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# Fraction Division Representation - Experience in a Teacher Education Course Focused on the Reference Unit

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## Abstract

This study focuses on the knowledge revealed and developed by Elementary Mathematics teachers, in a teacher education course related to the representation of fraction division and the flexibility of the reference unit. The teachers solved a task aimed at mobilizing (and accessing) their knowledge related to their approaches to the sense of division, representation, and reference unit regarding fraction division. The results suggest that teachers face challenges when representing and justifying fraction divisions using pictorial models, especially when the divisor is a non-unit fraction. This is based in a gap regarding the flexibility of the reference unit to which the numbers refer in their representations, as well as a challenge concerning the sense of fraction division and the different forms of representation. With this research we intend to contribute to reducing the scarcity of empirical studies in the area and the importance of this specialized teachers' knowledge to deal with this topic.

## Keywords

Specialized knowledge, teacher education, fractions, division.

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# Representación de División de Fracciones: Experiencia en un Curso de Formación Docente Centrado en la Unidad de Referencia

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## Resumen

Este estudio se centra en los conocimientos revelados y desarrollados por profesores de Matemáticas de Primaria, en un curso de formación de profesores relacionados con la representación de la división de fracciones y la flexibilidad de la unidad de referencia. Los docentes resolvieron una tarea destinada a movilizar (y acceder) a sus conocimientos relacionados con sus enfoques del sentido de la división, representación y unidad de referencia respecto a la división de fracciones. Los resultados sugieren que los maestros enfrentan desafíos al representar y justificar divisiones de fracciones usando modelos pictóricos, especialmente cuando el divisor es una fracción no unitaria. Esto se fundamenta en un vacío en cuanto a la flexibilidad de la unidad de referencia a la que se refieren los números en sus representaciones, así como un desafío en cuanto al sentido de la división de fracciones y las diferentes formas de representación. Con esta investigación, pretendemos contribuir a reducir la escasez de estudios empíricos en el área y la importancia del conocimiento de los docentes especializados para tratar este tema.

## Palabras clave

Conocimiento especializado, formación docente, fracciones, división.

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In this study, we address the understanding of fraction division. One of the reasons is due to the significant role it plays in the field of numbers, operations, and algebra (Norton & Hackenberg, 2010). The number of publications on fraction divisions has increased and although much studies address teachers' knowledge (Lo & Luo, 2012; Wahyu, et al. 2020), fewer studies address their knowledge in the context of representations and concepts. The existing studies on the representation of fraction division usually present the different types and frequencies of models designed by teachers, but do not analyze whether such's models focus on reference units; moreover, few studies have investigated the teachers' understanding of reference units and representations (Izsák, 2008; Lee, 2017).

Thus, we intend to address the teachers' knowledge of the representation of fraction division seeking to contribute to the understanding of concepts, representations, and comprehension for teaching purposes. To do so, our research is based on the following questions: How do teachers represent the division of fractions? What meaning do they attribute to their representation? How do the numbers (quantity and its representation) involved in their representations relate to the reference unit?

## Fraction Division – Sense, representation, and reference unit

Among the four operations with fractions, division is considered the most difficult and problematic for students and teachers (Fuentes & Olmos, 2019; Lo & Luo, 2012; Ma, 1999), and this may be related to the fact that the algorithm is the starting point in mathematics teaching. Many authors show that teachers have difficulties in using visual representations to explain fraction division (Borko et al., 1992; Jansen & Hohensee, 2016) and tend to use the algorithm disconnected from the sense of fraction division (Ball, 1990; Zembat, 2004).

As for the division' significance (Simon, 1993), we assume as a starting point the partition and the measurement (also referred to as quotative by some authors, such as Fischbein et al., 1985), as there are studies that present the teachers' difficulties in relation to these concepts. The concept of measurement is associated with the comparison between quantities that express magnitudes of the same nature and with quantification. In the meaning of division as measurement, the number of items in each group is known. Difficulties regarding division of fractions as measurement (Ma, 1999; Jansen & Hohensee, 2016) are related, for example, to identifying the involved quantities (i.e., the divisor and the dividend) and thinking about the relations between the associated units (Tzur & Hunt, 2015), in addition to using visual representations to solve fraction division problems (Lo & Luo, 2012).

Other researchers' approach focuses on the relative emphasis on developing the understanding of fraction division through partitioning experiments (Tzur & Hunt, 2015), resulting in fraction unit experiments. However, such experiments are quite limited (Shin & Lee, 2018), which reinforces the mistaken notion that partition conceptualization is inefficient when the divisor is a fraction (Sinicrope et al., 2002). This supports the fact that the division of fractions as partition is more difficult to understand than as measurement (Ball, 1988). The partition sense is significant in contexts in which, given a number of elements of a set (dividend), it is aimed to partition (distribute), in an equitable way, a quantity among a certain number of sets (divisor). Partition division is necessary, as it is related to other topics such as ratio and proportional reasoning (Empson et al., 2005). Those teachers whose knowledge of partition division is deficient may not understand that the operation generates a unit rate, involving iteration when the divisor is a fraction and not an integer (Jansen & Hohensee, 2016).

Difficulties related to the concepts of division affect not only teaching, but also the way teachers understand fraction division (Tirosh, 2000). Thus, developing an understanding of the concepts of partition (Ball,1990) and measurement is important, as they facilitate the understanding of fraction division tasks, especially when the divisor is a fraction, considering that it poses a greater challenge to teachers (Zembat, 2004; Tirosh, 2000; Rizvi & Lawson, 2001).

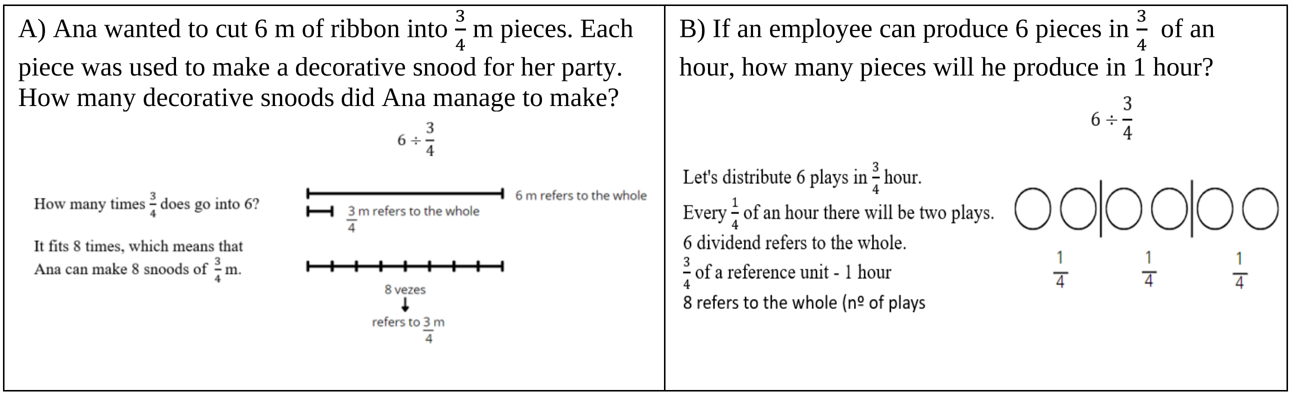
To represent fraction division using pictorial models, teachers must have knowledge regarding the reference unit flexibility, which is the ability to control the unit to which a fraction refers and change their understanding about quantity as the reference changes (Lee et al., 2011). Therefore, to represent fraction divisions, teachers must understand the units to which the numbers refer in their representations; that is, a unit is a standard for measurement, which can be an integer, e.g.: 1 in.; a part that is contained in a measurement standard, e.g., $\frac{1}{3}$ in.; or a value that contains a standard for measurement, e.g., 2 in. (Lee, 2017). Understanding these reference units is essential to solve fraction division problems and operations and produce their representations. Hence, the flexibility of the reference unit helps in understanding the representation of fractions (Lee et al., 2011; Stohlmann, et al 2019).

Representation records, such as pictorial and numerical ones, are key elements for the interpretation of division in the context of solving problems, as they contribute to the internal organization of knowledge through a cognitive process (Izsák, 2008). Therefore, it is necessary to consider the context, as it enables to interpret the division with a partition or measurement sense, that is, based on a problem, it will be possible to identify which sense of division is being evoked (Fischbein et al., 1985).

For instance, let us take operation $6 \div \frac{3}{4}$ and produce the representation, considering the partition and measurement sense to present the flexibility with which we are dealing.

**Figure 1**

*Representation of operation $6 \div \frac{3}{4}$ — measurement (A) and partition (B) senses*




In example A (Fig. 1), in the sense of measurement, we can see that the dividend and the divisor refer to the unit of 1m; therefore, they are related to the whole. We sought to find how many times $\frac{3}{4}$ fits in 6; thus, the quotient, 8, refers to $\frac{3}{4}$ and not to the whole, because Ana can make 8 ribbons of $\frac{3}{4}$ m each.

Conversely, in example B (Fig. 1), in the partition sense, we sought to divide 6 pieces in $\frac{3}{4}$ of an hour, so each ¼ will receive 2 pieces, with 8 being the total pieces produced in 1 hour. Accordingly, the dividend and the quotient refer to the whole, that is, to the total number of pieces; and the quotient is an isolated reference unit, related to 1 hour.

In the partition sense, with the divisor being a fraction, there is the process of interaction and not just partitioning, as occurs when the divisor is a natural number or a unit fraction. Hence, a unit rate is generated, once you find how much is in $\frac{1}{4}$ and then how much is in the whole. It is important for teachers to have this specialized knowledge1 so that, when doing a division, regardless of the type of divisor, they are aware of and know the processes that occur in the partition division, as well as the control of the reference unit to which a fraction refers, that is, the flexibility of the reference unit.

The ability to work using representations in teaching situations is a knowledge component that mathematics teachers must have (Turner,2008). Using multiple representations accurately demonstrates mathematical comprehension; however, teachers do not tend to use these representations in their classes (Lee, 2017) and have difficulties using representations properly when fraction division is the focus (Lamon, 1999; Lo & Luo, 2012). Even when they use representations, they do so to illustrate solutions rather than to promote the students' mathematical understanding of fractions (Izsák, 2008). We believe that a teacher's knowledge of representation must include understanding not only how to work with representations, but also the relationship between representations and conceptualizations or constructs associated with the domain (Adu-Gyamfi et al., 2019).

The textbook is often the main support for teachers, and this is reflected in the classroom, as teachers end up adopting the book's content in their practices (Wijaya & Doorman, 2015; Bütüner, 2021). In other words, usually, in terms of fraction division, practice means rules, repetitive and operative exercises. The teaching of fraction divisions is based on memorization and procedures; hence, it loses its meaning, as the concept is related to division and is confused with the algorithm itself, which is why the teaching of division is often based on rules, on the algorithm (Ma, 1999). We consider that pictorial representation can help to correct mistakes, providing a variety of partitioning experiences, especially when the unit to be partitioned is a fraction (Lamon, 1999, p. 75-109), as well as knowledge of the flexibilization of the reference unit and the fraction concept.

## Context and Method

This study focuses on the knowledge revealed and mobilized by Brazilian Elementary School teachers, who work with students aged 7 to 14 years old, through a task developed by the authors in relation to the concept, representation, and reference unit in the context of fraction divisions. The information was gathered during an online course for teachers, in two three hours meetings using google meet (and a WhatsApp group). The meetings were recorded and the teachers' productions to the tasks have been collected – the participants took photos and send them through the WhatsApp group.

Here we developed a case study with a group of teachers during the teacher education activity. The teachers did not know each other and belonged to different cities, so the group was formed by two mathematic teachers with more than five teaching years of experience (Bruno e Ana), two prospective teachers who were in the last year of their mathematics education courses (Dina e Carlos), and two primary teachers with more than three years of experience (Célia e Eva).

Within this context, participants were asked to respond to a task for teacher education conceptualized aiming at promoting the development of their knowledge regarding the representation and the concept of fraction division, enabling to carry out a qualitative analysis of the participants productions. Such analysis focused on teachers' use of representations and flexibility with the reference unit. The Task for Teacher Education (Mellone et al. 2017; Ribeiro et al., 2021) was composed of several questions contextualized in teachers' practices, but here we focus on two of those questions: questions 1) and 3) (Fig. 2 and Fig. 3).

**Figure 2.**

*Question 1 "Represent, at least in two different ways".*

| |
|---|
| 1) Represent, at least in two different ways, each of the operations a), b), c), and d). Justify your answer.<br>a) $5 \div 2$ b) $\frac{2}{5} \div 4$ c) $7 \div \frac{1}{2}$ d) $\frac{12}{15} \div \frac{3}{5}$ |

This question was intended to access (and develop) teachers' knowledge regarding the concept of differences, checking whether they were aware of different representations for fraction division, that is, different ways of externalizing a mental image as pictorial (drawings) and symbolic (known symbols). Through this task, the intention is to access the teachers' knowledge regarding fraction division in terms of representations used and ways of solving. Moreover, many studies point out teachers' difficulties in representing a fraction division by drawings, i.e., pictorially (Li & Kulm, 2008; Ma, 1999; Rizvi & Lawson, 2007).

In question 3, teachers must justify the fraction division algorithm, in addition to providing a representation for such understanding, as what is presented in the algorithm must correspond to the representation and to the mathematical language — considering that students do not understand division, although this does not indicate that they do not know how to divide, as representing is intuitive, and the algorithm is not.

**Figure 3.**

*Question 3 "What would you to from the students' comments"*

| |
|---|
| 3) The student Gabriela, from the seventh grade, when doing the task, asked:<br>— Teacher, why do we have to multiply to do a fraction division? I mean, why, when dividing a fraction, do we invert the divisor and multiply to get the result?<br>a) What would you answer Gabriela? Justify your answer.<br>b) What representation can we use to assign meaning to Gabriela's comment? |

It is important for mathematical language to be associated with what is represented. Only then will we have an effective understanding of what is done. Thus, we understand that it is important for teachers to promote meaningful teaching about understanding the concepts of both fractions and operations with fractions, using teaching strategies that favor the development of conceptual and procedural knowledge[2] of fractions (Fazio & Siegler, 2011; Wu, 1999).

The task allowed accessing and mobilizing teachers' knowledge of the representation of division with fractions, that is, if they were able to represent the requested operations, in addition to justifying them with comprehension. At first, teachers had to perform the task individually and then collectively, using whatever representations and/or fraction division concepts of their choice.

This dynamic was adopted because research shows that (prospective) teachers have better performance in posing problems that involve practice in training contexts as a group than when written tests are applied (Green et al., 2008).

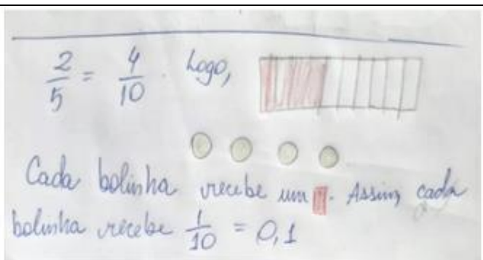
## Results

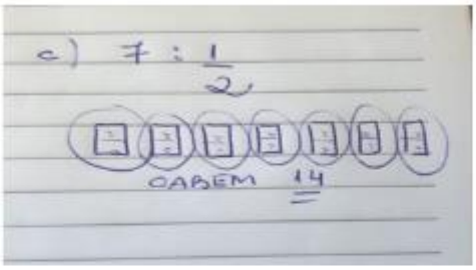
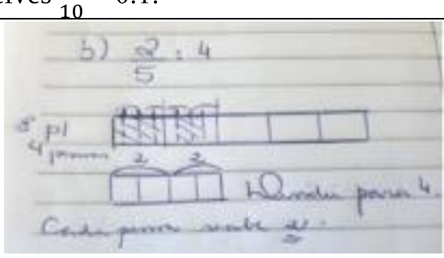
Regarding task 1 (Fig. 2), teachers were mostly and particularly successful in representing the operations a), b), and c) pictorially in rectangular form, using the partition concept as well as the invert-and-multiply algorithm (Fig. 4).

Most teachers claimed not to work with fractional operations through pictorial representation in their teaching practice and that they did not experience, as students, the representation of fraction division. Célia, one of the participating teachers, stated that: "[...] we always produce representations by drawings for fractions, but not to represent their operations."

**Figure 4.**

*Teachers' production for operations a), b) and c)*

<table>
<tr><td>

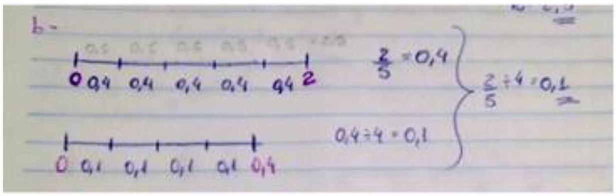
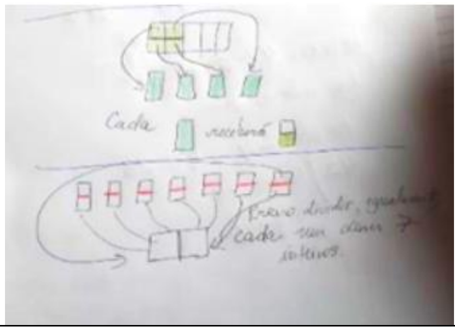
Carlos</td><td>Bruno 
</td></tr>
<tr><td>"Soon, each ball has a red part. Thus, each marble receives $\frac{1}{10}$ = 0.1."</td><td>"Fit 14"</td></tr>
<tr><td>Célia 
</td><td>Ana 
</td></tr>
<tr><td>"Divide by 4, each person gets 2."</td><td>"2 parts plus a half part of the remainder for each."</td></tr>
</table>

The teachers were unfamiliar with the meaning of partitive division; hence, they did not have a broad understanding of the division concept. The challenges encountered by them in producing pictorial representations, therefore, were related to the sense of fraction division and their own understanding of pictorial representations. This is because they did not have extensive knowledge of representations (such as continuous and discrete representations) and how to use them to help students develop mathematical concepts regarding fraction division.

For example, Dina (Fig. 5) uses decimals to represent $\frac{2}{5} \div 4$ and first represents the dividend on the number line and then the result of $0{,}4 \div 4$, taking 0.4 and dividing it into 4 parts. Note that she does not adequately present the scale in the representation. Eva (Fig. 5), for operations $\frac{2}{5} \div 4$ and $7 \div \frac{1}{2}$, uses the rectangular and continuous representation with partition sense is used by most teachers. Teachers seem more comfortable with representations in which the divisor or dividend are integers, as they mostly presented the continuous rather than the discrete model, and always in rectangular form. We perceived, therefore, that teachers have knowledge associated with representation records that consists of two ways of demonstrating a number (decimal and fractional) and their conversion.

**Figure 5.**

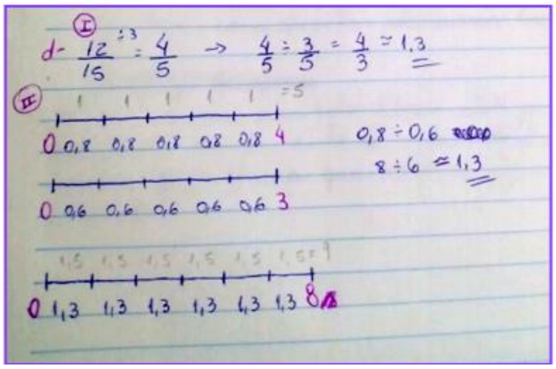
*Teachers' production (operations b, $\frac{2}{5} \div 4$, and c, 7÷ $\frac{1}{2}$)*

| Dina: | Eva: |
|---|---|
|  |  |
| | "Each green part will receive a quarter of the yellow part."<br>"I need to divide each of these integers equally." |

The operation that indicated both the numerator and the fractional denominator $\frac{12}{15} \div \frac{3}{5}$ (operation d) presented a greater challenge to teachers, in terms of relating the reference unit to the whole. They needed more time to do it and were not able, to individually, produce an accurate pictorial representation that was associated with the requested operation. Hence, they solved the fraction division only by the invert-and-multiply algorithm, often disconnected from the sense of fraction division, as already pointed out in research conducted by Ball (1990) and Zembat (2004).

**Figure 6.**

*Dina's production (operation d, $\frac{12}{15} \div \frac{3}{5}$)*



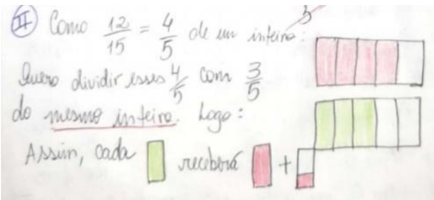
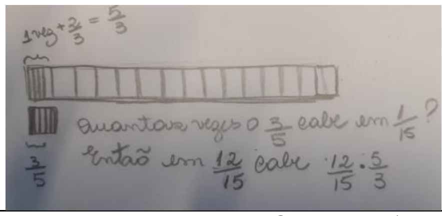

We must use the IM rule[3] when we have a fraction that is not easy to pictorially represent, such as $\frac{12}{15}$; so, it's easier to use the invert and multiply rule. It's hard to represent these numbers, so it's easier to simplify it beforehand or use the rule (Dina).

We understand, then, that the teacher uses a strategy to explain what was requested with the support of the number line; but, to do so, she makes use of a previous process, which is the simplification of fraction, because she claims to be more difficult to do the rectangular division form with fraction $\frac{12}{15}$.

It is noteworthy that this difficulty is not conceptual, nor procedural, but computational. It was only together and after the discussions conducted on the fraction division that the teachers attempted to represent the operation d, as shown in Fig. 7.

**Figure 7.**

*Carlos and Ana's productions (operation d, $\frac{12}{15} \div \frac{3}{5}$)*

| Carlos: | Ana: |
|---|---|
|  |  |
| Like $\frac{12}{15} = \frac{4}{5}$ of an integer. I want to divide $\frac{4}{5}$ with $\frac{3}{5}$. Therefore: Thus, each green part will receive one and a third of the red part | How many times does $\frac{3}{5}$ go into $\frac{1}{15}$? then in $\frac{12}{15}$ head $\frac{12}{15} \times \frac{5}{3}$ |

Carlos, despite making a partitive representation of fraction division — in his words: "I want to divide $\frac{4}{5}$ by $\frac{3}{5}$, so each one will receive it" —, when trying to explain it, he verbalizes it as follows: "I will distribute as many times as $\frac{3}{5}$ fits in $\frac{4}{5}$, and I see that it fits once and $\frac{1}{3}$ in $\frac{4}{5}$." Thus, the teacher gets confused and mixes the two concepts of division, if distributing and observing how many times it fits are synonyms. So, we perceive a weakness in the teacher's knowledge regarding the concept of fraction division, as he tries to explain his partitive representation using measurement verbalization and is unable to give an adequate final explanation of the produced representation. During the workshop, the trainer questioned the participants with the following question:

> Teacher educator: How will I divide $\frac{3}{5}$ by $\frac{4}{5}$? And how do I divide 5 by 2 or 5 by $\frac{1}{2}$? Is it possible?
> Eva: I think so, yeah, it's possible.
> Carlos: I don't know... I know that it's possible to do the division, I do it using the algorithm (Dialogue between the teacher and the research participants).

This difficulty encountered by the teacher involves the lack of flexibility of the reference unit, because, in the partitive representation, the dividend and the quotient have the same unit referring to the whole, as shown in the teacher's representation; the quotient colored in red is related to the whole, and the divisor has a different unit of measurement, the $\frac{3}{5}$ (colored in green). However, the teacher, when trying to explain his representation by verbalizing the concept of measurement (unconsciously), "it fits 1 time and $\frac{1}{3}$ in $\frac{4}{5}$," (Carlos) would have to consider that the dividend, the divisor, and the remainder would all have the same unit referring to the whole, but the quotient $\frac{4}{3}$ (1 time and $\frac{1}{3}$) would have the reference unit in relation to $\frac{3}{5}$, and not $\frac{4}{5}$, as mentioned by the teacher.

Thus, the teacher does not seem to have this flexibility of reference unit, considering that he is unable to provide an adequate final explanation for his representation, mixing the concept of measurement and partition. In addition, he seems not to have a deep knowledge of the partitive division of fractions, because, although indicating a partitive representation, he seems to be unaware of the unit rate, and does not associate the interaction with the division operation (Jansen & Hohensee, 2016; Lee, 2017).

Even though the teacher uses measurement verbalization, he is not aware of the concept of fraction division. The same occurs with the other teachers, as we can see in the statements of two of them after the group discussions during the training course:

> Célia: We usually ask how many times they fit, but without being aware if it is partition or measurement.
> Eva: We're instructed to ask this question to younger students in the teacher materials at the school where I work, but I didn't know it was because of the sense of measurement, and I believe that many teachers don't know it either.
> Dina: This is new for me, cause dividing was just partitioning, distributing, I didn't know about division as measurement, and now it makes perfect sense when thinking about fractions (Research participants' affirmations).

After discussions about the sense of division, the teachers tried to represent the fraction by the sense of measurement. According to Bruno: "with smaller numbers, it's easier; with these, I don't know if I can do it." Ana (Fig. 7) produced a representation in which she considers the reference unit in relation to $\frac{1}{15}$, and not in relation to the whole, $\frac{15}{15}$; therefore, her representation is inaccurate and inconsistent. In addition, there is the problem of scale about $\frac{3}{5}$ and $\frac{1}{15}$, because, considering the scale of numbers, such representation would be unfeasible.

When discussing representation, Bruno states that: “for me, this discussion of representation was important, cause now I understand the division, I didn’t before, I just used the IM rule.” For some teachers, the representation became very relevant with smaller numbers such as with $7 \div \frac{1}{2}$. Conversely, with $\frac{12}{15} \div \frac{3}{5}$, they deemed the understanding as more complex, but this concerns a difficulty of their own. As presented by the teachers’ statements:

> Ana: With $7 \div \frac{1}{2}$, I understood it right away and I loved it, ‘because I understood it. But with $\frac{12}{15} \div \frac{3}{5}$, I thought it was complicated.
>
> Dina: Now it’s clear for me, ‘because I use the IM, now I know how to explain it to students and, once they understand it, we use the rule to calculate these numbers (referring to $\frac{12}{15} \div \frac{3}{5}$), but then they’ll have understood why we use the IM (Research participants’ affirmations).

Hence, we observe that there is a challenge for teachers to understand the pictorial representation in relation to fractions other than the unit fraction; therefore, we observed that it is not enough to produce the representation with numerator 1 and then generalize it, as there is a difficulty in doing the geometric procedures of division by fraction, making teachers use a teaching approach that they master such as algorithms or a pictorial representation in which only the numerator is 1.

This difficulty with pictorial representation other than a unit fraction is expressed in Professor Bruno’s statement: “[…] producing the pictorial representation with $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$ is easy […] I always see this representation in textbooks, for example, but I don’t usually see this representation when the numerator is other than 1.”

Therefore, it is a challenge for teachers to teach conceptually, without ever having experienced this type of learning (Lubinski et al., 1998). To this end, we deem necessary to develop a meaningful learning of fraction division in which the answer to students is not “because it is the rule,” as stated by the teachers:

> Carlos: I’ve never thought of it that way, we always produce representation by drawings to explain fractions, but not their operations.
>
> Ana: [...] now I understand why the algorithm works and what to do with it, because I’ve always used the IM, and I’ve never stopped to think about what it meant.
>
> Bruno: Many of my students ask me why I have to do this to divide, and I’ve always said that it was the rule, because I didn’t know how to explain this to them. When I learned it, I learned it by the rule and I’ve never questioned it, I don’t remember having an explanation about the rule (Research participants’ affirmations).

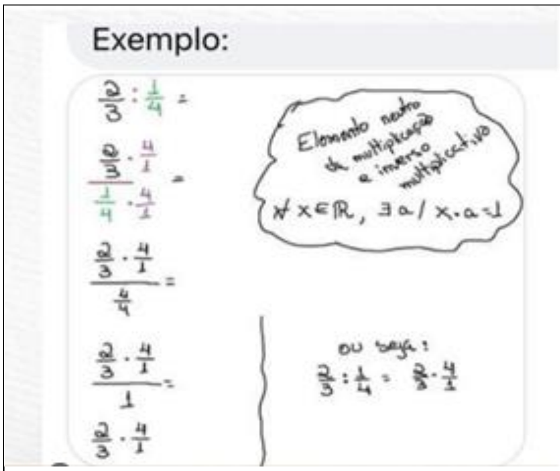
Thus, teachers demonstrated proficiency regarding procedures of fraction division, but not in their meaning, using procedures — such as the invert and multiply — without knowing why. As illustrated by Professor Bruno's statement: "for me, this discussion of representation was remarkable and important, because I'm used to the algorithm, to using the rule, only that."

The teachers, for presenting these gaps and challenges, were not able to provide an explanation for Gabriela's comment, in question 3 (Fig. 3) as well as an explanation that would attribute meaning to the student's comment or to the algorithm.

After the group reflections on these questions, we discussed the production of a teacher (not participating in the course) concerning this question to justify the IM, wondering whether this would be a good discussion to hold with students to understand the IM.

**Figure 8.**

*Production of a teacher not participating in the investigation*




"Explicar que multiplicar ou dividir o numerador e denominador da fração pelo mesmo número, a fração continua equivalente. Daí utiliza-se o inverso multiplicativo no denominador para gerar o elemento neutro da multiplicação. Qualquer número dividido por 1 dá ele mesmo, daí concluímos a 'regra'".

"Example: "Neutral element of multiplication and multiplicative inverse $\forall x \in R, \exists\ a/x.a = 1$."
"To explain that when multiplying or dividing the numerator and denominator of the fraction by the same number, the fraction remains equivalent. Then, the invert and multiply is used in the denominator to generate the neutral element of the multiplication. Any number divided by 1 is itself, hence the 'rule."

The teachers stated that this way would be more difficult for students in the final years of Elementary School to understand it, and that the best solution would be the pictorial representation, as it presents a broader understanding of the operation than the algebraic or arithmetic part.

> Carlos: For students of this age, 13 and 14 years old, it's difficult to justify with the algebraic part, because they don't have much sense and often cannot generalize it.
> Eva: I also think it's complicated with the algebraic part; they're not interested and cannot understand it; with the drawings, I think it's easier.
> Bruno: I think we can have this discussion with 7th graders, but it's complicated for most of them, as it involves arithmetic and neutral element properties. These are things that would be more complicated for them.
> Ana: For me that doesn't help much, it's just more numbers for them. I still think that the representation by drawings is the best, although I don't use it when I teach fraction division (Research participants' affirmations).

Based on the teachers' statements, we understand that the students' difficulties pointed out by the teachers are also theirs in relation to the justification and comprehension by the algebraic part and with representation other than with the unit fraction, considering that they did not use this resource to justify the IM in their productions or during group discussions.

## Conclusion

We sought to understand how the teachers who participated in the teacher education course represented and justified the different fraction divisions. The study contributes to the field by providing an initial presentation of teachers' knowledge of reference unit flexibility, sense of division, and representation by showing an analysis of how teachers make sense of fraction division using pictorial representations. It corroborates investigations that show that teachers have difficulties in using visual representations to explain fraction division (Borko et al., 1992; Jansen & Hohensee, 2016); tend to use the algorithm disconnected from the fraction division senses (Ball, 1990; Zembat, 2014) and face greater challenges when the divisor is a non-unit fraction.

Also, this study is consistent with research by Lee et al. (2011), who states that few teachers use representation to model concepts or solve problems and have difficulties with the reference unit in their representations (Lo & Luo, 2012). Thus, this study shows the relevance of knowing and understanding the sense of fraction division, the different representations, and the flexibility of the reference unit as necessary knowledge for teachers to teach fraction division with comprehension as well as to interpret the students' reasoning related to this topic (Turner, 2008; Lee et al., 2017).

In order to acquire flexibility regarding the unit of reference, teachers must have experience with multiple representations, in addition to the connection between representation and symbolic notation. Hence the importance of training courses that use tasks, such as the one presented in our study, focusing, simultaneously, on the sense of division assigned to fraction divisions and the reference unit to better understand the teachers' knowledge of these issues and their teaching applications.

Our findings suggest that reducing the gap requires both an improvement in mathematics teaching and in prospective and practice teacher education programs that address mathematical knowledge aimed at teaching, in such a way to develop proficiency in effectively using diagrams to illustrate the reasoning behind a certain solution.

Future research may focus on performing tasks for teachers to produce different representations of the same operation in different contexts, such as measurement, partition, and continuous and discrete representations, to acquire teachers' specialized knowledge so that they can contribute to teaching.

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## Notes

[1]This specialized knowledge is understood from the perspective of Mathematics Teachers Specialized Knowledge – MTSK (CARRILLO et al., 2018), which is specific to the teacher and specialized in teaching and learning mathematics.
[2]Conceptual knowledge has the meaning of fractions, their magnitudes, and the understanding of procedures. Procedural knowledge concerns undergoing a series of steps to solve the problem and is formulated by instructions (Fazio & Siegler, 2011).
[3]IM (invert and multiply) refers to the algorithm in which the fraction corresponding to the divisor is inverted, and the dividend is then multiplied by this new fraction. It refers to the algorithm expressed in the relation: abcd = abdc, where "a," "b," "c," and "d" are integers and "b" and "d" are not null.

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