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The didactic transposition of the fundamental theorem of calculus

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The Didactic Transposition of the Fundamental Theorem of Calculus

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Abstract

Using the tools of praxeological analysis and didactical transposition analysis, the treatments of the Fundamental Theorem of Calculus in one Norwegian, Grade 13 textbook is analysed, with a particular focus on the development of the logos block of the FTC. The terms *structure*, *functioning* and *utility*, first introduced by Chevallard in 2022, is further to describe different dimensions of the mathematical object at stake. Through the analysis, a lack in the logos relating to the concept of integrability is identified in the textbook, and consequences of this is explored in relation to a set of tasks found in the book.

Keywords: Didactic Transposition, Grade 13 Textbooks, Praxeological Analysis, the Fundamental Theorem of Calculus.

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La Transposición Didáctica del Teorema Fundamental del Cálculo

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Resumen

Utilizando las herramientas del análisis praxeológico y el análisis de transposición didáctica, se analizan los tratamientos del Teorema Fundamental del Cálculo en un libro de texto noruego de grado 13, con un enfoque particular en el desarrollo del bloque logos de la FTC. Los términos estructura, funcionamiento y utilidad, introducidos por primera vez por Chevallard en 2022, describen además diferentes dimensiones del objeto matemático en juego. A través del análisis, se identifica en el libro de texto una carencia en los logos relacionada con el concepto de integrabilidad, y se exploran las consecuencias de esto en relación con un conjunto de tareas que se encuentran en el libro.

Palabras clave: Transposición Didáctica, Libros de Texto de Grado 13, Análisis Praxeológico, el Teorema Fundamental del Cálculo.

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The movement, transformation, and incorporation of knowledge from one institution, where it is created, into the activity of other institutions (typically educational institutions) has been studied in mathematics and stem education in general (Freudenthal, 1983/2002; Bosch & Gascón, 2006) and in calculus specifically (e.g., Petropoulou et al., 2016; Strømskag & Chevallard, 2022) In the Anthropological Theory of the Didactic (ATD), this process of transposing an object of knowledge from one institution of knowledge to another institution is modelled by the concept of a *didactic transposition* (Bosch & Gascón, 2006). The transposition of the scholarly concept of integral analysis to the techniques and concepts of integrals found in upper secondary mathematics courses is an example of this. In this paper, a study and analysis of a Norwegian textbook, *Matematikk R2* (Borge et al., 2022), for upper secondary school Grade 13 (hereafter simply Grade 13) mathematics is presented, focusing on the Fundamental Theorem of Calculus (FTC) and the didactic transposition of this theme.

The textbook selected is a part of the resources produced for the recent curriculum reform in Norway, *Kunnskapsløftet 20* (Directorate of Education and Training, 2020). The reform was implemented for Grade 13 in 2022, after Grade 12 in 2021 and Grade 11 in 2020 and consists of a substantial reorganisation of the curricula and their contents. The reform introduced more specific goals for learning integral calculus. Students is now expected to be able to “account for the fundamental theorem of calculus, and account for consequences of the theorem”. The previous reform, *Kunnskapsløftet 06*, did not mention the FTC (Directorate of Education and Training, 2006).

Much weight is put on integral calculus in higher mathematics education, and students has been shown to have numerous difficulties in understanding the concept (e.g., Orton, 1983; Thompson & Harel, 2021; Burgos et al., 2021). A previous study (Topphol & Strømskag, 2022), identified a difficulty in relating the *indefinite integral* (which will be defined later) and the antiderivative, namely the definite integral, $\int_a^x f(t)dt$ (*essentially* the first part of the FTC) and showed that this difficulty could be traced back to the textbooks they had used in upper secondary. One textbook (Heir et al., 2016), the previous edition of *Matematikk R2*, written for *Kunnskapsløftet 06*, was examined specifically. *Matematikk R2* (Borge et al., 2022), was also the first textbook written for *Kunnskapsløftet 20* that was available to the author of this article.

With the advent of a new curriculum in mathematics, it is therefore of interest to investigate how the theme of integration is treated under the new curriculum. More concretely, I investigate the question of how integral calculus is presented in *Matematikk R2* (Borge et al., 2022), and what consequences there might be. Specifically, I seek to answer the questions:

1. What sort of changes have been made during the transposition of the theme of integration, and particularly the FTC, from scholarly knowledge to knowledge to be taught at Grade 13, as presented in *Matematikk R2*?
2. In case of any unused potential in the presentation of the FTC, with regard to strengthening its logos in Matematikk R2, what does this potential consist of?

Theoretical tools

This study is conducted with theoretical tools from the Anthropological Theory of the Didactic (ATD; Chevallard, 2019).

Knowledge is within the ATD modelled in terms of a praxeology, $\wp$, consisting of four components: type(s) of tasks, T, a technique, τ (or set of techniques), used to solve the tasks, a technology, θ, used to describe and explain the techniques, and a theory, Θ, used to justify the technology. The types of tasks and the techniques make up the praxis block of the praxeology, and the technology and the theory make up the logos block (Chevallard, 2019). Schematically, it is commonly written as $\wp = [T / \tau / \theta / \Theta]$.

Subscripts u (for *university*) and s (for *school* or *secondary*) respectively, are used to distinguish the praxeological elements. Thus, T_u is the types of tasks found in university mathematics textbooks, while T_s are types of tasks in the Grade 13 textbook.

The concept of a didactical transposition refers to a process, where an object of knowledge is transformed, from scholarly knowledge, through its selection by the noosphere to become knowledge to be taught, until it is actually taught, and becomes available to the students, in the teaching institutions (Chevallard & Bosch, 2014) (Figure 1).

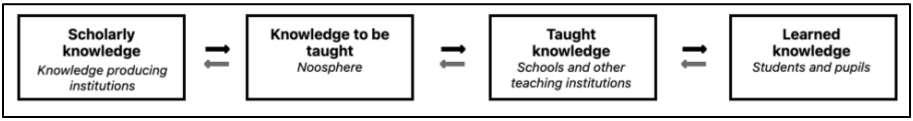


Figure 1. The Didactic Transposition Process (Adapted from Chevallard & Bosch, 2014, p. 171)

As explained by Strømskag and Chevallard:

> A praxeology $\mathscr{p}$ is usually the product of the activity of an institution or a collective of institutions I. It is often a result of an institutional transposition of a praxeology $\mathscr{p}$* living in a collective of institutions I* to a praxeology $\mathscr{p}$ that has to live within I and thus has to satisfy a set of conditions and constraints specific to I (Chevallard, 2020). This is the case when I is a collective of "didactic" institutions, that is, institutions declaring to teach some bodies of knowledge, such as secondary school for example. This is referred to as didactic transposition of I* into I. (Strømskag & Chevallard, 2022)

In the study of a mathematical object, $\mathscr{o}$, here the FTC, one can talk about the object's *structure*, *functioning*, and *utility* (Chevallard, 2022). Structure refers to what $\mathscr{o}$ consists of, or what elements the object ties together. Functioning refers to how $\mathscr{o}$ *works* to tie the elements together. Utility refers to what $\mathscr{o}$ *can be used for*. I distinguish between *intra mathematical utility*, or utility to mathematics itself, and *extra mathematical utility*, or utility to fields outside of mathematics.

Methodology

The methodological approach is a *didactic transposition analysis* (Chevallard, 1989; Chevallard & Bosch, 2014), where a reference praxeological model is constructed, and used to analyse the Grade 13 textbook (see e.g., Wijayanti & Winsløw, 2017). A *reference praxeological model* of the theme of integration is first created, a model where the researchers expose their own perspectives on the body of knowledge at hand. Then, an analysis of the Grade 13 textbook is conducted where praxeological elements are identified. At last, the reference model and the Grade 13 textbooks are compared. In all three steps I will structure the descriptions around the notions of *structure*, *functioning*, and *utility* of the mathematical object, adding to the method of Wijayanti and

Winsløw (2017). I focus mainly on the *intra mathematical utility*, in addition to structure and functioning of the mathematical object.

The reference praxeological model is partially based on *Calculus: A Complete Course* (Adams & Essex, 2018), from here on referenced to as *Calculus*. This book was chosen because of its use in many of the early mathematics courses in my own home university, the widespread international audience, and the authors' long experience in writing calculus textbooks. An article by Botsko (1991), presenting a more general form of the FTC than is found in *Calculus*, and the Norwegian calculus book *Kalkulus* (Lindstrøm, 2016), are used as supplementary sources. Because a single textbook is itself a result of a didactic transposition (Winsløw, 2022), it does not in general suffice alone as a description of scholarly knowledge.

A Reference Praxeological Model for the FTC

The FTC connects the concepts of *antiderivatives*, the *indefinite integral*, and the *definite integral*, defined as *Riemann integrals* (see e.g., Adams & Essex, 2018, pp. 302–307). By FTC establishing the *Newton-Leibniz formula*,

$$\int_a^b f(x)dx = F(b) - F(a),$$

and what I will call the *derivative-integral formula*,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

the FTC provides results that allows for calculations of areas that are not easily measured through simpler geometric means, and for doing calculations on accumulation. These results have numerous applications in other fields (for examples, see any university level calculus textbook, e.g., Adams & Essex, 2018, pp. 393–458; Lindstrøm, 2016, pp. 439–459). Through extensions and generalisations, like Fourier analysis and differential equations, based on improper integrals, it has proved indispensable in our technology-driven world.

Conditions for Riemann Integrability

Integrability and continuity are the main conditions for the FTC to work. For a function to be Riemann integrable, the integrand function must be bounded, and the upper and lower Riemann sums must exist. For an integral of a function over a closed interval, continuity is sufficient, but not a necessary condition. A detailed discussion of Riemann sums, integrability, and boundedness, can be found in *Calculus'* Appendix Sections III and IV (Adams & Essex, 2018, A-21 – A-31).

Definitions

An *antiderivative* of $f(x)$ on an interval *I*, is defined as a function, $F(x)$, such that $F'(x) = f(x)$ for all $x \in I$.

An *indefinite integral* of *f* on an interval *I*, defined as

$$\int f(x)dx = F(x) + C \quad \text{on } I,$$

where *F* is an antiderivative of *f* for all x $\in I$, and *C* is a real valued constant. The addition of the constant *C* makes it possible to use the indefinite integral to represent all antiderivatives in one expression.

A definition of the definite integral can now be stated (Adams & Essex, 2018, p. 304):

> Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have
>
> $$L(f, P) \leq I \leq U(f, P).$$
>
> Then we say that the function f is **integrable** on $[a, b]$, and we call *I* the **definite integral** of *f* on $[a, b]$. The definite integral is denoted by the symbol
>
> $$I = \int_a^b f(x)dx.$$

L and *U* are the lower and upper Riemann sums for a partition of the interval $[a, b]$. Boundedness plays a role in the existence of lower and upper Riemann sums. If the function f is not bounded on the interval, then either a lower or an upper Riemann sum cannot exist (details can be found in Adams & Essex, 2018, A-28–A-29).

Theorems which the FTC Builds on

Three theorems will be used in proving the FTC. The derivative of a constant function is zero (Theorem 13, Adams & Essex, 2018, p 142). A zero-width integral has result zero, and integrals have the *additivity property* (Theorem 3, Adams & Essex, 2018, p 308). The Mean-Value Theorem for Integrals (Theorem 4, Adams & Essex, 2018, p 310). Additivity will also prove significant as it provides a basis for a common technique used for, for example, area calculations.

The Statement of the FTC

A statement of the FTC is seen in *Calculus* (Adams & Essex, 2018, pp. 313–314):

> Suppose that the function f is continuous on an interval I containing the point a.
>
> PART I. Let the function F be defined on I by
>
> $$F(x) = \int_a^x f(t)dt.$$
>
> Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:
>
> $$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$
>
> PART II. If $G(x)$ is any antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I, we have
>
> $$\int_a^b f(x)dx = G(b) - G(a).$$

A similar statement can be found in *Kalkulus* (Lindstrøm, 2016, p. 416). Part II is there referred to as a corollary. In both treatments, continuity of the integrand is assumed both in Part I and Part II. This is also a necessary condition for the conclusion in Part I of the FTC.

However, there is a version of the FTC Part II, which is instead based on an integrand bounded on the interval of integration, allowing a countable (possibly countably infinite and possibly zero) number of discontinuities (i.e., the conditions for Riemann integrability). Such a function is called *continuous almost everywhere*. Similarly, a function *G* which is the derivative of another function *f* everywhere, except for a countable number of points is said to be *derivative of f almost everywhere*.

In other words, there exists a version of the FTC Part II which can be applied to all Riemann integrable functions (Botsko, 1991). The FTC Part II can be restated:

> PART II. If $f(x)$ is a Riemann integrable function, and if $G(x)$ is a continuous function for which $G'(x) = f(x)$ almost everywhere on *I*, then for any *a* and *b* in *I*, we have

$$\int_a^b f(x)dx = G(b) - G(a).$$

The condition that $G(x)$ is continuous is important, and the lack of this condition would have some consequences (see e.g., Pavlyk, 2008).

Now, it is not obvious why this is relevant for an upper secondary calculus textbook. I do also not expect secondary students to learn this version of the FTC. But the existence of this form of the theorem illustrates two important points. First, the difference between the two formulations of the definite integral, the definition based on Riemann sums, and the calculational formulation based on antiderivatives, often do have different conditions for their validity, in their forms expressed in typical textbooks. This difference is not always clearly communicated. And second, it illustrates one effect of the condition of boundedness. This, as will be demonstrated, is another crucial point that is not communicated in the textbook examined in this article.

Proving the FTC

A proof of Part I can be found in *Calculus* (Adams & Essex, 2018):

> Using the definition of the derivative, we calculate
>
> $$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt \quad \text{by Theorem 3(d)} \\ &= \lim_{h\to 0}\frac{1}{h}hf(c) \quad \begin{array}{l}\text{for some } c = c(h) \text{(depending on } h) \\ \text{between } x \text{ and } x+h \text{ (Theorem 4)}\end{array} \\ &= \lim_{c\to x} f(c) \quad \text{since } c \to x \text{ as } h \to 0 \\ &= f(x) \quad \text{since } f \text{ is continuous.} \end{aligned}$$
>
> Also, if $G'(x) = f(x)$, then $F(x) = G(x) + C$ on I for some constant C (by Theorem 13 of section 2.8). Hence,
>
> $$\int_a^x f(t)dt = F(x) = G(x) + C.$$

A proof of the FTC Part II, applying to all Riemann integrable functions, can be found in Botsko (1991).

Types of Tasks and Techniques from University Textbooks

The intra-mathematical utility of the FTC can be seen in the types of tasks it provides the foundation for. Based on tasks found in the two textbooks *Calculus* by Adams and Essex (2018) and *Kalkulus* by Lindstrøm (2016), seven types of tasks can be identified, and they make up the bulk of T_u:

Table 1.
Types of tasks relating to the FTC

Type of tasks	**Technique required (τ)**
t_1: Evaluate a definite integral	τ_1: - Find an antiderivative and apply Newton-Leibniz formula.
t_2: Find the area of a bounded region	τ_2: - Find all zeros of the integrand on the interval of integration. - Evaluate the definite integral over each subinterval. Negate value if the area lies below the abscissa. - Add the resulting integrals.
t_3: Derivative of functions defined by an integral with variable integration limit.	τ_3: - Apply the derivative-integral formula.
t_4: Find the average value of a function	τ_4: - Find the area of a bounded region (τ_2). - Divide by length of integration interval.
t_5: Integral equation	τ_5: - Apply the derivative-integral formula (τ_3). - Solve resulting algebraic equation.
t_6: Approximating a sum using an integral	τ_6: - Recognize the sum as a Riemann sum. - Find a non-discrete real valued function, f, corresponding to the expression in the sum. - Find an antiderivative of f and apply Newton-Leibniz formula.
t_7: Approximating/calculating an integral using a Riemann sum	τ_7: - Calculating function values for the integrand in each subinterval. - Approximate area over each subinterval using rectangles. - Approximate integral by summation of rectangles.

With the possibility of applying the FTC to discontinuous integrands, all these types of tasks can be extended. In several cases, solving the discontinuous versions do require extra techniques.

Reference example 1

Calculate the integral $\int_{-1}^{2} f(x)dx$ for

$$f(x) = \text{sign(x)} = \begin{cases} -1 \;\forall\; x < 0 \\ 0 \;\forall\; x = 0 \\ 1 \;\forall\; x > 0. \end{cases}$$

Two techniques can be used. For the first technique, observe that $F(x) = |x|$, is an antiderivative of $f(x)$ everywhere except $x = 0$, and F is continuous. Using this antiderivative,

$$\int_{-1}^{2} f(x)dx = |x|\Big|_{-1}^{2} = 2 - 1 = 1.$$

Note that this technique is the same as for tasks of type t_1. This is, however, not general, and only works for certain cases of discontinuous, bounded functions.

The second technique is more general. The interval of integration is subdivided, such that *f* is continuous on each of the subintervals. The integral is calculated over each subinterval separately, and then added, which is possible due to the additivity of integrals. This technique yields

$$\int_{-1}^{2} f(x)dx = \int_{-1}^{0} -xdx + \int_{0}^{2} xdx = -1 + 2 = 1.$$

Note the similarity between this technique and τ_2. The main difference is that one does not need to negate the area that lies below the abscissa.

Reference example 2: Examples of improper integrals

One example and one task from *Calculus* provide an interesting case. Example 6 (Adams & Essex, 2018, p. 316) starts with the function $f(x) = \frac{1}{x}$, and explains that the integral of $f(x)$ from -1 to 1 diverges. The book does not present the full argument at this point and instead refers to this integral being of a type called and *improper* integral. But the key reason for why $\int_{-1}^{1} \frac{1}{x} dx = 0$ is false, is that the function is not defined, has no limit in $x = 0$, and is not integrable on neither $[-1,0]$ nor $[0,1]$. The FTC does therefore not apply. Note the significance of the criterion of *integrability*.

The consequence of the lack of integrability is clearer in Task 49 (Adams & Essex, 2018, p. 319), a classical example (see e.g. Orton, 1983; Rubio & Gómez-Chacón, 2011). Here, the erroneous calculation

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1 + \frac{1}{-1} = -2$$

is to be criticized.

To see the solution, note first that since the function is strictly positive, the integral is also expected to be positive, and the answer -2 is clearly wrong. Also, since the function $1/x^2$ is not defined at $x = 0$, and is in fact unbounded on any interval including $x = 0$, and therefore not integrable, the FTC does not apply to the interval $[-1,1]$.

Praxeological Analysis of the FTC in Matematikk R2

I here describe the treatment of the FTC found in the textbook *Matematikk R2* (Borge et al., 2022)[1, 2]. In doing so, I examine the mathematical organization of integration and the FTC. This will then be used as a foundation for the praxeological analysis. The examination therefore has a focus on how the techniques are developed and justified and then applied to tasks.

Organisation of Integration and the FCT in Matematikk R2

Aschehoug's Matematikk R2 divides the treatment of the integral into six chapters. They deal with the definite integral (Chapter 2A, pp. 90–103), numerical integration and Riemann sums (Chapter 2B, pp. 104–112), different uses of the definite integral (Chapter 2C, pp. 113–127), the FTC (Chapter 2D, pp. 128–142), methods of integration (Chapter 2E, pp. 1143–153), and some volume and surface integrals (Chapter 2F, pp. 154–169). I will mainly focus on Chapter 2A, 2B and 2D, but one example is also taken from Chapter 2F.

Two important features of the organisation of the book are the activities called *explore*[3] and *talk*[4]. These are activities intended to help students explore and talk about these themes collectively and are often placed strategically as part of the theoretical treatment of the themes. These tasks are not uniquely named, and I will therefore give them reference names here, which do not correspond to any naming found in the textbook itself.

The Logos Elements of Matematikk R2

The notions of *limits*, *continuity* and *existence* of functions are discussed in the Grade 12 mathematics textbook *Matematikk R1*, providing a foundation for integral and differential calculus (Borgan et al., 2021). I cannot provide any detailed account of this here, but it suffices to say that the treatment is based on intuitive notions of what it means for a function to *tend to* a limit, and what it means for a function or a value to *tend to infinity*. The distinction of bounded and unbounded functions is not made, but different sorts of discontinuity are discussed and demonstrated.

The Definition of the Definite Integral

Chapter 2A starts with an *explore* task (Borge et al., 2022, p. 90), named *Explore-Task 1* henceforth. The students are tasked with examining the area under two graphs by making a lower and upper approximation of the area under $f(x) = x^2$ and $g(x) = 5x$, using rectangles with equal width. The terms *upper* and *lower staircase sums*, a simplification of Riemann sums, not to be confused with the step function, are then defined. "The collected area of the rectangles below and above the graph we call a *lower staircase sum*, *N*, and an *upper staircase sum*, *Ø*, respectively" (Borge et al., 2022, p. 91). The true area under the graph lies between these two staircase sums.

Then, a definition of the integral based on staircase sums is presented. Starting with an area, *A*, between the values $x = a$ and $x = b$, and under the graph of a continuous function *f*, defined on the interval $[a, b]$, on which $f(x) \geq 0$ for all $x \in [a, b]$ (see Figure 2). The interval is divided in n equal subintervals. Points from $x_0 = a$ to $x_n = b$, with distance $\Delta x = \frac{b-a}{n}$, such that $x_i - x_{i-1} = \Delta x$, are marked on the x-axis. The i-th subinterval is $[x_{i-1}, x_i]$ (see Figure 3). For one subinterval, a pair of rectangles are defined, one with height equal to the lowest function value, and one with height equal to the highest function value on the interval. This process is repeated for all *n* subintervals. To get the lower staircase sum, N_n, the smallest rectangles for each subinterval are selected, and correspondingly, the largest rectangles for the upper staircase sum, $Ø_n$.

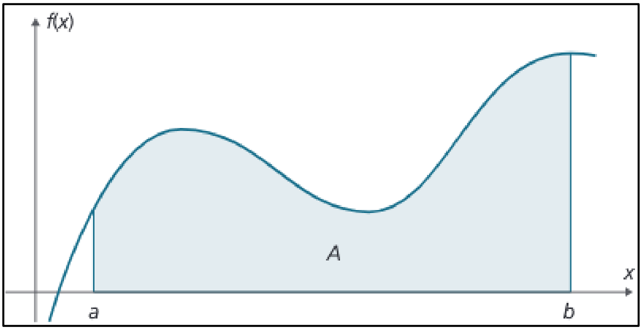


Figure 2. Area under the graph of f(x) (taken from Borge et al., 2022, p. 95)

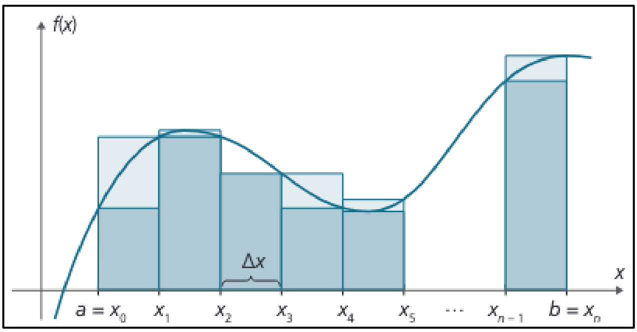


Figure 3. Subdivision of the area under the graph of f(x) (taken from Borge et al., 2022, p. 95)

Then the book explains the convergence of staircase sum:

> We let $n \to \infty$, so that $\Delta x \to 0$. We say that the sequence of staircase sums, {Nn} and {Øn}, converge towards a limit value if Nn and Øn gets closer and closer to that value when $n \to \infty$. When the two sequences converge toward the same limit, we call this limit the definite integral of f on the interval [a,b], and write $\int_a^b f(x)dx$. We read this as "the definite integral of f from a to b". (Borge et al., 2022, p. 95)

After presenting the definition, a note about integrability is given.

> If the two limits are equal, the definite integral is equal to the area A. This is always the case for continuous functions, and we say that f is integrable on the interval $[a, b]$.
> If the limits are different, f is not integrable.
> All the functions you will meet in R2 are integrable. (Borge et al., 2022, p. 96)

The note about integrability does not seem to serve much purpose, besides reassuring the students that they will not need to deal with this topic in depth, since all functions in the following sections are promised to be integrable. However, as will be seen, this is not true, for at least one case, and can potentially lead to false justifications.

Riemann Sums

In Chapter 2B, the concept of staircase sums is expanded upon to define Riemann sums. First, the selection of height of the rectangles in the interval $[x_{i-1}, x_i]$, is changed from the strictly highest and strictly lowest in the interval, to an arbitrary value. A value, $x_i^* \in [x_{i-1}, x_i]$, in each subinterval is selected, and the function value for each x_i^* is calculated. The book then notes that both the upper and lower staircase sums are on the form

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x,$$

called a Riemann sum.

It is noted that the Riemann sums require that f is defined on a closed interval $[a, b]$, and that the n subintervals can have varying width, but in Grade 13 mathematics they will consider only cases with constant width. The fact that continuous functions are integrable is reiterated, but again without mentioning why. The definite integral is then defined as a sequence of Riemann sums:

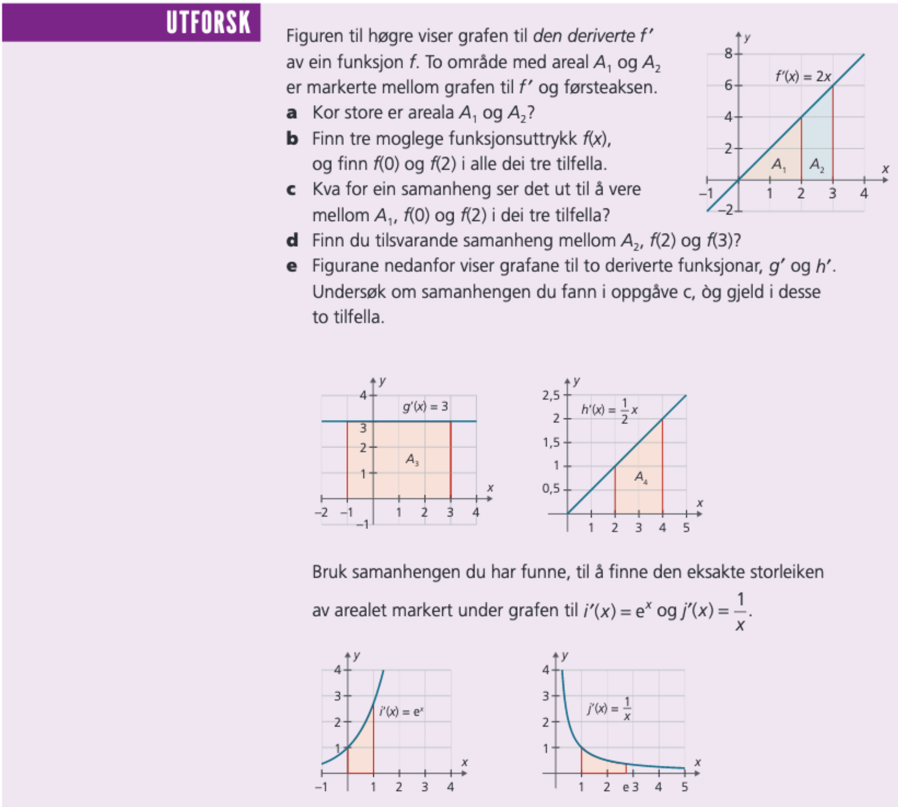
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}.$$

The following pages of Chapter 2B present different types of numerical integration, and the following Chapter 2C illustrates different uses of the integral, with a focus on techniques for area calculations.

Antiderivatives and Indefinite Integrals

Chapter 2D begins by presenting antiderivatives and indefinite integrals. First, an *explore-task* is presented (*Explore-Task 2*), where the derivative of a function $f'(x) = 2x$ is given, together with its graph (see Figure 4). Two areas under the graph, A_1 between $x = 0$ and $x = 2$, and A_2 between $x = 2$ and $x = 3$, are shown in the graph, and five tasks are given (Borge et al., 2022, p. 128):

a) How big are the two areas A_1 and A_2?
b) Find three possible $f(x)$, and calculate $f(0)$ and $f(2)$ in all three cases.
c) What connection does it appear to be between A_1, $f(0)$ and $f(2)$ in the three cases?
d) Can you find a corresponding connection between A_2, $f(2)$ and $f(3)$ in the three cases?
e) The figures below (see Figure 4) show the graphs of two derivatives $g'(x)$ and $h'(x)$. Examine whether the connection you found in task c) also holds for these two cases.
Use the same technique to calculate the exact areas under the function $i'(x) = e^x$ and under the function $j'(x) = \frac{1}{x}$.



UTFORSK

Figuren til høgre viser grafen til *den deriverte f′* av ein funksjon *f*. To område med areal A_1 og A_2 er markerte mellom grafen til *f′* og førsteaksen.

a Kor store er areala A_1 og A_2?

b Finn tre moglege funksjonsuttrykk $f(x)$, og finn $f(0)$ og $f(2)$ i alle dei tre tilfella.

c Kva for ein samanheng ser det ut til å vere mellom A_1, $f(0)$ og $f(2)$ i dei tre tilfella?

d Finn du tilsvarande samanheng mellom A_2, $f(2)$ og $f(3)$?

e Figurane nedanfor viser grafane til to deriverte funksjonar, g' og h'. Undersøk om samanhengen du fann i oppgåve c, òg gjeld i desse to tilfella.

Bruk samanhengen du har funne, til å finne den eksakte storleiken av arealet markert under grafen til $i'(x) = e^x$ og $j'(x) = \frac{1}{x}$.

Figure 4. Explore task about the indefinite integral (Borge et al., 2022, p. 128)

The technique of finding an $f(x)$ when you know $f'(x)$ is named *to find the antiderivative*. The book observes that the only difference between the three functions found in Question *b*) is a constant, justifying the introduction of a general constant, *C*. Antiderivatives are then defined: “If $K'(x) = f(x)$, we say that *K* is one antiderivative of *f*. All antiderivatives of *f* are then given as $K(x) + C$, where $C \in \mathbb{R}$.” (Borge et al., 2022, p. 129). This is called an *indefinite integral* and defined as $\int f(x)dx = K(x) + C$, where $K'(x) = f(x)$ and $C \in \mathbb{R}$. The process of finding an indefinite integral is called *to integrate*.

Note that a connection between the antiderivative, the indefinite integral, and the area under a graph is communicated, constituting an attempt at sharing the burden of the work done by the later proof of the FTC.

The Fundamental Theorem of Calculus

Another *explore-task* immediately precedes the FTC (*Explore-Task 3*). The students are given the function $f(x) = 2x + 3$ and a graph of *f*, and they are asked to use the formula of a trapezoid to explain why $F(x) = x^2 + 3x$ describes the area under the graph, from $x = 0$ to an arbitrary *x*-value greater than 0. Continuing, the students are asked to use the area function to explain why the area under the graph from $x = 2$ to $x = 5$, becomes $A = F(5) - F(2) = 40 - 10 = 30$, and why this implies

$$\int_2^5 f(x)dx = F(5) - F(2).$$

A proof, or rather a demonstration, is presented. The book does not explicitly call it a proof, but claims to be demonstrating the *carrying idea* of what could become a proof:

> We shall show the carrying idea in the proof for the Fundamental Theorem of Calculus, using the figure below (see Figure 5).

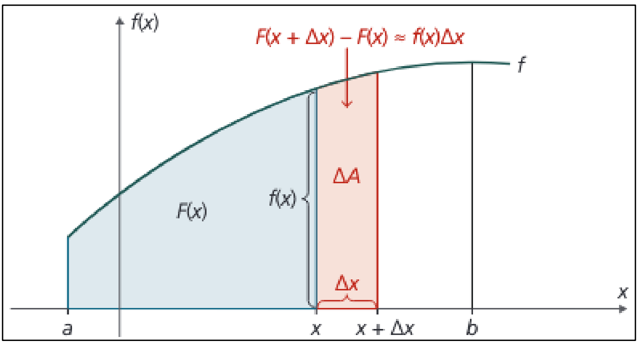


Figure 5. The definite integral of $f(x)$ from $x = a$ to $x = b$ (taken from Borge et al., 2022, p. 136)

We call the area of the blue region $F(x)$. This area corresponds to the definite integral that we defined using Riemann sums

$$F(x) = \int_a^x f(t)dt.$$

The area of the pink region, ΔA, is a small additional area. The sum of the two area corresponds to the definite integral

$$F(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt.$$

The area of only the pink region is therefore the difference between the two area above.

$$\Delta A = F(x + \Delta x) - F(x).$$

An approximation for ΔA is a rectangle of width Δx and height $f(x)$

$$\Delta A \approx \Delta x \cdot f(x).$$

As with the Riemann sums, the approximation is more accurate the narrower the rectangle is, that is, the smaller Δx is.
We set the two expressions for ΔA equal to each other, and recalculate

$$F(x + \Delta x) - F(x) \approx \Delta x \cdot f(x)$$

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} \approx f(x)$$

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

$$F'(x) = f(x).$$

So F is therefore the antiderivative of f.

> The step from the approximation value on line 2 to the limit on line 3 of the calculation on the previous page demands a formal proof, which we will not enter in R2, but this step is the carrying idea in the proof. (Borge et al., 2022, p. 136)

After this, the book provides an example and a few tasks where the FTC is used to differentiate functions defined by definite integrals. The Newton-Leibniz formula is then proved:

> Starting with the FTC, we can now develop a useful result.
> $\int_a^a f(x)dx = F(a) = 0$ because we do not have a region with area when the upper and the lower limits of the integral are equal.
> Now, let K be an arbitrary antiderivative of f. Then $F(x) = K(x) + C$. Thus
>
> $$\int_a^b f(x)dx = F(b) = F(b) - F(a)$$
> $$= (K(b) + C) - (K(a) + C)$$
> $$= K(b) - K(a)$$
> $$= [K(x)]_a^b$$
>
> Here $F(b) = F(b) - F(a)$ since $F(a) = 0$.
> $[K(x)]_a^b$ is a shorthand for $K(b) - K(a)$. (Borge et al., 2022, p 138)

Tasks and Techniques in Matematikk R2

The theory is then used as foundation for what types of tasks can be given. In the textbook, tasks of all the seven types defined in the reference model were found. The significance of this is that the technology, θ_s, seems to be relatively similar to θ_u. One significant difference can, however, be seen in three specific tasks.

One of the tasks is an *examine* tasks and one is a *talk* tasks. Since the textbook provides no solutions, students and teachers are left with the option to either argue well enough to be convinced, or to seek answers from external sources. In some cases, answers are implied in the following text, but not in all cases.

Towards the end of the section, two *examine*-tasks are given. Both are motivated simply by stating the mathematical problem, and the book does not provide any reason for the utility of the techniques demonstrated in these tasks. The second of these tasks demonstrates a technique relevant to the discussion. I call this task *Explore-Task 4.*

Explore-Task 4 shows a calculation,

$$\int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} (x - x^3) dx + \int_{1}^{2} (x^3 - x) dx = \frac{11}{4},$$

and asks why this calculation holds. Note the similarity between the technique used to solve *Explore-Task 4*, to the second technique used in *Reference Example 1*. Dividing the area of integration, as a technique, is well within the scope of the textbook, and not restricted to examples with calculations of areas. There are, however, no similar tasks later, and the task seems therefore to serve a purpose as a mathematical curiosity. The utility of the FTC to this task, and possibly similar types of tasks, is not examined.

The *talk* task, from now on called *Talk-Task 1*, that comes after the introduction of the Newton-Leibniz formula is also worth some attention (Borge et al., 2022, p. 140). Here the students are asked to discuss an erroneous result. “Discuss what is wrong with this calculation:”

$$\int_{-1}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$$

Note the similarity between this task, and Task 49 from *Reference Example 2*. *Matematikk R2* does not present any solution to the task. However, the intended solution is likely to be related to the area interpretation, given θ_s. Boundedness as a condition for integrability is not part of θ_s, and only the area analogy is present in detail in the preceding theoretical discussion. Since the task does not have a solution presented in the textbook, it is therefore unlikely that students would discover the significance of the criteria of integrability and boundedness.

Task 2.121 is a third task relevant to the discussion (Borge et al., 2022, p. 168). This task presents the famous *Gabriel’s Horn*. The function $f(x) = 1/x$ is given, and the students are asked to define the integrals of the volume and surface of revolution, $V(a)$ and $A(a)$ respectively, about the abscissa from $x = 1$ to $x = a$, where $a > 1$. Then, by letting $a \to \infty$, they are tasked with examining whether the limits $\lim_{a\to\infty} V(a)$ and $\lim_{a\to\infty} A(a)$ exist. To solve the task, the limits

$$\lim_{a\to\infty} V(a) = \lim_{a\to\infty} \pi \int_1^a \frac{dx}{x^2} = \pi,$$

and

$$\lim_{a\to\infty} A(a) = \lim_{a\to\infty} 2\pi \int_1^a \frac{1}{x}\sqrt{1 + (\ln x)^2}dx \to \infty,$$

are calculated.

This task demonstrates a type of improper integral with integration limits that tend to infinity, that is, it breaks the criterion of a closed interval of integration. It also demonstrates that some integrals of this type can be calculated to a concrete value, while others cannot. *Task* 2.121 is the only instance of such a task and seems to be another case of a mathematical curiosity. The technique used in this task is not used for anything else, nor are any later uses for the techniques mentioned.

A common theme of these three tasks is that of examining the very limits of the FTC. More specifically, they illustrate what sort of functions are permissible as integrands in a definite integral. And with the addition of *Task* 2.121, it illustrates how one can handle cases where the FTC cannot be applied directly, but where it needs to be modified in certain ways. The connection between them is, however, not explicitly made.

Elements of the Didactic Transposition

In this section I will compare the praxeological organisation found in the Grade 13 book *Matematikk R2* with the reference model. In this way I will be able to describe the didactic transposition from scholarly knowledge to the Grade 13 noosphere. I do it element by element first, and then, in the following section I discuss implications and answer the research questions concretely.

Didactical Transposition – Elements of the Logos

Comparing the demonstration of the FTC in *Matematikk R2* with the reference proof for the FTC Part I, we first see some similarities. The premises are the same, that of a continuous integrand, and they therefore have the same applicability. As the reference proof, it also starts by defining the function $F(x)$ using an integral, and both have the goal of proving that $F'(x) = f(x)$. But we do see some major differences. Whereas the reference model states the theorem formally first, *Matematikk R2* presents the proof *before* the formal statement of the theorem. As a result, it is less clear in the beginning of the proof what to expect as the end goal. By stating the goal in the beginning, the reference proof start by using the definition of the derivative, and directly show that by rewriting $F'(x)$, we will end up with $F'(x) = f(x)$.

The reference proof also bases its argument on previously proven results, which in turn are based on formal definitions, making the proofs rigorous. The argument in *Matematikk R2*, is instead based on graphical representations, and justifies the algebraic expressions it later manipulates using this graph. What it does reference, and therefore lends its legitimacy to, is the definition of the definite integral and Riemann sums. It is therefore crucial that these are defined properly for the FTC to be properly justified.

In *Matematikk R2*, there is also one major step within the proof that is not explained. For the argument to become rigorous, the step from

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} \approx f(x)$$

to the equality

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x),$$

needs to be argued. This is not done in the proof, nor does it reference any previously proven theorems. The book does, however, not claim to present a formal proof. They instead call this step the *carrying idea* and foreshadows a more complete proof to be found later in the students' journey towards knowledge.

While a rigorous approach would base the argument on previously proven theorems, founded on the formal definition of limits, *Matematikk R2* bases its argument on intuitions and algebraic manipulations. The *importance* of rigor can, however, be seen. *Matematikk R2* shows that by referring to the fact that a more rigorous proof exists, which the students will possibly encounter somewhere later along their trajectory of learning.

One structural change which is consequential, is the dependence on a correct definition of the integral. *Matematikk R2* does have a definition that is useable in most cases encountered in the textbook, but not, as claimed, in all cases. By basing the definition of a definite integral on continuous functions, and not contending with what it means for a function to be integrable, more than in a passing note, it leaves out a crucial piece of information.

Didactic Transposition – Tasks

The *intra-mathematical utility* of the FTC presented in *Matematikk R2* seems to be quite similar to that of the university textbooks. Much of the same types of tasks available in the university textbooks are also available using θ_s.
The exception is when integrability is at stake. The case of Task 49 from *Calculus* and *Talk-Task 1* in *Matematikk R2* exemplifies this. Although *Matematikk R2* has a logos that can provide support for justifying *that* the calculation fails, through arguing that the area cannot be negative, it does not have a means of explaining *why* the calculation fails. The fact that boundedness is a criterion for integrability is not discussed, and neither is the fact that it is precisely because of boundedness that a function which is continuous on a closed interval is also always integrable. A pertinent question related to this would then be "what would the students have made of the task if the interval of integration were $[0,1]$ rather than $[-1,1]$?" Certainly, the function *looks* continuous on the whole interval.

Furthermore, *Examine Task 2*, and *Task 2.121* show the use of techniques and themes that could have been useful in a more thorough treatment of integrability. The technique of dividing the area of integration into subintervals, seen in *Examine Task 2*, which can also be used for piecewise continuous functions, as seen in *Reference Example 1*, could be instrumental in providing examples of integrable non-continuous functions. In that way, the importance of boundedness could be illustrated. Task 2.121 is an example of an improper integral. The fact that this task is included, does show the willingness of the textbook to include integrals that are not proper definite integrals, but which are nevertheless *extensions* of the concept of definite integrals. With relatively few modifications, a discussion about other types of improper integrals, for example of the type where the integrand itself tends to infinity rather than the independent variable, could be included.

Thus, in these three examples, one can see a potential for a deepening of the understanding of the FTC, and particularly for the premises for its application. For that to be possible, a more precise notion of integrability is needed, which also includes the criterion of boundedness. The connection is, however, not made clear in *Matematikk R2*, and the three tasks stand as separate examples of *mathematic curiosities* rather than providing justification for further theoretical developments. The lack of this distinction in some form is therefore a major constraint.

Concluding remarks

On this background, the didactic transposition can be summarised. It is first important to note the clear similarities between the organisation seen in *Matematikk R2* and the one identified in the referenced university textbooks. The treatment of the FTC, and not only the *Newton-Leibniz* formula, allows for a broader range of tasks and techniques. In particular, the inclusion of, and focus on Part I of the FTC presents conditions that allow for a closer connection between the analogies of accumulation and area, well known to be a difficulty for students (Thompson & Harel, 2021; Burgos et al., 2021).

The main concern, however, is that the notion of integrability is undeveloped. Although the term is used once, it is never defined properly, and the condition of boundedness is never mentioned or described.

The importance of boundedness is most apparent in cases where the integrand is either *not* bounded, or the function is not continuous but still integrable. Didactic implications of boundedness, and of closure of the interval of integration, in relation to improper integrals has been examined in several publications (e.g., González-Martín & Camacho, 2004; González-Martín & Correira de Sá, 2007; Rúbio & Gómez-Chacón, 2011), showing that first-year university students have great difficulty in comprehending the importance and significance of these two criteria, and even seem to be generally unaware of this importance.

It is therefore, in my opinion, a disservice to the Grade 13 students to not discuss what significance boundedness has, while at the same time include tasks that could clearly benefit from such a discussion. It is also likely that a discussion about boundedness and integrability could strengthen the conceptions of continuity of functions in general, another area of calculus that has proved difficult conceptually for students (Hanke, 2018; Lankeit & Biehler, 2020). Thus, by not including boundedness, an important part of the FTC's utility is left out, reducing the scope of both the set of available techniques, τ_s, and types of tasks, T_s.

The observations in this study and research of the organisation of the FTC in *Matematikk R2* illustrates well the challenge of including new material in a textbook. The praxis block has clearly been strengthened by an explicit inclusion of the FTC, and not only the Newton-Leibniz formula. But with this inclusion, new challenges arrive. Because of an undeveloped notion of integrability, the textbook does not provide students with the resources to know the conditions for when the FTC can be applied, and why the conditions are as they are. The consequences can be seen in three tasks, which without a concept of integrability which includes boundedness, cannot be connected, and therefore remain as mathematical curiosities, instead of contributing to the FTC's intra-mathematical and extra-mathematical utility.

However, the choices of the textbook authors are, just as the activities of students and teachers, formed by their own conditions and constraints. In this case, through a new curriculum reform, the requirement of introducing a more concrete treatment of the FTC, a constraint, was introduced, but the underlying concepts of integrability and boundedness has not been given the same attention. And with the time constraint put on the school system (Leong & Chick, 2011; Teig et al., 2019), balancing the size and content of the curriculum, and consequently also textbooks' contents, is not an easy task. If one adds something, another thing must often go. In this case, though, I claim it is sensible to include boundedness as a criterion for integrability, since it provides both a more solid foundation for the FTC, and because of the insight it might provide into details about the concept of continuity.

Since the analysis here focuses on one textbook, two immediate questions remain. How is the concept of the FTC treated in other Grade 13 textbooks in Norway? What impact does this change in curriculum, and the consequent change in the textbooks have on students' learning and readiness for further mathematics studies? The last of these questions may only be answered in a few years, when the first students that have been taught using this textbook, under the new curriculum, have arrived at the universities.

Notes

[1]Figures taken from the textbook by Borge et al. (2022) are reproduced with permission from the publisher, Aschehoug. All figures are designed by Eirek Engmark at "Framnes Tekst & Bilde AS".

[2]All translations from Norwegian to English are made by the author of this article.

[3] *Utforsk* in Norwegian.

[4] *Snakk* in Norwegian.

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