{dt}\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2 =&2(\boldsymbol{Q}\boldsymbol{V}^a, A^{\frac{1}{2}}\boldsymbol{V}^a)_{E_1}+2(\sigma A^{\frac{1}{2}}u^a z(\theta_t\omega), Au^a)\\ &+2\left(\sigma(\mu+2\varepsilon-\alpha A^{\frac{1}{2}}-\sigma z(\theta_t\omega))u^a z(\theta_t\omega)-\sigma v^a z(\theta_t\omega), A^{\frac{1}{2}}v^a\right).\end{aligned} \tag{66}$$

Substituting (41) and (42) into (66),, we infer the following by combining with Lemma 3

$$\begin{aligned}\frac{d}{dt}\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2 \leq &-\frac{\varepsilon}{2}\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2+\left(\frac{4}{\varepsilon}+\frac{\varepsilon}{4}\right)\sigma^2|z(\theta_t\omega)|^2\|A^{\frac{1}{4}}v^a\|^2\\ &+\left(\frac{8(\mu+2\varepsilon)^2}{\varepsilon\lambda_1^{\frac{1}{2}}}+\frac{8\alpha^2}{\varepsilon}+\frac{4}{\varepsilon\lambda_1^{\frac{1}{2}}}+\frac{2}{\varepsilon\lambda_1^{\frac{1}{2}}}\right)\sigma^2|z(\theta_t\omega)|^2\|A^{\frac{3}{4}}u^a\|^2,\end{aligned}$$

hence

$$\frac{d}{dt}\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2 \leq \left(-\frac{\varepsilon}{2}+\sigma^2|z(\theta_t\omega)|^2 M_0\right)\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2,$$

where M_0 is given by (46). Thus, we have

$$\|A^{\frac{1}{4}}\boldsymbol{V}^a\|_{E_1}^2 \leq e^{\int_{-t}^0 -\frac{\varepsilon}{2}t+\sigma^2|z(\theta_s\omega)|^2 M_0 ds}\|A^{\frac{1}{4}}\boldsymbol{V}^a(-t,-t,\omega)\|_{E_1}^2.$$

Since the random variable $z(\theta_t\omega)$ is tempered, along with Lemma 1 states

$$e^{\int_{-t}^0 -\frac{\varepsilon}{2}t+\sigma^2|z(\theta_s\omega)|^2 M_0 ds}\|A^{\frac{1}{4}}\boldsymbol{V}^a(-t,-t,\omega)\|_{E_1}^2 \to 0, t\to+\infty,$$

Thus, it can be obtained that

$$\lim_{t\to+\infty}\left\|A^{\frac{1}{4}}\boldsymbol{U}^a\right\|(t,\theta_{-t}\omega)_{E_1}=0.$$

Proof for Lemma 6 : Taking the inner product $(\cdot,\cdot)_{E_1}$ of (19) with $\boldsymbol{V}^b$ gives

$$\frac{d}{dt}\|\boldsymbol{V}^b\|_{E_1}^2 \leq \Upsilon^{(1)}+\Upsilon^{(2)}, \tag{67}$$

here

$$\begin{aligned}\Upsilon^{(1)} =&2(\boldsymbol{Q}\boldsymbol{V}^b,\boldsymbol{V}^b)+2\left(\sigma A^{\frac{1}{2}}u^b z(\theta_t\omega), A^{\frac{1}{2}}u^b\right)+2\left(-\left[\beta\left\|A^{\frac{1}{4}}u^b\right\|^2-p\right]A^{\frac{1}{2}}u^b, v^b\right)\\ &+2\left(\sigma\left(\mu+2\varepsilon-\alpha A^{\frac{1}{2}}-\sigma z(\theta_t\omega)\right)u^b z(\theta_t\omega)-\sigma v z(\theta_t\omega), v^b\right),\\ \Upsilon^{(2)} =&2\left(\left[\beta\left\|A^{\frac{1}{4}}u^b\right\|^2-p\right]A^{\frac{1}{2}}u^b, v^b\right)+2\left(-\left[\beta\left\|A^{\frac{1}{4}}u\right\|^2-p\right]A^{\frac{1}{2}}u, v^b\right),\end{aligned} \tag{68}$$

in which $\Upsilon^{(1)}$ is bounded. On the other hand

$$\begin{aligned}\Upsilon^{(2)} =& 2\left[\beta\left\|A^{\frac{1}{4}}u-A^{\frac{1}{4}}u^a\right\|^2-p\right]\left(A^{\frac{1}{2}}u,\frac{1}{2}v^b\right)+2\left[\beta\left\|A^{\frac{1}{4}}u-A^{\frac{1}{4}}u^a\right\|^2-p\right]\left(-A^{\frac{1}{2}}u^a,\frac{1}{2}v^b\right)\\ &+2\left(-\left[\beta\left\|A^{\frac{1}{4}}u\right\|^2-p\right]A^{\frac{1}{2}}u, v^b\right)\\ \leq& \frac{\beta}{\lambda_1^{\frac{1}{4}}}\left(2\|\boldsymbol{V}\|^2+2\|\boldsymbol{V}^a\|^2+\frac{|p|\lambda_1^{\frac{1}{4}}}{\beta}\right)(\|\boldsymbol{V}\|^2+\|\boldsymbol{V}^a\|^2)+\left(\frac{\beta}{\lambda_1^{\frac{1}{4}}}\|\boldsymbol{V}\|^2+|p|\right)\|\boldsymbol{V}\|^2\\ &+\frac{\beta}{2\lambda_1^{\frac{1}{4}}}\left(4\|\boldsymbol{V}\|^2+2\|\boldsymbol{V}^a\|^2+\frac{3|p|\lambda_1^{\frac{1}{4}}}{\beta}\right)\overline{\mathcal{H}}(u^b,v^b).\end{aligned} \tag{69}$$

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Set

$$\mathcal{H}^{'}(t,\tau,\omega)=\overline{\mathcal{H}}(u^b,v^b)=\frac{1}{2\beta}\left[\beta\left\|A^{\frac{1}{4}}u^b\right\|^2-p\right]^2+\left\|A^{\frac{1}{2}}u^b\right\|^2+\|v^b\|^2,$$

taking account into (67), (68) and (69), we can get

$$\begin{aligned}\frac{d}{dt}\overline{\mathcal{H}}(u^b,v^b)\leq&-\frac{\varepsilon}{2}\overline{\mathcal{H}}(u^b,v^b)+\left(\frac{\beta}{\lambda_1^{\frac{1}{4}}}\|\boldsymbol{V}\|^2+|p|\right)\|\boldsymbol{V}\|^2+\frac{\varepsilon p^2}{4\beta}\\&+\left(M\sigma^2|z(\theta_t\omega)|^2+\frac{2\beta}{\lambda_1^{\frac{1}{4}}}\|\boldsymbol{V}\|^2+\frac{\beta}{\lambda_1^{\frac{1}{4}}}\|\boldsymbol{V}^a\|^2+3|p|\right)\overline{\mathcal{H}}(u^b,v^b)\\&+\frac{2\beta\|\boldsymbol{V}\|^2+2\beta\|\boldsymbol{V}^a\|^2+|p|\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}}(\|\boldsymbol{V}\|^2+\|\boldsymbol{V}^a\|^2).\end{aligned}\tag{70}$$

Thus,

$$\begin{aligned}\mathcal{H}^{'}(0,-t,\omega)\leq\;&e^{-\frac{\varepsilon}{2}t+\int_{-t}^{0}M\sigma^2|z(\theta_s\omega)|^2+\frac{\beta}{2\lambda_1^{\frac{1}{4}}}\left(4\|\boldsymbol{V}\|^2+2\|\boldsymbol{V}^a\|^2+\frac{3|p|\lambda_1^{\frac{1}{4}}}{\beta}\right)ds}\mathcal{H}^{'}(-t,-t,\omega)\\&+\int_{-t}^{0}e^{-\frac{\varepsilon}{2}s+\int_s^0 M\sigma^2|z(\theta_k\omega)|^2+\frac{\beta}{2\lambda_1^{\frac{1}{4}}}\left(4\|\boldsymbol{V}\|^2+2\|\boldsymbol{V}^a\|^2+\frac{3|p|\lambda_1^{\frac{1}{4}}}{\beta}\right)dk}\\&\times\left(\frac{2\beta\|\boldsymbol{V}\|^2+2\beta\|\boldsymbol{V}^a\|^2+|p|\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}}(\|\boldsymbol{V}\|^2+\|\boldsymbol{V}^a\|^2)+\frac{\beta\|\boldsymbol{V}\|^4}{\lambda_1^{\frac{1}{4}}}+|p|\|\boldsymbol{V}\|^2+\frac{\varepsilon p^2}{4\beta}\right)ds.\end{aligned}$$

Applying Lemma 4 and Lemma 5, we obtain

$$\begin{aligned}\mathcal{H}^{'}(0,-t,\omega)\leq\;&e^{-\frac{\varepsilon}{2}t+\int_{-t}^{0}M\sigma^2|z(\theta_s\omega)|^2+M_1ds}\mathcal{H}^{'}(-t,-t,\omega)\\&+M_2\int_{-t}^{0}e^{-\frac{\varepsilon}{2}s+\int_s^0 M\sigma^2|z(\theta_k\omega)|^2+M_1dk}ds,\end{aligned}$$

where $M_1,\ M_2$ are defined by (48) and (49) correspondingly. Since the random variable $z(\theta_t\omega)$ is tempered, together with Lemma 1, we find

$$e^{-\frac{\varepsilon}{2}t+\int_{-t}^{0}M\sigma^2|z(\theta_s\omega)|^2+M_1ds}\mathcal{H}^{'}(-t,-t,\omega)\to 0,t\to+\infty,$$

and

$$M_2\int_{-t}^{0}e^{-\frac{\varepsilon}{2}s+\int_s^0 M\sigma^2|z(\theta_k\omega)|^2+M_1dk}ds<\infty.$$

Let

$$\rho_2(\omega)=M_2\int_{-t}^{0}e^{-\frac{\varepsilon}{2}s+\int_s^0 M\sigma^2|z(\theta_k\omega)|^2+M_1dk}ds.\tag{71}$$

As indicated above, we obtain that there exists $T_2(\omega)>0$ such that

$$\|\boldsymbol{U}\|_{E_1}(t,\theta_{-t}\omega)=\|\boldsymbol{U}\|_{E_1}(0,-t,\omega)\leq r_2(\omega),\quad\forall t\geq T_2(\omega),$$

where $r_2(\omega)=\sqrt{\rho_2(\omega)}$.

Proof for Lemma 7 : Taking the inner product $(\cdot,\cdot)_{E_1}$ of (19) with $A^{\frac{1}{2}}\boldsymbol{V}^b$ gives

$$\begin{aligned}\frac{d}{dt}\|A^{\frac{1}{4}}\boldsymbol{V}^b\|_{E_1}^2\leq&2(\boldsymbol{Q}\boldsymbol{V}^b,A^{\frac{1}{2}}\boldsymbol{V}^b)+2\left(\sigma A^{\frac{1}{2}}u^bz\left(\theta_t\omega\right),Au^b\right)\\&+2\left(-\left[\beta\left\|A^{\frac{1}{4}}u\right\|^2-p\right]A^{\frac{1}{2}}u,A^{\frac{1}{2}}v^b\right)\\&+2\left(\sigma\left(\mu+2\varepsilon-\alpha A^{\frac{1}{2}}-\sigma z\left(\theta_t\omega\right)\right)u^bz\left(\theta_t\omega\right)-\sigma vz\left(\theta_t\omega\right),A^{\frac{1}{2}}v^b\right).\end{aligned}\tag{72}$$

https://doi.org/10.17993/3ctecno.2023.v12n1e43.386-412

Set

$$\mathcal{H}_1(t,\tau,\omega)=\overline{\mathcal{H}}_1(u,v)=\left[\beta\|A^{\frac{1}{4}}u\|^2-p-2\right]\|A^{\frac{1}{2}}u^b\|^2+\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}, \tag{73}$$

we find

$$\begin{aligned}\mathcal{H}_1(t,\tau,\omega)&=\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}+\beta\|A^{\frac{1}{2}}u^b\|^2\|A^{\frac{1}{4}}u\|^2-p\|A^{\frac{1}{2}}u^b\|^2-2\|A^{\frac{1}{2}}u^b\|^2\\ &\geq\|A^{\frac{1}{4}}u^b\|^2+\beta\|A^{\frac{1}{2}}u^b\|^2\|A^{\frac{1}{4}}u\|^2+(1-\frac{p+2}{\lambda_1^{\frac{1}{4}}})\|A^{\frac{3}{4}}u^b\|^2\\ &\geq\|A^{\frac{1}{4}}u^b\|^2+\beta\|A^{\frac{1}{2}}u^b\|^2\|A^{\frac{1}{4}}u\|^2+C(p)\|A^{\frac{3}{4}}u^b\|^2\\ &\geq 0,\end{aligned}$$

where $C(p)$ is given by (52), which along with (43), (44) and (72) states that

$$\begin{aligned}\frac{d}{dt}\overline{\mathcal{H}}_1(u,v)\leq&-\frac{\varepsilon}{2}\overline{\mathcal{H}}_1(u,v)+\overline{\mathcal{H}}_1(u,v)+\widehat{M}(t,\tau,\omega)\\ &+\sigma^2|z(\theta_t\omega)|^2\left[\beta\|A^{\frac{1}{4}}u\|^2-p-2\right]\|A^{\frac{1}{2}}u^b\|^2\\ &+\left(\frac{4(\mu+2\varepsilon)^2+2}{\varepsilon\lambda_1^{\frac{1}{2}}}+\frac{4\alpha^2}{\varepsilon}+\frac{2\,|p|+4}{\lambda_1^{\frac{1}{4}}}+\frac{\varepsilon}{4\lambda_1^{\frac{1}{2}}}\right)\sigma^2|z(\theta_t\omega)|^2\|A^{\frac{3}{4}}u^b\|^2\\ &+\left(\frac{2}{\varepsilon}+\frac{\varepsilon}{4}\right)\sigma^2|z(\theta_t\omega)|^2\|A^{\frac{1}{4}}v^b\|^2\\ &\leq-\frac{\varepsilon}{2}\overline{\mathcal{H}}_1(u,v)+\left(1+M_3\sigma^2|z(\theta_t\omega)|^2\right)\overline{\mathcal{H}}_1(u,v)+\widehat{M}(t,\tau,\omega),\end{aligned}$$

where $\widehat{M}(t,\tau,\omega)$ and M_3 are defined by (45) and (50) respectively. Then by Lemma 4, Lemma 5 and Lemma7, the following holds

$$\begin{aligned}\mathcal{H}_1(0,-t,\omega)\leq\ & e^{-\frac{\varepsilon}{2}t+\int_{-t}^0\left(1+M_3\sigma^2|z(\theta_s\omega)|^2\right)ds}\mathcal{H}_1(-t,-t,\omega)\\ &+r\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds.\end{aligned}$$

Since the random variable $z(\theta_t\omega)$ is tempered, applying Lemma 1, we find

$$e^{-\frac{\varepsilon}{2}t+\int_{-t}^0\left(1+M_3\sigma^2|z(\theta_s\omega)|^2\right)ds}\mathcal{H}_1(-t,-t,\omega)\to 0,\, t\to+\infty,$$

$$r\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds<\infty.$$

then we get

$$\mathcal{H}_1(0,-t,\omega)\leq\rho_{30}(\omega),$$

where $\rho_{30}(\omega)=r\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds.$
Exploiting (61) and (73), we can obtain

$$\begin{aligned}&\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}(0,-t)+\beta\|A^{\frac{1}{2}}u^b\|^2(0,-t)\,\|A^{\frac{1}{4}}u\|^2(0,-t)\\ &\leq\rho_{30}(\omega)+\frac{p+2}{\lambda_1^{\frac{1}{4}}}\|A^{\frac{3}{4}}u^b\|^2(0-t)\\ &\leq\rho_{30}(\omega)+\frac{p+2}{\lambda_1^{\frac{1}{4}}}\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}(0,-t),\end{aligned}$$

then

$$(1-\frac{p+2}{\lambda_1^{\frac{1}{4}}})\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}(0,-t)\leq\rho_{30}(\omega),$$

utilizing (52), we get

$$\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}(0,-t)\leq\frac{\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}-p-2}\rho_{30}(\omega). \tag{74}$$

https://doi.org/10.17993/3ctecno.2023.v12n1e43.386-412

By (72)-(74), we obtain that there exists $T_{30}(\omega) > 0$, such that

$$\|A^{\frac{1}{4}}\boldsymbol{V}^b\|^2_{E_1}(0,-t) \leq \rho_3(\omega), \quad \forall\ t \geq T_{30}(\omega), \tag{75}$$

where, $\rho_3(\omega) = \frac{\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}-p+2}\rho_{30}(\omega)$. Since the relation between $\|A^{\frac{1}{4}}\boldsymbol{U}^b\|_{E_1}$ and $\|A^{\frac{1}{4}}\boldsymbol{V}^b\|_{E_1}$,

$$\begin{aligned}
&\|A^{\frac{1}{4}}\boldsymbol{U}^b\|^2_{E_1}(0,-t,\omega)\\
\leq&2\left(1+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\|A^{\frac{1}{4}}\boldsymbol{V}^b(0,-t,\omega)\|^2_{E_1}\\
\leq&\frac{2r\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}-p+2}\left(1+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds.
\end{aligned} \tag{76}$$

Since the random variable $z(\theta_t\omega)$ is tempered, we get the following holds by Lemma 1,

$$\frac{2r\sigma^2}{\lambda_1^{\frac{1}{4}}-p+2}|z(\theta_t\omega)|^2\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds \to 0, t \to +\infty,$$

and

$$\frac{2r\lambda_1^{\frac{1}{4}}}{\lambda_1^{\frac{1}{4}}-p+2}\int_{-t}^0 e^{-\frac{\varepsilon}{2}s+\int_s^0(1+M_3\sigma^2|z(\theta_k\omega)|^2)dk}ds < \infty.$$

Thus, there exists a constant $T_3(\omega) > 0$ such that

$$\left\|A^{\frac{1}{4}}\boldsymbol{U}^b\right\|_{E_1}(t,\theta_{-t}\omega) = \left\|A^{\frac{1}{4}}\boldsymbol{U}^b\right\|_{E_1}(0,-t,\omega) \leq r_3(\omega),\ \forall\ t > T_3(\omega),$$

Proof for Lemma 8 : For $\forall\ \kappa \in \mathcal{D}(A^{\frac{1}{2}})$, we find that

$$\begin{aligned}
&\left\|\left(\widehat{X}_{32}(u_1)-\widehat{X}_{32}(u_2)\right)\kappa\right\|\\
\leq&\left\|\left(\beta\|\nabla u_1\|^2-\beta\|\nabla u_2\|^2\right)(-\Delta)\kappa\right\| + \|2\beta(\nabla u_1,\nabla\kappa)(-\Delta)u_1 - 2\beta(\nabla u_2,\nabla\kappa)(-\Delta)u_2\|\\
\leq&\beta\left(\|\nabla u_1\|-\|\nabla u_2\|\right)\left(\|\nabla u_1\|+\|\nabla u_2\|\right)\|(-\Delta)\kappa\|\\
&+2\beta\|(\nabla u_1,\nabla\kappa)\|\,\|(-\Delta)u_1-(-\Delta)u_2\| + 2\beta\,|\,(\nabla u_1-\nabla u_2,\nabla\kappa)\,|\|\|(-\Delta)u_2\|\\
\leq&\beta\left(\|\nabla u_1\|-\|\nabla u_2\|\right)\left(\|\nabla u_1\|+\|\nabla u_2\|\right)\|(-\Delta)\kappa\|\\
&+2\beta\|\nabla u_1\|\,\|\nabla\kappa\|\,\|(-\Delta)u_1-(-\Delta)u_2\| + 2\beta\|\nabla u_1-\nabla u_2\|\,\|\nabla\kappa\|\,\|(-\Delta)u_2\|,
\end{aligned}$$

Merging Lemma 4 with (51), we find

$$\begin{aligned}
&\left\|\left(\widehat{X}_{32}(u_1)-\widehat{X}_{32}(u_2)\right)\kappa\right\|\\
\leq&\beta\left(\|\nabla u_1\|-\|\nabla u_2\|\right)\left(\|\nabla u_1\|+\|\nabla u_2\|\right)\|(-\Delta)\kappa\|\\
\leq&\frac{\beta}{\lambda_1^{\frac{1}{4}}}\left(\|\Delta u_1\|-\|\Delta u_2\|\right)2r_1(\omega)\|(-\Delta)\kappa\|\\
&+\frac{2\beta}{\lambda_1^{\frac{1}{4}}}r_1(\omega)\|\Delta\kappa\|\,\|(-\Delta)u_1+\Delta u_2\| + \frac{2\beta}{\lambda_1^{\frac{1}{4}}}\|\Delta u_1-\Delta u_2\|\,\|\Delta\kappa\|r_1(\omega)\\
\leq&c_1(\omega)\|\Delta u_1-\Delta u_2\|\,\|\Delta\kappa\|, \quad \forall u_1,u_2\in\mathcal{D}(A^{\frac{1}{2}}),
\end{aligned}$$

along with Lemma 4 demonstrates $\mathbb{E}\left(c_1(\omega)\right) \leq \infty$, hence

$$\begin{aligned}
\left\|\widehat{X}_{32}(u_1)-\widehat{X}_{32}(u_2)\right\|_{\mathscr{L}\left(\mathcal{D}(A^{\frac{1}{2}}),L^2(D)\right)} =& \sup_{\kappa\in\mathcal{D}(A^{\frac{1}{2}})}\frac{\|\left(\widehat{X}_{32}(u_1)-\widehat{X}_{32}(u_2)\right)\kappa\|}{\|\kappa\|_{\mathcal{D}(A^{\frac{1}{2}})}}\\
\leq& c_1(\omega)\left\|A^{\frac{1}{2}}u_1-A^{\frac{1}{2}}u_2\right\|.
\end{aligned}$$

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On the other hand, since

$$\begin{aligned}\left\|\widehat{X}_{32}(u)\right\|_{\mathscr{L}\left(\mathcal{D}(A^{\frac{1}{2}}),L^2(D)\right)} &= \sup_{\kappa\in\mathcal{D}(A^{\frac{1}{2}})}\frac{\|\widehat{X}_{32}(u)\,\kappa\|}{\|\kappa\|_{\mathcal{D}(A^{\frac{1}{2}})}}\\ &= \frac{\|\left[\beta\|\nabla u\|^2-p\right](-\Delta)\kappa+2\beta(\nabla u,\nabla\kappa)(-\Delta)u\|}{\|\Delta\kappa\|}\\ &\leq \frac{\|\left[\beta\|\nabla u\|^2-p\right](-\Delta)\kappa\|+\|2\beta(\nabla u,\nabla\kappa)(-\Delta)u\|}{\|\Delta\kappa\|}\\ &\leq [\beta\|\nabla u\|^2-p]^2+\frac{2\beta}{\lambda_1^{\frac{1}{8}}}\|\Delta u\|^2.\end{aligned}$$

In term of Lemma 4 and (51), we find

$$\left\|\widehat{X}_{32}(u)\right\|_{\mathscr{L}\left(\mathcal{D}(A^{\frac{1}{2}}),L^2(D)\right)} \leq \frac{\overline{r}^2(\omega)}{2}+\frac{2\beta}{\lambda_1^{\frac{1}{8}}}r_1(\omega)=c_2(\omega),$$

and

$$\mathbb{E}\left(c_2(\omega)\right)\leq\infty.$$

Proof for Lemma 9 : Taking the inner product of (23) by $\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}$ in E_1 illustrates

$$\begin{aligned}\frac{1}{2}\frac{d\left\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right\|_{E_1}^2}{dt} &= \left(\boldsymbol{Q}\left(\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right),\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right)_{E_1}\\ &+\left(\boldsymbol{X}_2\left(\boldsymbol{V}^{(1)}\right)-\boldsymbol{X}_2\left(\boldsymbol{V}^{(2)}\right),\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right)_{E_1}\\ &+\left(X_{32}\left(\omega,u_1\right)-X_{32}\left(\omega,u_2\right),v_1-v_2\right),\end{aligned}\tag{77}$$

Merging with (24), we have

$$\|X_{32}(\omega,u_1)-X_{32}(\omega,u_2)\|\leq c_2\left\|A^{\frac{1}{2}}\left(u_1-u_2\right)\right\|.\tag{78}$$

Furthermore, we get

$$\begin{aligned}\left(X_{32}(\omega,u_1)-X_{32}(\omega,u_2),v_1-v_2\right) &\leq \|X_{32}(\omega,u_1)-X_{32}(\omega,u_2)\|\,\|v_1-v_2\|\\ &\leq c_2(\omega)\left\|A^{\frac{1}{2}}\left(u_1-u_2\right)\right\|\,\|v_1-v_2\|\\ &\leq \frac{c_2(\omega)}{2}\left\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right\|_{E_1}^2,\end{aligned}\tag{79}$$

and

$$\begin{aligned}&\left(\boldsymbol{X}_2\left(\boldsymbol{V}^{(1)}\right)-\boldsymbol{X}_2\left(\boldsymbol{V}^{(2)}\right),\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right)_{E_1}\\ &\leq\frac{|\sigma z(\theta_t\omega)(\mu+2\varepsilon-\sigma z(\theta_t\omega))|+2\lambda_1^{\frac{1}{2}}|\sigma z(\theta_t\omega)|}{2\lambda_1^{\frac{1}{2}}}\left(\|u^n-u^m\|_{H^2}^2+\|v^n-v^m\|^2\right)\\ &+\frac{|\alpha\sigma z(\theta_t\omega)|}{2}\left(\left\|A^{\frac{1}{2}}(u^n-u^m)\right\|^2+\|v^n-v^m\|^2\right)\\ &\leq\frac{|\sigma z(\theta_t\omega)(q-\sigma z(\theta_t\omega))|+2\lambda_1^{\frac{1}{2}}|\sigma z(\theta_t\omega)|}{2\lambda_1^{\frac{1}{2}}}\|\boldsymbol{V}_n-\boldsymbol{V}_m\|_{E_1}^2\\ &\leq\left(q|z(\theta_t\omega)|+\frac{\sigma^2|z(\theta_t\omega)|^2}{2\lambda_1^{\frac{1}{2}}}\right)\|\boldsymbol{V}_n-\boldsymbol{V}_m\|_{E_1}^2,\end{aligned}\tag{80}$$

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here q is defined by (53), and

$$\left(\boldsymbol{Q}\left(\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right),\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right)\leq-\frac{\varepsilon}{2}\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\|_{E_1}^2-\frac{\varepsilon}{4}\|v_1-v_2\|^2. \tag{81}$$

Substituting (79), (80) and (81) into (77), we get

$$\frac{d\left\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right\|_{E_1}^2}{dt}\leq\left(-\varepsilon+c_2(\omega)+2q|z(\theta_t\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z(\theta_t\omega)|^2\right)\left\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right\|_{E_1}^2.$$

thus, we have

$$\left\|\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\right\|_{E_1}^2(0,-t,\omega)\leq e^{\int_{-t}^0-M_4+2q|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z(\theta_s\omega)|^2ds}\|\boldsymbol{I}\|_{E_1}^2, \tag{82}$$

taking into account the between $\left\|\widehat{\boldsymbol{V}}\right\|_{E_1}^2$ and $\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2$,

$$\left\|\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}\right\|_{E_1}^2\leq2\left(1+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)e^{\int_{-t}^0-M_4+2q|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z(\theta_s\omega)|^2ds}\|\boldsymbol{I}\|_{E_1}^2,$$

here $M_4>0$ is formulated by (54). Combine assertion that random variable $z(\theta_t\omega)$ is tempered with Lemma 1, we find

$$\frac{2\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2e^{\int_{-t}^0-M_4+2q|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z(\theta_s\omega)|^2ds}\to0,t\to+\infty,$$

and

$$2e^{\int_{-t}^0-M_4+2q|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{2}}}|z(\theta_s\omega)|^2ds}<\infty,$$

then, $\exists\, t_0', \forall\, t\geq t_0'$, $\left\|\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}\right\|_{E_1}^2\leq p_1'$, set $p_1=\max\{1,p_1'\}$, we obtain

$$\left\|\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}\right\|_{E_1}^2(0,-t,\omega)=\left\|\boldsymbol{U}^{(1)}-\boldsymbol{U}^{(2)}\right\|_{E_1}^2(t,\theta_{-t}\omega)\leq p_1\|\boldsymbol{I}\|_{E_1}.$$

and

$$p_1(\omega)\geq1,\mathbb{E}(p_1(\omega))<\infty,\mathbb{E}\left(\ln\left(p_1(\omega)\right)\right)<\infty. \tag{83}$$

On the other hand, taking the inner product of (21) by $\widehat{\boldsymbol{V}}$ in E_1, which together with Lemma 3 shows

$$(\boldsymbol{Q}\widehat{\boldsymbol{V}},\widehat{\boldsymbol{V}})_{E_1}\leq-\frac{\varepsilon}{2}\|\widehat{\boldsymbol{V}}\|_{E_1}^2-\frac{\varepsilon}{4}\left\|\widehat{V}_2\right\|^2, \tag{84}$$

and

$$\begin{aligned}&\left(\widehat{\boldsymbol{X}}_3\widehat{\boldsymbol{V}},\widehat{\boldsymbol{V}}\right)_{E_1}\\ =&-\left(\left[\beta\|\nabla u\|^2-p\right](-\Delta)\widehat{V}_1,\widehat{V}_2\right)-\left(2\beta\left(\nabla u,\nabla\widehat{V}_1\right)(-\Delta)u,\widehat{V}_2\right)\\ \leq&|\beta\|\nabla u\|^2-p|\left\|\Delta\widehat{V}_1\right\|\left\|\widehat{V}_2\right\|+\frac{2\beta}{\lambda_1^{\frac{1}{4}}}\|\Delta u\|^2\left\|\Delta\widehat{V}_1\right\|\left\|\widehat{V}_2\right\|\\ \leq&\frac{8}{\varepsilon}\left(|\beta\|\nabla u\|^2-p|\left\|\Delta\widehat{V}_1\right\|\right)^2+\frac{8}{\varepsilon}\left(\frac{2\beta}{\lambda_1^{\frac{1}{4}}}\|\Delta u\|^2\left\|\Delta\widehat{V}_1\right\|\right)^2+\frac{\varepsilon}{4}\left\|\widehat{V}_2\right\|^2\\ \leq&\frac{8}{\varepsilon}\left(\beta\bar{r}^2+|p|\right)^2\left\|\Delta\widehat{V}_1\right\|^2+\frac{64\beta^2\bar{r}^4}{\varepsilon\lambda_1^{\frac{1}{2}}}\left\|\Delta\widehat{V}_1\right\|^2+\frac{\varepsilon}{4}\left\|\widehat{V}_2\right\|^2,\end{aligned} \tag{85}$$

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with the similar calculation of (80), we have

$$\left(\widehat{\boldsymbol{X}}_2\widehat{\boldsymbol{V}},\widehat{\boldsymbol{V}}\right)\leq\left(M_5|z\left(\theta_t\omega\right)|+\frac{\sigma^2}{2\lambda_1^{\frac{1}{4}}}|z\left(\theta_t\omega\right)|^2\right)\left\|\widehat{\boldsymbol{V}}\right\|_{E_1}^2, \tag{86}$$

where M_5 is given by (55). From (84), (85) and (86), we get

$$\begin{aligned}\frac{d\|\widehat{\boldsymbol{V}}\|_{E_1}^2}{dt}&\leq-\varepsilon\|\widehat{\boldsymbol{V}}\|_{E_1}^2+2\left(\left(M_5|z\left(\theta_t\omega\right)|+\frac{\sigma^2}{2\lambda_1^{\frac{1}{4}}}|z\left(\theta_t\omega\right)|^2\right)\left\|\widehat{\boldsymbol{V}}\right\|_{E_1}^2\right)\\&+2\left(\frac{8}{\varepsilon}\left(\beta\overline{r}^2+|p|\right)^2\left\|\Delta\widehat{V}_1\right\|^2+\frac{64\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{2}}}\left\|\Delta\widehat{V}_1\right\|^2\right)\\&\leq\left(-M_6+2M_5|z\left(\theta_t\omega\right)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z\left(\theta_t\omega\right)|^2\right)\left\|\widehat{\boldsymbol{V}}\right\|_{E_1}^2,\end{aligned} \tag{87}$$

where $M_6>0$ is defined by (56). Thus

$$\|\widehat{\boldsymbol{V}}\|_{E_1}\leq e^{\int_{-t}^0-M_6+2M_5|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_s\omega)|^2ds}\|\boldsymbol{I}\|_{E_1},$$

by the relation between $\left\|\widehat{\boldsymbol{V}}\right\|_{E_1}^2$ and $\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2$, we have

$$\begin{aligned}\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2&\leq2\left(1+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)\|\widehat{\boldsymbol{V}}\|_{E_1}^2\\&\leq2\left(1+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2\right)e^{\int_{-t}^0-M_6+2M_5|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_s\omega)|^2ds}\|\boldsymbol{I}\|_{E_1}.\end{aligned}$$

The random variable $z(\theta_t\omega)$ is tempered, which together with Lemma 1 gives that

$$\frac{2\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_t\omega)|^2e^{\int_{-t}^0-M_6+2M_5|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_s\omega)|^2ds}\to0,t\to+\infty,$$

and

$$2e^{\int_{-t}^0-M_6+2M_5|z(\theta_s\omega)|+\frac{\sigma^2}{\lambda_1^{\frac{1}{4}}}|z(\theta_s\omega)|^2ds}<\infty,$$

then, $\exists\ t_2'$, $\forall\ t\geq t_2'$, $\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2\leq p_2'$, set $p_2=\max\{1,p_2'\}$, we obtain

$$\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2(0,-t,\omega)=\left\|\widehat{\boldsymbol{U}}\right\|_{E_1}^2(t,0,\theta_{-t}\omega)\leq p_2\|\boldsymbol{I}\|_{E_1}, \tag{88}$$

then we can get

$$\mathbb{E}(p_2(\omega))<\infty. \tag{89}$$

In the rest paper, the value of $\widehat{\boldsymbol{U}}$ at $t=1$ is still denoted by $\widehat{\boldsymbol{U}}$, and $\widehat{\boldsymbol{U}}(1)=\widehat{DS}_\varepsilon\left(\omega,\boldsymbol{U}^{(1)}\right)\boldsymbol{I}$, where $\widehat{DS}_\varepsilon\left(\omega,\boldsymbol{U}^{(1)}\right)$ is the linear solution mapping of system (20).

According to (84) and (89), we get that

$$\left\|\widehat{DS}_\varepsilon\left(\omega,\boldsymbol{U}^{(1)}\right)\right\|_{\mathscr{L}(E_1,E_1)}=\sup_{I\in E_1}\frac{\|\widehat{DS}_\varepsilon\left(\omega,\boldsymbol{U}^{(1)}\right)\boldsymbol{I}\|_{E_1}}{\|\boldsymbol{I}\|_{E_1}}\leq p_2(\omega), \tag{90}$$

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and

$$p_2(\omega) \geq 1, \mathbb{E}\left(\ln\left(p_2(\omega)\right)\right) < \infty. \tag{91}$$

On the other hand, Γ satisfies

$$\begin{aligned}
\|\Gamma(\omega)\|_{E_1} &\leq \|\boldsymbol{U}^{(1)} - \boldsymbol{U}^{(2)}\|_{E_1} + \| - \widehat{\boldsymbol{U}}\|_{E_1} \\
&\leq (p_1 + p_2)\,\|\boldsymbol{I}\|_{E_1} \\
&\leq c(\omega)\|\boldsymbol{I}\|_{E_1},
\end{aligned}$$

where $c(\omega) = \max\{1, p_1 + p_2\}$. Obviously, when $t = 1$

$$\|\Gamma(1, \omega)\|_{E_1} \leq c(\omega)\|\boldsymbol{I}\|_{E_1}.$$

Combining (91) with (89), we have

$$\mathbb{E}\left(\ln c(\omega)\right) < \infty, \mathbb{E}\left(c(\omega)\right) < \infty.$$

Since $S_\varepsilon(\omega) := S_\varepsilon(1, \omega)$, merging with (88) and (25), we can conclude that $S_\varepsilon(\omega)$ is almost surly uniform differentiable on $\mathscr{A}(\omega)$.

CONCLUSIONS

This paper consider global stochastic stability of the Euler-Bernoulli beam equations excited by multiplicative white noise. The system can induce a RDS which owns global random attractors, moreover, Hausdorff dimension of the attractor is finite. Specially, when $\lambda_1^{-\frac{1}{2}} \leq \frac{\frac{\varepsilon}{2} - \frac{8}{\varepsilon}\left(\beta\overline{r}^2 + |p|\right)^2 - \frac{2M_7}{\sqrt{\pi\mu}} - \frac{2M_8}{2\mu}}{\frac{16\beta^2\overline{r}^4}{\varepsilon\lambda_1^{\frac{1}{4}}}}$, the Hausdorff dimension is 0, which indicates that the stochastic Euler-Bernoulli beam possesses a random fixed point which is global stochastic stability.

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