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Khalil Conformable fractional derivative and its applications to population growth and body cooling models

Derivada Conformable de Khalil y sus aplicaciones a modelos de crecimiento poblacional y enfriamiento de los cuerpos

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Abstract
The objective of this article is to develop some results on conformable fractional derivatives, specifically the one known as Khalil's conformable fractional derivative. Its origin, properties, comparisons with other fractional derivatives and some applications on population grow and Newton law of cooling models are studied.
Keywords . Fractional derivatives, Khalil fractional derivative, fractional differential equations.

Resumen
El objetivo de este artículo es desarrollar algunos resultados sobre derivadas fraccionarias conformes, específicamente la conocida como derivada fraccionaria conforme de Khalil. Se estudia su origen, propiedades, comparaciones con otras derivadas fraccionarias y algunas aplicaciones en modelos de crecimiento poblacionales y la ley de enfriamiento de Newton.
Palabras clave. Derivadas fraccionarias, derivada fraccionaria de Khalil, ecuaciones diferenciales fraccionarias.

1. Introduction. Historically it is known that the origin of the Fractional Calculus begins with correspondences between Leibniz and L'Hôpital [10], the first answering the question of whether it was possible that the order of the derivative was of fractional type in the expression

$$\frac{d^n y}{dx^n}.$$

Clearly, this question is paradoxical, understanding that the order of the derivative indicates the number of times in which the differential operator is applied, that is, it counts the repetition of the differentiation process in its entirety.

After this origin, outstanding mathematicians added contributions, among them, Euler, Lacroix, Laplace and others. One of the most interesting, extensive and well-founded works is the one by Riemman-Liouville [12]; the corresponding fractional model of it starts with two forms of fractional differential operators

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha+1-n}}dt, \quad n-1<\alpha<n\in\mathbb{N}, \tag{1.1}$$

$$D^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}}dt, \quad n-1<\alpha<n\in\mathbb{N}, \tag{1.2}$$

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where $\alpha > 0, x > a, \alpha, a, b, x \in \mathbb{R}$, and in the case $\alpha = n \in \mathbb{N}$ then

$$D^n_{a+}f(x) = \frac{d^n f(x)}{dx^n} \quad \text{and} \quad D^n_{b-}f(x) = (-1)^n \frac{d^n f(x)}{dx^n}.$$

Over time, other researchers such as Weyl [19] and Riesz [13] proposed models which differ from the previous one in the kernel used in the integrand

$$(I^\alpha_\pm f)(x) = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{\infty}\frac{e^{ik(x-y)}}{(\pm ik)^\alpha}\right)f(y)dy \quad (Weyl),$$

$$(I^\alpha f)(x) = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\frac{f(y)}{|x-y|^{1-\alpha}}dy \quad (Riesz).$$

From the beginning of the 20th century, the Fractional Calculus has been developed extensively with multiple publications in which new definitions are proposed, among them, that of Michelle Caputo:

$$({}^C D^\alpha_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,$$

$$({}^C D^\alpha_{b-}f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt.$$

In this definition the considered functions are absolutely continuous, α can be complex with a real part greater than or equal to zero. Completing this Caputo definition states that

$$({}^C D^\alpha_{a+}f)(x) = f^{(n)}(x) \quad \text{y} \quad ({}^C D^\alpha_{b-}f)(x) = (-1)^n f^{(n)}(x).$$

Interested readers in knowing more about the historical development of the Fractional Calculus can consult the article by B. Ross [14].

These fractional derivatives have limitations and strengths. In the case of the Riemman-Liouville fractional derivative, it is not necessary for the considered function to be differentiable and it could even be non-continuous at the origin or at the starting point a or at the terminal b; in the case of the Caputo fractional derivative, it is known that it allows the use of initial or boundary conditions in differential equation models. On the other hand, as constraints, the Riemman-Liouville derivative of a constant is not zero and the Caputo derivative requires strong differentiability conditions and therefore both of continuity for the considered function.

New definitions of fractional derivatives have emerged in the second half of the last century and have been studied from a theoretical point of view and their corresponding applications. These are called ”conformable”. Among these are the conformable fractional derivative of Khalil [8] and Atangana [2]. Several investigations have been carried out by other authors in this direction [16]. Other investigations have directed their goals towards fractional derivatives with kernels involving special functions [5, 6, 17, 18].

The objective of this paper is to expose the definition of the Khalil conformable derivative, develop the basic properties that it fulfills and give some examples of applications in population growth and body cooling models.

2. Preliminaries. From Differential Calculus we know that the derivative of a function f at a given point $x = a$ of its domain is defined by the following limit

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

if it limits exists, and we say that f is differentiable function on $x = a$, using the notation $f'(a)$.

It is known that if f is differentiable at this point then it is also continuous at the same point.

In [8] we find a simpler and more efficient answer to the question: is it possible to define a fractional derivative, that is, of order , by means of a limit? The authors define the following model for such a response.

Definition 2.1. *Given a function $f : [0,\infty) \to \mathbb{R}$ then the conformable fractional derivative of order $\alpha \in (0, 1]$ is defined by*

$$(T_\alpha f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$. We will use the notation $f^{(\alpha)}$ instead of $T_\alpha f$. If the limit exist for some $x = t > 0$ then it is said that f is $\alpha-$differentiable in that point, similarly, is said to be $\alpha-$differentiable on an interval

$[a,b] \subset [0,\infty)$ *if it is* $\alpha-$*differentiable in each point of* $[a,b]$*, and it is said that* f *is* $\alpha-$*differentiable if its* $\alpha-$*derivative exists in each point of its domain. In particular, if* f *is* $\alpha-$*differentiable on some interval* $(0,a)$ *with* $a>0$ *and* $\lim_{t\to 0^+} f^{(\alpha)}(t)$ *then*

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Note that if $h=\varepsilon t^{1-\alpha}$ then $h\to 0$ when $\varepsilon\to 0$, so we have that

$$f^{(\alpha)}(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} = t^{1-\alpha}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = t^{1-\alpha}f'(t), \qquad (2.2)$$

if f is differentiable. With Definition 2.1 and remark (2.2) we will show below some results of interest for this fractional derivative. We start with the well-known property that the differentiability of a function at a point implies its continuity in the neighborhood from said point.

Theorem 2.1. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a* $\alpha-$*differentiable function in a point* $t=a>0$ *and* $\alpha\in(0,1]$, *then* f *is continuous in* $t=a$.

Proof: Let the difference $f(a+\varepsilon a^{1-\alpha})-f(a)$; multiplying and dividing by $\varepsilon\ >0$, then we have

$$f(a+\varepsilon a^{1-\alpha})-f(a) = \frac{f(a+\varepsilon a^{1-\alpha})-f(a)}{\varepsilon}\cdot\varepsilon.$$

It follows that

$$\begin{aligned}\lim_{\varepsilon\to 0} f(a+\varepsilon a^{1-\alpha})-f(a) &= \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon a^{1-\alpha})-f(a)}{\varepsilon}\cdot\lim_{\varepsilon\to 0}\varepsilon\\ &= f^{(\alpha)}(a)\cdot\lim_{\varepsilon\to 0}\varepsilon.\end{aligned}$$

If we make then change $h=\varepsilon a^{1-\alpha}$, the we can observe that $h\to 0$ when $\varepsilon\to 0$, so we have

$$\lim_{h\to 0} f(a+h)-f(a) = f^{(\alpha)}(a)\cdot 0 = 0,$$

i.e., f is continuous in $t=a$. □

The following results show the conformable derivatives of a constant function and the potential function.

Theorem 2.2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a function such that* $f(t)=k\in\mathbb{R}$ *where* k *is arbitrary an fixed, and* $\alpha\in(0,1]$. *Then* $f^{(\alpha)}(t)=0$ *for all* $t>0$.

Proof: From the definition of the function we have

$$f(t+\varepsilon t^{1-\alpha}) = k \text{ y } f(t)=k,$$

for all $t>0$. Using definition 2.1 we obtain that

$$f^{(\alpha)}(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{0}{\varepsilon} = 0,$$

which is the desired result. □

Theorem 2.3. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a function defined by* $f(t)=t^p, p\in\mathbb{R}$, *y* $\alpha\in(0,1]$. *Then* $f^{(\alpha)}(t)=pt^{p-\alpha}$ *for all* $t>0$.

Proof: Using the remark (2.2) we have

$$f^{(\alpha)}(t) = t^{1-\alpha}f'(t) = t^{1-\alpha}pt^{p-1} = pt^{p-\alpha}.$$

The proof is complete. □

The following result shows that the conformable fractional derivative of a linear combination of functions is the same linear combination of the conformable fractional derivatives of the considered functions (if they exist).

Theorem 2.4. *Let* $f,g:[0,\infty)\to\mathbb{R}$ *be* $\alpha-$*diferenciable functions in a given point* $t=a$, $\alpha\in(0,1]$ *and* $\theta,\beta\in\mathbb{R}$. *Then the following equality holds*

$$(\theta f+\beta g)^{(\alpha)}(t) = \theta f^{(\alpha)}(t)+\beta g^{(\alpha)}(t).$$

Proof: First we proof that for any $\alpha-$diferenciable function f in an arbitrary point $t=a>0$, and any real constant θ we have

$$(\theta f)^{(\alpha)}(a)=\theta f^{(\alpha)}(a).$$

Indeed, let $h(t)=\theta f(t)$, then

$$\begin{aligned}
h^{(\alpha)}(a) &= \lim_{\varepsilon\to 0}\frac{h(a+\varepsilon a^{1-\alpha})-h(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{(\theta f)(a+\varepsilon a^{1-\alpha})-(\theta f)(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\theta f(a+\varepsilon a^{1-\alpha})-\theta f(a)}{\varepsilon}\\
&= \theta\lim_{\varepsilon\to 0}\frac{f(a+\varepsilon a^{1-\alpha})-f(a)}{\varepsilon}=\theta f^{(\alpha)}(a).
\end{aligned}$$

On the other hand, we prove that for any two $\alpha-$differentiable functions f and g, in $t=a$ we have

$$(f+g)^{(\alpha)}(a)=f^{(\alpha)}(a)+g^{(\alpha)}(a).$$

So, if $h(t)=(f+g)(t)=f(t)+g(t)$ we have

$$\begin{aligned}
h^{(\alpha)}(a) &= \lim_{\varepsilon\to 0}\frac{h(a+\varepsilon a^{1-\alpha})-h(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{(f+g)(a+\varepsilon a^{1-\alpha})-(f+g)(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon a^{1-\alpha})+g(a+\varepsilon a^{1-\alpha})-f(a)+g(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon a^{1-\alpha})-f(a)}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{g(a+\varepsilon a^{1-\alpha})-g(a)}{\varepsilon}\\
&= f^{(\alpha)}(a)+g^{(\alpha)}(a).
\end{aligned}$$

therefore, from the previous obtained results we attained the desired result. □

Obviously the previous result is valid for functions $\alpha-$differentiable on some interval or on its entire domain.

Now, we will see the rule for the product of functions.

Theorem 2.5. *Let $f,g:[0,\infty)\to\mathbb{R}$ be two $\alpha-$differenciable functions in some point $t=a>0$, and $\alpha\in(0,1]$. Then*

$$(f\cdot g)^{(\alpha)}(a)=f^{(\alpha)}(a)g(a)+f(a)g^{(\alpha)}(a).$$

Proof: Let f and g be functions as in the statement. Then

$$\begin{aligned}
(f\cdot g)\left(a+\varepsilon a^{1-\alpha}\right)-(f\cdot g)(a) &= (f\cdot g)\left(a+\varepsilon a^{1-\alpha}\right)-(f\cdot g)(a)\\
&\quad +g\left(a+\varepsilon a^{1-\alpha}\right)f(a)-g\left(a+\varepsilon a^{1-\alpha}\right)f(a)\\
&= f\left(a+\varepsilon a^{1-\alpha}\right)g\left(a+\varepsilon a^{1-\alpha}\right)-f(a)\,g(a)\\
&\quad +g\left(a+\varepsilon a^{1-\alpha}\right)f(a)-g\left(a+\varepsilon a^{1-\alpha}\right)f(a);
\end{aligned}$$

conveniently grouping we have

$$\begin{aligned}
&(f\cdot g)\left(a+\varepsilon a^{1-\alpha}\right)-(f\cdot g)(a)=\\
&\qquad =\left[f\left(a+\varepsilon a^{1-\alpha}\right)-f(a)\right]g\left(a+\varepsilon a^{1-\alpha}\right)+f(a)\left[g\left(a+\varepsilon a^{1-\alpha}\right)-g(a)\right],
\end{aligned}$$

so, it follows

$$\begin{aligned}
(f\cdot g)^{(\alpha)}(a) &= \lim_{\varepsilon\to 0}\frac{(f\cdot g)\left(a+\varepsilon a^{1-\alpha}\right)-(f\cdot g)(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\left[f\left(a+\varepsilon a^{1-\alpha}\right)-f(a)\right]g\left(a+\varepsilon a^{1-\alpha}\right)+f(a)\left[g\left(a+\varepsilon a^{1-\alpha}\right)-g(a)\right]}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f\left(a+\varepsilon a^{1-\alpha}\right)-f(a)}{\varepsilon}\cdot\lim_{\varepsilon\to 0}g\left(a+\varepsilon a^{1-\alpha}\right)+f(a)\lim_{\varepsilon\to 0}\frac{g\left(a+\varepsilon a^{1-\alpha}\right)-g(a)}{\varepsilon}\\
&= f^{(\alpha)}(a)g(a)+f(a)g^{(\alpha)}(a).
\end{aligned}$$

The proof is complete. □

Now we will show the Khalil fractional derivative for the reciprocal multiplicative function.

Theorem 2.6. *Let* $f : [0, \infty) \to \mathbb{R}$ *be a* $\alpha-$*differenciable function in some point* $t = a > 0$ *such that* $f(a) \neq 0$, *and* $\alpha \in (0, 1]$. *Then*

$$\left(f^{-1}\right)^{(\alpha)}(a) = -\frac{f^{(\alpha)}(a)}{f^2(a)}.$$

Proof: Let f as in the statement. Then

$$\begin{aligned} f^{-1}\left(a + \varepsilon a^{1-\alpha}\right) - f^{-1}(a) &= \frac{1}{f\left(a + \varepsilon a^{1-\alpha}\right)} - \frac{1}{f(a)} \\ &= \frac{f(a) - f\left(a + \varepsilon a^{1-\alpha}\right)}{f\left(a + \varepsilon a^{1-\alpha}\right) f(a)} \\ &= -\frac{f\left(a + \varepsilon a^{1-\alpha}\right) - f(a)}{f\left(a + \varepsilon a^{1-\alpha}\right) f(a)}, \end{aligned}$$

so,

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{f^{-1}\left(a + \varepsilon a^{1-\alpha}\right) - f^{-1}(a)}{\varepsilon} &= -\lim_{\varepsilon \to 0} \frac{f\left(a + \varepsilon a^{1-\alpha}\right) - f(a)}{\varepsilon\left[f\left(a + \varepsilon a^{1-\alpha}\right) f(a)\right]} \\ &= -\lim_{\varepsilon \to 0} \frac{f\left(a + \varepsilon a^{1-\alpha}\right) - f(a)}{\varepsilon} \cdot \lim_{\varepsilon \to 0} \frac{1}{f\left(a + \varepsilon a^{1-\alpha}\right) f(a)} \\ &= -f^{(\alpha)}(a) \cdot \frac{1}{f^2(a)}, \end{aligned}$$

therefore

$$\left(f^{-1}\right)^{(\alpha)}(a) = -\frac{f^{(\alpha)}(a)}{f^2(a)}.$$

The proof is complete. □

With this last result and the multiplication rule for the conformable fractional derivative, we obtain the proof for the quotient rule.

Theorem 2.7. *Letn* $f, g : [0, \infty) \to \mathbb{R}$ *be two* $\alpha-$*diferenciable functions in some point* $t = a > 0$ *and such that* $g(a) \neq 0$, *additionally* $\alpha \in (0, 1]$. *Then*

$$\left(\frac{f}{g}\right)^{(\alpha)}(a) = \frac{f^{(\alpha)}(a)g(a) - f(a)g^{(\alpha)}(a)}{g^2(a)}.$$

Proof: Let f and g as in the statement. It is clear that

$$\frac{f}{g} = fg^{-1}.$$

Appliying Theorem 2.5 we obtain

$$\left(fg^{-1}\right)^{(\alpha)}(a) = f^{(\alpha)}(a)g^{-1}(a) + f(a)\left(g^{-1}\right)^{(\alpha)}(a),$$

and using theorem 2.6 we have

$$\begin{aligned} \left(fg^{-1}\right)^{(\alpha)}(a) &= \frac{f^{(\alpha)}(a)}{g(a)} - \frac{f(a)g^{(\alpha)}(a)}{g^2(a)} \\ &= \frac{f^{(\alpha)}(a)g(a) - f(a)g^{(\alpha)}(a)}{g^2(a)}. \end{aligned}$$

The proof is complete. □

3. Applications to Physics.

3.1. Population Grow. It is known that the population growth problem follows a differential model given by

$$\frac{dy}{dt} = ky(t), \quad y(0) = M,$$

where k is a proportionality constant, y is de population function and M is the initial value of the population.

This problem has the following solution $y(t) = Me^{kt}$.

Now, we write this problem in terms of the Khalil conformable derivative

$$y^{(\alpha)}(t) = ky(t), \quad y(0) = M.$$

From (2.2) we have

$$t^{1-\alpha}y'(t) = ky(t),$$

so, we can pass to the following expression

$$y'(t) = kt^{\alpha-1}y(t) \qquad \text{or} \qquad \frac{dy(t)}{dt} = kt^{\alpha-1}y(t).$$

Applying separation of variables we obtain

$$\frac{dy(t)}{y(t)} = kt^{\alpha-1}dt, \tag{3.1}$$

now by integration we find

$$\ln y(t) = k\frac{t^{\alpha}}{\alpha} + C,$$

applying the exponential function in both sides of the equality we get

$$y(t) = e^{C}e^{k\frac{t^{\alpha}}{\alpha}}.$$

Using the initial condition we obtain

$$y(t) = Me^{k\frac{t^{\alpha}}{\alpha}}. \tag{3.2}$$

It is clearly observed that if $\alpha = 1$ then we obtain the solution through ordinary derivatives.

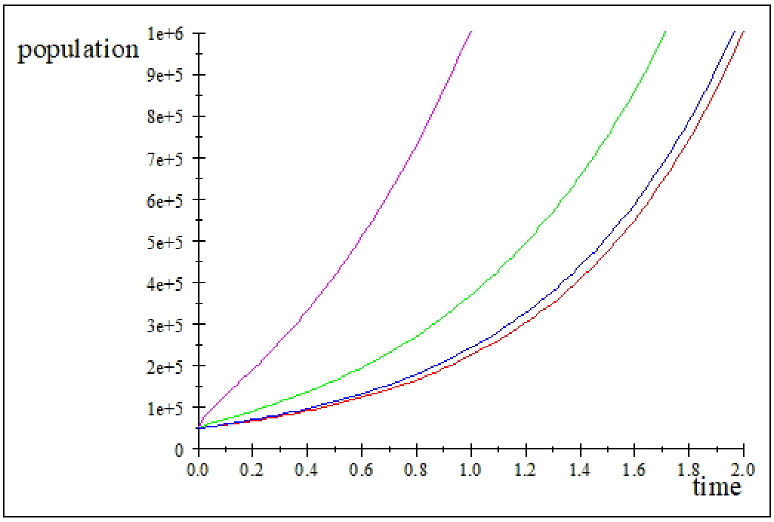

Figure 3.1: Comparative solutions for several values of α.

The figure 3.1 shows the solutions obtained using the Khalil conformable derivative for values of α: $\alpha = 0.5$ (magenta), $\alpha = 0.75$ (green) and $\alpha = 0.95$ (blue), and the solution obtained using ordinary derivatives

We can observe in the graph how for values of α close to 1, the behavior of the solutions obtained through Khalil's fractional derivative is close to the solution obtained through ordinary derivatives.

3.2. Body cooling law. Newton's law of cooling states that the rate of heat loss from a body is proportional to the temperature difference between the body and its surroundings. Heat transfer is important in physical processes because it is a type of energy that is in motion due to temperature change, for example, is present in processes of condensation, vaporization, crystallization, climatic changes and others.

Specifically, Newton's law of cooling states that the cooling of a body is directly proportional to the difference between the initial temperature of a body $T(t), t > 0$, and that of the environment T_a, and follows the following differential model

$$\frac{dT}{dt} = -k\left(T(t) - T_a\right), \;\; T(0) = T_0.$$

where T_0 is the initial temperature in $t = 0$. The solution shows how cooling of a body follows a law of exponential decay of the form

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

The following figure 3.2 shows this solution

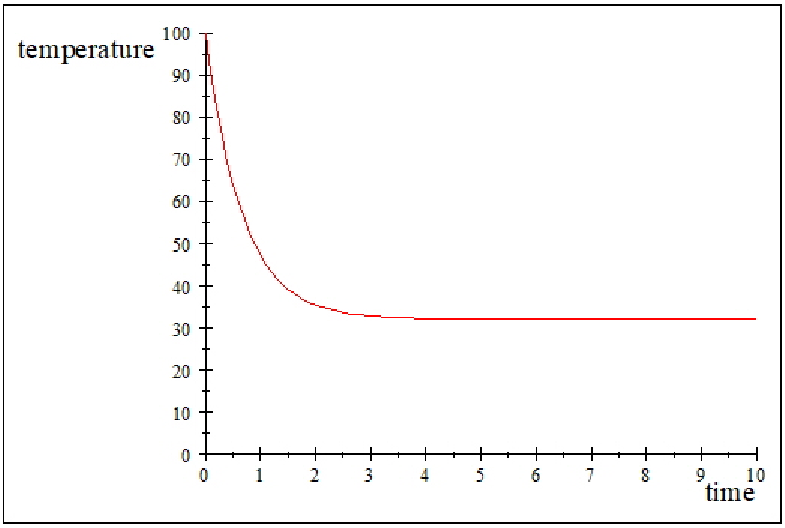

Figure 3.2: Graph of differential equations solutions with ordinary derivative.

Again, using (2.2) we obtain a fractional differential equation for this problem

$$\frac{d^{\alpha}T(t)}{dt^{\alpha}} = -k\left(T(t) - T_a\right),$$

so,

$$t^{1-\alpha}T'(t) = -k\left(T(t) - T_a\right) \Rightarrow \frac{dT(t)}{dt} = -t^{\alpha-1}k\left(T(t) - T_a\right) \Rightarrow \frac{dT(t)}{T(t) - T_a} = -t^{\alpha-1}k,$$

then we have

$$Ln(T(t) - T_a) = \frac{-t^{\alpha}k}{\alpha} + C,$$

and using the exponential function we get

$$T(t) = T_a + e^{C}e^{-t^{\alpha}k/\alpha}.$$

Applying the initial condition we obtain

$$T(t) = T_a + \left(T_0 - T_a\right)e^{-t^{\alpha}k/\alpha}.$$

The graphical representation of this solution for the values of α: $\alpha = 0.5$ (magenta), $\alpha = 0.75$ (green), $\alpha = 0.95$ (blue) , and the solution obtained by ordinary derivatives is as follows:

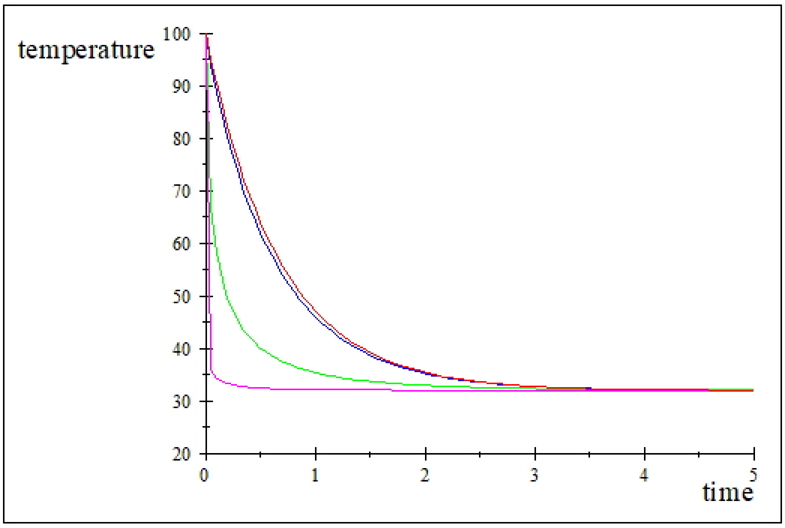

Figure 3.3: Comparative solutions for several values of α.

4. Conclusions. This informative article deals with Khalil's conformable derivative, its definition, basic properties as a conformable differential operator, and some applications to population growth problems and Newton's Law of Cooling. The contributions of this article, in terms of its applications, are not registered in any other publication. In addition, certain graphic representations are presented that show how, for different values of the parameter α, the solutions of the presented fractional differential equations approximate those obtained by means of ordinary derivatives.

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