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A new notion of convergence on ideal topological spaces

Una nueva noción de convergencia sobre espacios topológicos ideales

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Abstract
In this article, we use the notions of b-open and b-I-open sets to introduce the idea of b-I-convergence which we will denoted by b-I-convergence, we also show some of its properties. Besides, some basic properties of b-I-Fréchet-Urysohn space is shown. Moreover, notions related to b-I-sequential and b-I-sequentially are proved. Furthermore, we show some relations of b-I-irresolute functions between preserving b-I-convergence functions and b-I-covering functions.

Keywords . b-I-convergence, b-I-irresolute functions, preserving b-I-convergence functions, b-I-sequentially open, b-I-sequential spaces, b-I-covering functions, b-I-Fréchet-Urysohn spaces.

Resumen
En este artículo, usamos las nociones de conjuntos b-abierto y b-I-abierto para introducir la idea de b-I-convergencia la cual vamos a denotar por b-I-convergencia, también mostramos algunas de sus propiedades. Además, algunas propiedades básicas del espacio b-I-Fréchet-Urysohn son mostradas. Adicionalmente, nociones relativas a espacios pre-I-secuenciales y pre-I-secuencialmente abiertos son probadas. Además, mostramos algunas relaciones entre funciones b-I- irresolutas, funciones que preservan b-I-convergencia y funciones de b-I-cobertura.

Palabras clave. b-I-convergencia, funciones b-I- irresolutas, funciones que preservan b-I-convergencia, b-I-secuencialmente abierto, espacios b-I-secuenciales, funciones de b-I-cobertura, espacios b-I-Fréchet-Urysohn.

1. Introduction. The notion of ideal was introduced by Kuratowski in 1933 on [5], an ideal I on a space X is a collection of elements of X which satisfies: (1) $\emptyset \in I$; (2) If $A, B \in I$ then $A \cup B \in I$; and (3) if $B \subset I$ and $A \subset B$, then $A \in I$. This notion has been grown in several concepts of general topology. In 2019, Zhou and Lin on [7] used the notion of ideal on the set $\mathbb{N}$ to extend the notion of I-convergence, those results were useful for the developing of this paper. On the other hand, in 1996, Andrijevi on [1] introduced the concept of b-open sets in a topological spaces. A subset A of (X, τ) is said to be b-open if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. After that in 2004, Aysegul and Gulhan on [2] presented the idea of b-I-open sets and b-I-continuous functions in ideal topological spaces. The b-I-open sets were defined as: Let (X, τ, I) be an ideal topological space and let A be a subset of X, then A is said to be b-I-open if $A \subseteq Cl^\star(Int(A)) \cup Int(Cl^\star(A))$, where $Cl^\star(A) = A \cup A^\star$. $A^\star$ is called the local function of A respect to an ideal I and a topological space τ which was defined by [5]. The local function of A was defined as:

$$A^\star = \{x \in X : U \cap A \notin I \, for\, each\, U \in \tau(x)\} \text{ where } \tau(x) = \{U \in \tau : x \in U\}.$$

In this article, we took whole the notions mentioned above and we define other properties on b-I-convergence and we study the relation between b-I-sequentially open and b-I-sequential space. Moreover, we define and study some basic properties of preserving b-I-convergence functions and b-I-covering functions, furthermore we prove some relations with b-I-irresolute functions. Besides, the idea of b-I-Fréchet-Urysohn space is defined.

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Throughout this paper, the terms (X, τ) and (Y, σ) means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Additionally, we sometimes write X or Y instead of (X, τ) or (Y, σ), respectively. By other hand, $|A|$ will denote the cardinality of set A.

2. b-I-convergence. We first introduce some definitions.

Definition 2.1. *Let (X, τ) be a topological space, $A \subset X$ and $x \in X$. Then, A is said to be b-neighbourhood of x if and only if there exits a a b-open set B such that $x \in B \subset A$.*

Definition 2.2. *An ideal $I \subseteq \mathbb{N}$ is said to be non-trivial, if $I = \varnothing$ and $I \neq \mathbb{N}$. A non-trivial ideal $I \subseteq K$ is called admissible if $I \supseteq \{\{n\} : n \in \mathbb{N}\}$.*

Definition 2.3. *Let I be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called I-convergent to a point $x \in X$, provided for any neighbourhood V of x, it has $A_V = \{n \in \mathbb{N} : x_n \notin V\} \in I$, which is denoted by $I\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \to^I x$.*

Definition 2.4. *Let I be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called b-I-convergent to a point $x \in X$, provided for any b-neighbourhood V of x, it has $A_V = \{n \in \mathbb{N} : x_n \notin V\} \in I$, which is denoted by b-$I$-$\lim_{n\to\infty} x_n = x$ or $x_n \to^{bI} x$ and the point x is called the b-I-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.*

Lemma 2.1. *[1] Every open set of (X, τ) is a b-open set.*

Lemma 2.2. *b-I-convergence implies I-convergence.*
Proof: Let V an open set of (X, τ), then by the Lemma 2.1 V is a b-open set. Since $\{x_n\}$ is a b-I-convergent sequence, we have that $\{n \in \mathbb{N} : x_n \notin V\} \in I$. Therefore, by the Definition 2.3, $\{x_n\}$ is a I-convergent sequence.

Remark 1. *The converse of the above Lemma need not be true as can be seen in the following example:*

Let $\mathbb{R}$ be the set of real numbers with the usual topology and I be an admissible ideal and the sequence $(a_n)_{n\in\mathbb{N}}$ be defined by $a_n = b^n$, where $0 < b < 1$. It sees that the sequence $a_n = b^n$ is I-convergence to 0, since for any open set U containing 0, the set $\{n \in \mathbb{N} : a_n \notin U\}$ is finite. Now, take a b-open set $V = (-1, 0]$, we can see that V is a b-open set since that $(-1, 0] \subseteq [1, 0]$. Now, we can see that the set $\{n \in \mathbb{N} : a_n \notin V\}$ is equal to the set of natural numbers and then the sequence $a_n = b^n$ is not b-I-convergent to 0.

Remark 2. *Since every open set is a b-open, we proved that b-I-convergence implies I-convergence. If we would like to find an equivalent between them, (X, τ) should be a topological space with τ be the discrete topology. If τ is the discrete topology, every open set is a closed set and conversely. Let V be a b-open set, then*

$$\begin{aligned}
&V \subseteq Cl(Int(V)) \cup Int(Cl(V)),\\
&V = Cl(V) \cup Int(V),\\
&V = V \cup V,\\
&V = V.
\end{aligned}$$

And V is a subset of (X, τ), therefore if we have a sequence $\{x_n\}$ which is I-convergent, $\{x_n\}$ must be b-I-convergent.

Remark 3. *Taking into account the Remark 2,are there other conditions in which I-convergence implies b-I-convergence? This is an open problem.*

Definition 2.5. *Let (X, τ) be a topological space and $A \subset X$. Then, A is called b-I-sequentially open if and only if no sequence in $X - A$ has a b-I-limit in A. i.e. sequence can not b-I-converge out of a b-I-sequentially closed set.*

Definition 2.6. *Let I be an ideal on $\mathbb{N}$ and X be a topological space, then*

1. *A subset J of X is said to be b-I-closed if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq J$ with $x_n \to^{bI} x \in X$, then $x \in J$.*
2. *A subset V of X is said to be b-I-open if $X - V$ is b-I-closed.*
3. *X is said to be a b-I-sequential space if each b-I-closed set in X is closed.*

Remark 4. *The notion showed in the previous definition point 1 on b-I-closed set, this notion is equivalent of the notion showed by [2] about b-I-closed set.*
Let A be a b-I-closed and $\{x_n\} \subseteq A$ with $\{x_n\}$ a b-I-convergent to x. If $x \notin A$, then $x \in A^c$ and then we have that $\{x_n\} \in A^c$, and this is a contradiction. Therefore, $x \in A$.
The converse is proved similarly.

Definition 2.7. *Let (X, τ) be a topological space. Then, X is b-I-sequential when any set A is b-open if and only if it is b-I-sequentially open.*

Now, we show some results.

Lemma 2.3. *(cf. [7]) Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(x_n)_{n\in\mathbb{N}}$ I-converges to a point $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n)_{n\in\mathbb{N}}$ I-converges to $x \in X$*

Lemma 2.4. *(cf. [7]) Let $I \subseteq J$ be two ideals of $\mathbb{N}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X such that $x_n \to^I x$, then $x_n \to^J x$.*

Lemma 2.5. *Let (X, τ) be a topological space. Then, $B \subset X$ is b-I-sequentially open if and only if every sequence with b-I-limit in B has all but finitely many terms in B. Where the index set of the part in B of the sequence does not belong to I.*

Proof: Suppose that B is not a b-I-sequentially open, then there is a sequence with terms in $X - B$, but b-I-limit in B. Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence with infinitely many terms in $X - B$ such that b-I-converges to $y \in B$ and the index set of the part in B of the sequence does not belong to I. Then, $(x_n)_{n\in\mathbb{N}}$ has a subsequence in $X - B$ that has to still converges to $y \in B$ and so B is not b-I-sequentially open.

Lemma 2.6. *Let I and J be two ideals of $\mathbb{N}$ where $I \subseteq J$ and X is a topological space. If $V \subseteq X$ is b-J-open, then it is b-I-open.*

Proof: Let $V \subseteq X$ be b-I-open. Then, $X - V$ is pre-I-closed set, so every sequence $(x_n)_{n\in\mathbb{N}}$ in $X - V$ with $x_n \to^{bI} x$, hold that $x_n \to^{bJ} x$, by Lemma 2.4. So, $x \in X - V$ and therefore, V is b-J-open.

Corollary 2.1. *Let I and J be two ideals of $\mathbb{N}$, where $I \subseteq J$. If a topological space X is b-I-sequential, then it is b-J-sequential.*

Lemma 2.7. *Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(x_n)_{n\in\mathbb{N}}$ b-I-convergent to a point $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n)_{n\in\mathbb{N}}$ b-I-convergent to $x \in X$.*

Proof: The proof is followed by the Lemma 2.3 and Definition 2.4.

Lemma 2.8. *Let X be a topological space X, $A \subset X$ and I be an ideal on $\mathbb{N}$. Then, the following statements are equivalent.*

1. *A is b-I-open.*
2. *$\{n \in \mathbb{N} : x_n \in A\} \notin I$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{bI} x \in A$.*
3. *$|\{n \in \mathbb{N} : x_n \in A\}| = 0$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{bI} x \in A$.*

Proof: $(1) \Rightarrow (2)$: Suppose that A is a b-I-open set of X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X satisfying $x_n \to^{bI} x \in A$. Now, take $N_0 = \{n \in \mathbb{N} : x_n \in A\}$. If $N_0 \in I$, then $N_0 \neq \mathbb{N}$ and so $A \neq X$. Now, take a point $a \in X - A$ and define the sequence $(y_n)_{n\in\mathbb{N}}$ in X by

$$y_n = \begin{cases} a & \text{if} \quad n \in N_0 \\ x_n & \text{if} \quad n \notin N_0 \end{cases}.$$

By Lemma 2.7, the sequence $(y_n)_{n\in\mathbb{N}}$ b-I-converges to x. We can see that $X - A$ is b-I-closed and $(y_n)_{n\in\mathbb{N}} \subseteq X - A$, in consequence $x \in X - A$ and this is a contradiction. Therefore, $N_0 \notin I$.

The implication $(2) \Rightarrow (3)$ it follows form the notion that the ideal I is admissible.

Now, it shows the following implication. $(3) \Rightarrow (1)$: Let A not be b-I-open in X. Then, $X - A$ is no b-I-closed and there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X - A$ with $x_n \to^{bI} x \in A$ and this is a contradiction.

Theorem 2.1. *Every b-I-sequential space is hereditary with respect to b-I-open (b-I-closed) subspaces.*

Proof: Let X be a b-I-sequential space. Now, suppose that A is a b-I-open set of X. Then, A is b-open in X. Now, we can see that A is b-I-sequential. Let V be a pre-I-open set in A, thus V is b-open in X. Indeed, by the Definition 2.7, if we show that V is b-I-open in X, it will be sufficient.

Now, suppose that there is a point $y \in X - V$ and take an arbitrary $x \in V$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to^{bI} x$ in X. Since, A is b-open in X and $x \in A$, the set $\{n \in \mathbb{N} : x_n \notin A\} \in I$. We define the sequence $(y_n)_{n\in\mathbb{N}}$ in X by

$$y_n = \begin{cases} x_n & \text{if} \quad x_n \in A \\ y & \text{if} \quad x_n \notin A \end{cases}.$$

By the Lemma 2.7, the sequence $(y_n)_{n\in\mathbb{N}}$ b-I-converges to x. Since $|\{n \in \mathbb{N} : x_n \notin V\}| = |\{n \in \mathbb{N} : y_n \notin V\}|$ and by the Lemma 2.8, V is b-I-open in X.

Now, let A be a b-I-closed set of X. Then, A is pre-closed in X. For any b-I-closed set J of A. It has to show that J is b-closed in X. Since X is a b-I-sequential space, it is enough that J is b-I-closed in X. Hence, let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in J with $x_n \to^{bI} x \in X$. It obtains that $x \in J$. Indeed, since A is b-closed, it has that $x \in A$ and so $x \in J$ since J is a b-I-closed set of A.

Theorem 2.2. *b-I-sequential spaces are preserved by topological sums.*

Proof: Let $\{X_\delta\}_{\delta\in\Delta}$ be a family of b-I-sequential spaces. Take $X = \bigoplus_{\delta\in\Delta} X_\delta$, being the topological sum of $\{X_\delta\}_{\delta\in\Delta}$. Now, it will show that the topological sum is a b-I-sequential space. Let J be a b-I-closed set in X. For each $\delta \in \Delta$, since X_δ is b-closed in X, $J \cap X_\delta$ is b-I-closed in X. We can see that $J \cap X_\delta \subseteq X_\delta$ and $J \cap X_\delta$ is b-I-closed in X_δ. By the assumption, it has that $J \cap X_\delta$ is b-closed in X_δ. By the definition of topological sums, it gets that J is b-closed in X. Therefore, the topological sum X is a b-I-sequential space.

Remark: The union of a family of b-I-open sets of a topological space is b-I-open. Therefore, the intersection of finitely many sequentially b-I-open sets is sequentially b-I-open.

Definition 2.8. *(cf. [7]) Let I be an ideal on $\mathbb{N}$ and A be a subset of a topological space X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is I-eventually in A if there is $B \in I$ such that, for all $n \in \mathbb{N} - B$, $x_n \in A$.*

Proposition 2.1. *Let I be a maximal ideal on $\mathbb{N}$ and X be a topological space. Then, A is a subset of X where A is b-I-open if and only if each pre-I-convergent sequence in X, converging to a point of A is I-eventually in A.*

Proof: Let A be a b-I-open and $x_n \to^{bI} x \in A$. Since I is maximal, by the Lemma 2.8, $B = \{n \in \mathbb{N} : x_n \notin A\} \in I$. Therefore, for each $n \in \mathbb{N} - B, x_n \in A$, i.e., the sequence $(x_n)_{n\in\mathbb{N}}$ is I-eventually in A.

Theorem 2.3. *Let I be an ideal of $\mathbb{N}$ and X be a topological space. If V, W are two b-I-open sets of X, then $V \cap W$ is b-I-open.*

Proof: It will be shown that every b-I-convergent sequence converging to a point in $V \cap W$ is I-eventually in it. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to^{bI} x \in V \cap W$. There are $A, S \in I$ such that for each $n \in \mathbb{N} - A, x_n \in V$ and for each $n \in \mathbb{N} - S, x_n \in W$. Since $A \cup S \in I$ and for each $n \in \mathbb{N} - (A \cup S), x_n \in V \cap W$, therefore $V \cap W$ is a b-I-open set.

3. Further properties.

3.1. b-I-irresolute functions. In this part, it is introduced b-I-irresolute functions and it shows some relations among continuous and b-I-continuous functions.

Definition 3.1. *(cf. [3]) Let $f : (X, \tau) \to (Y, \sigma)$ be a functions. f is called sequentially continuous provided V is sequentially open in Y, then $f^{-1}(V)$ is sequentially open in X.*

Definition 3.2. *Let I be an ideal on $\mathbb{N}$, $(X, \tau), (Y, \sigma)$ be a topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function, then.*

1. *f is said to be preserving b-I-convergence provided for each sequences $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to^{bI} x$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ b-I-converges to $f(x)$.*
2. *f is said to be b-I-irresolute if for each b-I-open V in Y, then $f^{-1}(V)$ is b-I-open in X (cf. [2]).*

Lemma 3.1. *(cf. [2]) Every b-I-irresolute function is b-I-continuous.*

Theorem 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is continuous, then f preserves b-I-convergence.*

Proof: Suppose that f is continuous and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to^{bI} x \in X$. Now, let V be an arbitrary semi-neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(V)$ is a semi-neighbourhood of x. Therefore, it has that $\{n \in \mathbb{N} : x_n \notin f^{-1}(V)\} \in I$. We can see that $\{n \in \mathbb{N} : f(x_n) \notin V\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(V)\}$. This implies that $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$. Hence, $f(x_n) \to^{bI} f(x)$.

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves b-I-convergence, then f is b-I-irresolute.*

Proof: Suppose that f preserves b-I-convergence and J is an arbitrary b-I-closed set in Y. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $f^{-1}(J)$ such that $x_n \to^{bI} x \in X$ .By the assumption, it has that $f(x_n) \to^{bI} f(x)$. Since $(f(x_n))_{n\in\mathbb{N}} \subseteq J$ and J is b-I-closed in Y, hence $f(x) \in J$, i.e., $x \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is b-I-closed in X and then f is b-I-irresolute.

Proposition 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves b-I-convergence, then f is b-I-continuous.*

Proof: The proof is followed by the Lemma 3.1 and Theorem 3.2.

Theorem 3.3. *Let I be anideal on $\mathbb{N}$. Then, a function $f : (X, \tau) \to (Y, \sigma)$ is b-I-irresolute if and only if it preserves b-I-convergent sequences.*

Proof: Assume that f is b-I-irresolute and a sequence $x_n \to^{bI} x$ in X. It has to show that $f(x_n) \to^{bI} f(x)$ in Y. Now, let V a semi-neighbourhood of $f(x)$. Then, $x \in f^{-1}(V)$ is b-I-open in X, because V is b-I-open in Y. Hence, there is $B \in I$ such that for all $n \in \mathbb{N} - B, x_n \in f^{-1}(V)$. This means that for all $n \in \mathbb{N} - B, f(x_n) \in V$. Therefore, $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$ and hence $f(x_n) \to^{bI} f(x)$.

Theorem 3.4. *Let X be a b-I-sequential space and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent.*

1. *f is continuous.*
2. *f preserves b-I-convergence.*
3. *f is b-I-irresolute.*

Proof: $(1) \Leftrightarrow (2)$ was proved in the Theorems 3.1 and 3.2.

$(3) \Rightarrow (1)$: Let f be b-I-irresolute and J be an arbitrary b-closed set in Y. Then, J is b-I-closed in Y. Since f is b-I-irresolute, $f^{-1}(J)$ is b-I-closed in X. By assumption, it has that $f^{-1}(J)$ is b-closed in X. Therefore, f is continuous.

Proposition 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and X be a b-I-sequential space. Then, the following statements are equivalent.*

1. *f is continuous.*
2. *f is b-I-continuous.*

Proof: The proof is followed by the Proposition 3.1 and Theorem 3.4.

Lemma 3.2. *Let X be a b-I-sequential space, then the function $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is sequentially continuous.*

Proof: Let X be a b-I-sequential space, then every b-I-closed set is closed, by [3] who proved that f is continuous if and only if f is sequentially continuous, indeed we have completed the proof.

Corollary 3.1. *Let X be a b-Isequential space and for a function $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.*

1. *f is continuous.*
2. *f preserves b-I-convergence.*
3. *f is b-I-continuous.*
4. *f is sequentially continuous.*

Proof: $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ was proved in the Theorem 3.4, by the Lemma 3.2, we have $(1) \Leftrightarrow (4)$.

Lemma 3.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and X be a b-I-sequential space. Then, the following statements are equivalent.*

1. *f is sequentially continuous.*
2. *f is b-I-continuous.*

Proof: The proof is followed by the Proposition 3.2 and Corollary 3.1.

3.2. b-I-irresolute and b-I-covering functions. Continuity and sequentially continuity are ones of the most important tools for studying sequential spaces on [6]. In this part, it is defined the concept of b-I-covering functions and it is shown some of their properties.

Definition 3.3. *(cf. [3]) Let $f : (X, \tau) \to (Y, \sigma)$ be a topological space. Then, f is said to be sequentially continuous provided $f^{-1}(V)$ is sequentially open in X, then V is sequentially open in Y.*

Definition 3.4. *(cf. [3]) Let $f : (X, \tau) \to (Y, \sigma)$ be a topological space. Then, f is said to be sequence-covering if, whenever $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y covering to y in Y, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to x$.*

Definition 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, f is said to be b-I-covering if, whenever $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y, b-I-converging to y in Y, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to^{bI} x$.*

Theorem 3.5. *Every b-I-covering function is b-I-irresolute.*

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a function and f be a b-I-covering function. Now, assume that V is a non-b-I-closed in Y. Then, there exits a sequence $(y_n)_{n\in\mathbb{N}} \subseteq V$ such that $y_n \to^{bI} y \notin V$. Since f is b-I-covering, there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to^{bI} x$. Now, we can see that $(x_n)_{n\in\mathbb{N}} \subseteq f^{-1}(V)$ and so $x \notin f^{-1}(V)$, therefore $f^{-1}(V)$ is non-b-I-closed. In conclusion, f is b-I-irresolute.

Theorem 3.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, the following statements hold.*

1. *If X is a b-I-sequential space and f is continuous, then Y is a b-I-sequential space and b-I-irresolute.*
2. *If Y is a b-Y-sequential space and f is b-I-irresolute, then f is continuous.*

Proof:

1. Let X be a b-I-sequential space and f be continuous. Suppose that V is b-I-open in Y. Since f is a continuous function and X is a b-I-sequential space, take an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to^{bI} x \in f^{-1}(V)$ in X. Since f is a continuous function, by the Theorem 3.1, $f(x_n) \to^{bI} f(x) \in V$. Now, since V is pre-I-open in Y and by the Lemma 2.8, it has that $|\{n \in \mathbb{N} : f(x_n) \in V\}| = \theta$, i.e., $|\{n \in \mathbb{N} : x_n \in f^{-1}(V)\}| = \theta$. Therefore, $f^{-1}(V)$ is b-I-open in X.

Now, assume that $V \subseteq Y$ such that $f^{-1}(V)$ is b-I-open in X. Then, $f^{-1}(V)$ is a open set of X since X is b-I-sequential space. as well know that f is continuous, then V is open in Y. Hence, f is continuous.

2. Let Y be a b-I-sequential space and f be b-I-irresolute. If $f^{-1}(V)$ is a open set of X, then $f^{-1}(V)$ is a b-I-open set of X. Since f is b-I-irresolute, V is a b-I-open set of Y. Now, we know that Y is a b-I-sequential space and so V is an open set of Y. Therefore, f is continuous.

By the Theorems 3.4 and 3.6 it is had the following result.

Corollary 3.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function, then f is continuous if and only if f is b-I-irresolute and Y is a b-I-sequential space.*

3.3. b-I-Fréchet-Urysohn spaces. A topological space X is said to be Fréchet-Urysohn (cf. [4]) if for each $A \subseteq X$ and each $x \in Cl(A)$, there is a sequence in A converging to the point x in X. Now, in this part, it introduces the notion of b-I-Fréchet-Urysohn and it shows a short result.

Definition 3.6. *Let (X,τ) be a topological space. Then, X is said to be b-I-Fréchet-Urysohn or simply b-I-FU, if for each $A \subseteq X$ and each $x \in bCl(A)$, there exits a sequence in A b-I-converging to the point $x \in X$.*

Lemma 3.4. *For two ideals I and J on $\mathbb{N}$ where $I \subseteq J$, if X is a b-I-FU-space, then it is a b-J-FU-space.*

Proof: Let A be a subset of X and $x \in bCl(A)$. Since X is a b-I-FU-space, then there exits a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n \to^{bI} x$, in consequence $x_n \to^{bI} x$ in X, and so X is b-J-FU-space.

Theorem 3.7. *Let (X,τ) be a topological space. Then, X is a b-I-FU-space, then X is a b-I-sequential space.*

Proof: Let $\{A_\delta : \delta \in \Delta\}$ be a family of b-I-closed subsets of X where $\delta \in \Delta \in X$, since X is a b-I-FU-space, by the Definition 3.6 $A_\delta \subseteq X$ and each $x \in bCl(A_\delta)$. Now, since A_δ is b-I-closed $bCl(A_\delta) = A_\delta \in Cl(A)$, but by the Definition 3.6, there exits a b-I-converging to the point $x \in bCl(A) \in Cl(A) \in X$, therefore $\{A_\delta : \delta \in \Delta\}$ is a closed set of X. In consequence X is a b-I-sequential space.

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