A general format to find the generating function in a canonical Transformation

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Abstract

This paper gives a straightforward, elementary procedure for obtaining a generating function in canonical transformation. We hope the method suggested will help the students and the teachers to have a clear idea about canonical transformation procedure and the concept of generating function.

Keywords: Canonical transformation, Generating function, Harmonic oscillator.

Resumen

Este artículo proporciona un procedimiento sencillo y elemental para obtener una función generadora en transformación canónica. Esperamos que el método sugerido ayude a los estudiantes y profesores a tener una idea clara sobre el procedimiento de transformación canónica y el concepto de función generadora.

Palabras clave: Transformación canónica, Función generadora, Oscilador armónico.

I. INTRODUCTION

Jacobi developed a technique [1] for carrying out a transformation from (q, p) to a new set of coordinates say (Q, P) for which Hamilton's equations still hold and for which the integration of the equations of motion is trivial. Consequently, the problem of obtaining the integrated equations of motion is reduced to the problem of finding a "generating function" that will yield the desired transformation. The transformation of one set of co-ordinates and momenta (q_i, p_i) to a new set of (Q_i, P_i) are called Canonical transformation (CT). The transformation can be represented as,

$$Q_i = Q_i\,(q_i, p_i, t),$$

$$P_i = P_i\,(q_i, p_i, t). \quad (1)$$

We know

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \wedge \dot{p}_i = \frac{-\partial H}{\partial q_i}. \quad (2)$$

After the transformation (q, p) to (Q, P),

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \wedge \dot{P}_i = \frac{-\partial K}{\partial Q_i}, \quad (3)$$

where K is the new Hamiltonian, then we say that the transformation is canonical [2, 3, 4, 5]. The preservation of the canonical equations of motion assures that the solution of a dynamical problem in (q, p) will be mapped on to a solution of a dynamical problem in (Q, P) [6]. A generating function (GF) usually exists for canonical transformation [7, 8]. In the next section we will give a step by step procedure for obtaining GF followed by three examples.

II. GENERAL METHOD

1. From the Hamiltonian (H) obtain the equations for q and p.
2. Substitute these values back into the H to get the constant H.
3. Define a new convenient Hamiltonian K in terms of one new coordinate say P and find Q the new position coordinate.
4. Rearrange q and p in terms of P and Q.
5. Rearrange P and p in terms of q and Q.
6. Substitute p and P in the CT condition, $pdq - PdQ$, and find the exact differential function, which will be the generating function for the given system.

III. EXAMPLES

A. Free fall of a particle

We will follow the general procedure given above.

Step 1

Write the Hamiltonian

$$H = \frac{p^2}{2m} + mgq, \quad (4)$$

where p is the momentum, q is the position, m is the mass and g is the acceleration due to gravity. Find the equation of motion

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m},$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -mg,$$

$$\ddot{q} = \frac{\dot{p}}{m} = -g,$$

$$\ddot{q} + g = 0.$$

To find the solution we assume the solution to be

$$q = h - \frac{1}{2}gt^2, \quad (4)$$

$$\dot{q} = -\ \ddot{q} = -g.$$

So equation (4) can be a solution. So

$$q = h - \frac{1}{2}gt^2,$$

$$\dot{q} = p{-}mgt. \quad (5)$$

Step 2

Substitute q and p and find the constant Hamiltonian.

$$H = \frac{(-mgt)^2}{2m} + mg\left(h - \frac{1}{2}gt^2\right),$$

$$H = mgh.$$

Step 3

In canonical transformation, we are finding new coordinates which must be convenient in handling and also for plotting phase space diagrams. So let the new Hamiltonian be

$$K = mgP,$$

where P is the new momentum coordinate. With this Hamiltonian the phase space diagram will be a line parallel to the X-axis with Q as the new position coordinate. We can have a different form of Hamiltonian, but remember that the new Hamiltonian must be simpler than the original Hamiltonian. So let

$$K = mgh = mgP.$$

Hence $P = h = constant.$ For canonical transformation

$$\dot{Q} = \frac{\partial K}{\partial P} = mg,$$

$$Q = mgt.$$

where in the new coordinate system we take the initial displacement as zero.

Step 4

The condition for canonical transformation is [3]

$$pdq - PdQ = dF, \quad (6)$$

where F is the generating function, which must be an exact differential. So rewrite 'p' and 'q' in terms of 'P' and 'Q'. So Equation (4) becomes

$$q = P - \frac{Q^2}{2m^2g},$$

which gives

$$P = q + \frac{Q^2}{2m^2g},$$

and equation (5) becomes

$$p = -Q.$$

Substituting in (6) we get

$$dF = -Qdq - \left(q + \frac{Q^2}{2m^2g}\right)dQ,$$

$$dF = -d\left(Qq + \frac{Q^3}{6m^2g}\right),$$

which gives the generating function as

$$F = -\left(Qq + \frac{Q^3}{6m^2g}\right).$$

B. Harmonic oscillator

Step 1

For a Harmonic oscillator, the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2,$$

where ω is the angular frequency. We know the equation of motion is

$$\ddot{q} + \omega^2q = 0.$$

To find the solution let us assume the solution to be

$$q = Asin\omega t (7),$$

Then

$$p = mA\omega cos\omega t.$$

Step 2

Then we get

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}m\omega^2 A^2 = constant.$$

Step 3

Now let the new Hamiltonian be *K=P*

$$P = \frac{1}{2}m\omega^2 A^2,$$

$$P = \frac{1}{2}m\omega^2 A^2,$$

$$\dot{P} = 0.$$

For transformation to be canonical

$$\dot{Q} = \frac{\partial K}{\partial P} = 1.$$

Therefore $Q = t$.

Step 4

Rewrite 'p' and 'q' in terms of 'P' and 'Q'. So

$$q = Asin\omega Q = \left(\frac{2P}{m\omega^2}\right)^{\frac{1}{2}} sin\omega Q, \quad (8)$$

$$p = mA\omega cos\omega Q = m\omega\left(\frac{2P}{m\omega^2}\right)^{\frac{1}{2}} cos\omega Q. \quad (9)$$

On simplification we get

$$P = \frac{1}{2}m\omega^2 q^2 (csc\omega Q)^2,$$

$$p = m\omega q cot\omega Q,$$

$$pdq - PdQ = d\left(\frac{1}{2}m\omega q^2 cot\omega Q\right).$$

So the generating function is

$$F = \frac{1}{2}m\omega q^2 cot\omega Q.$$

C. A particle thrown upwards

Step 1

The Hamiltonian for a particle thrown upwards is

$$H = \frac{p^2}{2m} - mgq,$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m},$$

$$\dot{p} = -\frac{\partial H}{\partial q} = mg,$$

$$\ddot{q} = \frac{\dot{p}}{m} = g,$$

$$\ddot{q} - g = 0.$$

Let

$$q = h + \frac{1}{2}gt^2, \quad (10)$$

$$\dot{q} = gt,$$

$$\ddot{q} = g.$$

So equation (10) can be a solution. So

$$q = h + \frac{1}{2}gt^2.$$

And

$$pmgt. \quad (11)$$

Step 2

Substituting q and p, the new Hamiltonian.

$$H = \frac{(mgt)^2}{2m} - mg\left(h + \frac{1}{2}gt^2\right),$$

$$H = -mgh.$$

Step 3

Let

$$K = -mgP,$$

$$K = -mgh = -mgP.$$

Hence $P = h = constant.$

$$\dot{Q} = \frac{\partial K}{\partial P} = -mg,$$

$$Q = -mgt.$$

Step 4

So equation (10) becomes

$$q = P + \frac{Q^2}{2m^2 g},$$

which gives

$$P = q - \frac{Q^2}{2m^2g},$$

and equation (11) becomes

$$p = -Q.$$

Substituting in (6) we get

$$dF = -Qdq - \left(q - \frac{Q^2}{2m^2g}\right)dQ,$$

$$dF = -d\left(Qq - \frac{Q^3}{6m^2g}\right),$$

which gives the generating function as

$$F = -\left(Qq - \frac{Q^3}{6m^2g}\right).$$

IV. CONCLUSIONS

Usually for any classical system, the generating function is given directly in a textbook, not showing how it is obtained. This makes canonical transformation a puzzling situation for the teachers and students. We with the help of some simple exercises had shown that a generating function can be directly obtained for all problems.

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