Golden ratio and other ratios. The interaction of science and art

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Abstract
The golden ratio that appears in the Renaissance is the name given to the ratio that appears in Euclid's Elements as the division of a line into "extreme and mean ratio". The author himself did not attach much importance to this ratio, he first considered it as one of the steps in editing pentagons. In present paper, we deal with the question of whether this ratio is distinguished among the ratios, or it is simply just one of many others, and what potential it provides in physics teaching.

Keywords: Mathematics in physics, Golden ratio.

Resumen
La proporción áurea que aparece en el Renacimiento es el nombre que recibe la proporción que aparece en los Elementos de Euclides como la división de una línea en "proporción extrema y media". El propio autor no le dio mucha importancia a esta proporción; primero la consideró como uno de los pasos en la edición de pentágonos. En el presente artículo, abordamos la cuestión de si esta relación se distingue entre las razones o es simplemente una más entre muchas otras, y qué potencial ofrece en la enseñanza de la física.

Palabras clave: Matemáticas en física, Proporción áurea.

I. INTRODUCTION

"Proportion is not only found in numbers and measurements but also in sounds, weights, times, positions, and in whatsoever power there may be."

Leonardo da Vinci

In a general sense, two line segments can be in any proportion to each other, i.e. the ratio of the lengths of the two line segments can be any integer, rational or even irrational number, the ratio is determined only by the ratio of the lengths of the line segments.

For further investigations, we apply the concept of similarity, now "extended" to sections, rather than plane geometrical shapes! Two shapes are similar if they can be moved into each other by the composition of a magnification and a congruence transformation. Based on the definition, each line segment is therefore similar to the other (e.g. like all regular polygons). If, on the other hand, we interpret the similarity not between two but at least three line segments, then we can speak of "group similarity".

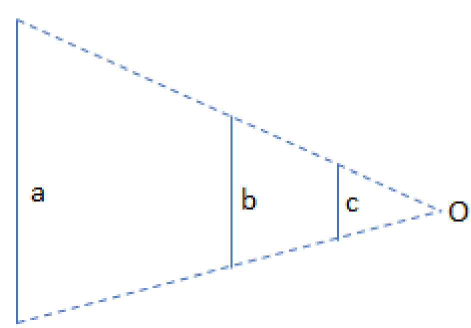


FIGURE 1. The ratios of the three line segments are: $\frac{a}{b}=\frac{b}{c}=\lambda=2$.

Thus, the three line segments shown in Fig. 1 can be said to be similar to each other (group similarity), since the ratios of their lengths are the same in pairs. This group can of course be expanded to more than three, i.e. to any number of line segments. If, in addition to group similarity, an additional condition is imposed by the operation of addition/subtraction ($b+c=a$ in Fig.1, i.e. the sum of the lengths of the two shorter line segments is the same as the length of the longer line segment), then the similarity ratio is:

$$\frac{b+c}{b}=\frac{b}{c}. \quad (1)$$

By multiplying both sides of equation (1) by *c/b*, we get a quadratic equation for *c/b*, the positive root of which is an irrational number:

$$\frac{c}{b}=\frac{1+\sqrt{5}}{2}=\Phi\approx 1{,}618. \qquad (2)$$

That is, the condition $a=b+c$ is valid for a single similarity ratio only if $\lambda=\frac{1}{\Phi}\approx 0{,}618$.

If we impose other conditions, e.g. $b\cdot c=a$, or $b^c=a$, then the similarity ratio is a variable depending on one of the parameters a, b and c, so we get an infinite number of similarity ratios as a solution.

Sorting the lengths of group-similar line segments by magnitude, a geometric sequence can be obtained. For instance, if λ =2, then: 1; 2; 4; … ; 2^{n-1}, and based on equations (1) and (2):

$$1;\ \frac{1+\sqrt{5}}{2};\ \frac{3+\sqrt{5}}{2};\ 2+\sqrt{5};\ \dots;\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}.$$

Ignoring the group similarity ratio Φ and only dealing with the additional condition ($a=b+c$, b=1, c=2), we get whole numbers instead of irrational terms (similarly to Fibonacci numbers):

1; 2; 3; 5; 8; 13; 21; … i.e. $a_n=a_{n-1}+a_{n-2}$, if $n\geq 3$.

Here we get the following numbers for the proportions of adjacent line segments: 2; 1.5; 1.67; 1.6; 1.61. It is noticeable that the similarity ratio is not constant, but tends quickly to *Φ*.

The first written trace of the ratio *Φ* appears in Euclid's Elements (ca. 300 BC), where he mentioned it as one of the steps in editing pentagons. Since then, a number os edits have been published, perhaps the most simple editing procedure can be seen in Figure 2.

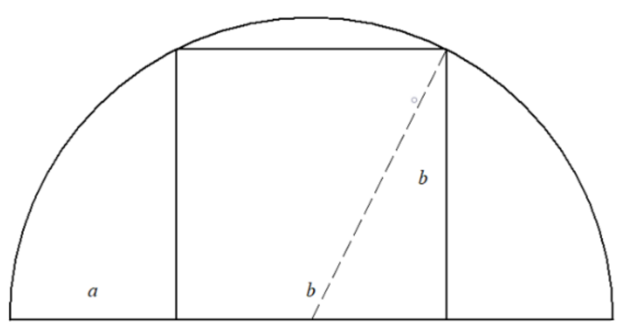


FIGURE 1. The diameter of the circle passing through the vertices of the opposite side from the side bisector of the square with side *b* is *2a+b*, where *b/a*= *Φ* [1].

The problem, known earlier as the "extreme and intermediate ratio", gained particular importance only much later: at the end of the 15th century, based on the work *Divina proportione* by Luca Pacioli, where figures were drawn by Leonardo da Vinci himself. Based on his theory, he presents a general law of nature in the exact mathematical formula. In his writing, the created order of the world and the mathematical rule of artistic beauty seem to be formulated. In the earlier writings of the Roman architect and military engineer Vitruvius (around 80-70 BC - after 15 BC), Pacioli believes he has already discovered the "divine proportion", and the golden ratio became soon one of the leading ideas in the Italian Renaissance. His thread clearly outlined that: if the main ratio of nature has been found, what is more natural is that the works of art of "true" artists are also created according to this kind of composition, so the possibility of objectively "classifying" works of art and artists is provided based on the golden ratio.

Physics, as the keystone of sciences, researches the rules of the behavior of nature and the Universe, and its discoveries have a great influence on the sciences and philosophy to present days. The discovery of regularities and directing principles of nature has generated the view of a created world against of randomness in several acknowledged physicists. The combination of several separate formulas and theories suggested the way to find the "single-line formula for the functioning of the Universe" [1].

For example, Kepler also enthusiastically sought the harmony of the world. He believed that regular shaped bodies could be inserted between the spheres carrying the six planets known at that time (Mercury, Venus, Earth, Mars, Jupiter, Saturn). Kepler saw the endless process of rebirth in the golden ratio: „I believe that this geometrical proportion served as idea to the Creator when He introduced the creation of likeness out of likeness, which also continues indefinitely." [1, 3].

This principle corresponds to the stages of crystal development as well, when the ratios of the unit cells are the same as the ratios of the macroscopic crystal itself [1]. (Another issue is that this ratio is not necessarily related to *Φ*.) The physical laws forming the crystal cell determine not only the form but also other physical and chemical properties (material quality) of the macroscopic body, while crystal defects are the result of the influence of the environment. We can find self-similar species in the flora, too (e.g. Daucus), or the arrangement of leaves, petals and fruits according to the Fibonacci sequence (e.g. sunflower, Romanesco broccoli) [4], but the influence of the environment is obviously more significant in case of the plants, so deviation from the ideal, the number of defects is significantly higher than in crystals. The most frequently mentioned examples in the fauna in connection with the golden ratio are the nautilus octopus and the five-armed starfish. Their growth is characterized by maintaining proportions in a non-strict sense, but the fairly strong environmental influence on the animal world leaves much deeper traces in the development of individual species than in plant populations, which could even be the development of easier movement coordination for the purpose of getting food more successfully. The Nautilus shell shape and its mathematical description arouse also Descartes' interest: in 1638, he defined the type of spiral where the angle between the radius and the corresponding tangent is constant, i.e. where the radii follow each other in an increasing geometric series.

The search for the "law of one" is also embodied in Einstein's "dream" according to wich there is a theoretical system that includes all fundamental physical interactions (forces), the so-called "Theory of all things", but the fine-structural constant, called a magic formula by Planck (1920), could also be mentioned (≈1/137). According to Dirac, the laws of physics must be mathematically beautiful, and according to Leibniz, our world is the best of all worlds.

II. ON THE PROPORTIONS OF THE HUMAN BODY AND RECTANGLES

The diversity of the human race has been determined by a series of genetics and historical factors to this day. The human body has a very complicated structure: it is made up of hundreds of bones and even more muscles. Consequently, the number of possible ratios is extremely large. The variety of body shapes is mainly due to genetic reasons. It would be debatable to proclaim a certain body shape or body proportion as ideal. However, if we were to analize the proportions of people considered beautiful by the majority, and these proportions were approximatelly the same based on a large number of samples, we would discover some regularity, that is, the mathematical formulation of beauty [2].

The concept of harmonious body proportions and beauty was already connected in the ancient Greek sculpture, since they did not create their statues depicting ideal men and women based on real models, but based on certain formulas (Fig. 3). This connection has accompanied the attitude of mind of many philosophers and artists on beauty up to the present day.

According to Thomas Aquinas: "Beauty is what pleases when contemplated" and "Beauty consists in radiance and proportion." Then, in the Renaissance, L. B. Alberti defined it as harmony and beautiful proportion: beauty depends on the harmonious arrangement of parts.

This connection of proportion and beauty clearly marks the aspiration to define the "most beautiful proportion". Of course, a mathematical ratio cannot be beautiful in itself, rather we can call it beautiful because of some kind of manifestation. Perhaps it should be sought in such a way that the mentioned ratio takes precedence over other ratios, e.g. when considered from the pont of view of practicality, exactly because of its simplicity: "it can be defined with fewer instructions", "it requires less initial data".

This is fulfilled for the golden ratio, as there is no need to specify the ratio, as e.g. 1:2, but applies the relationship of the two parts to the whole. In many cases, nature chooses processes that can be carried out with fewer instructions, but this does not proof that the golden ratio can be the only preferred proportion of beauty.

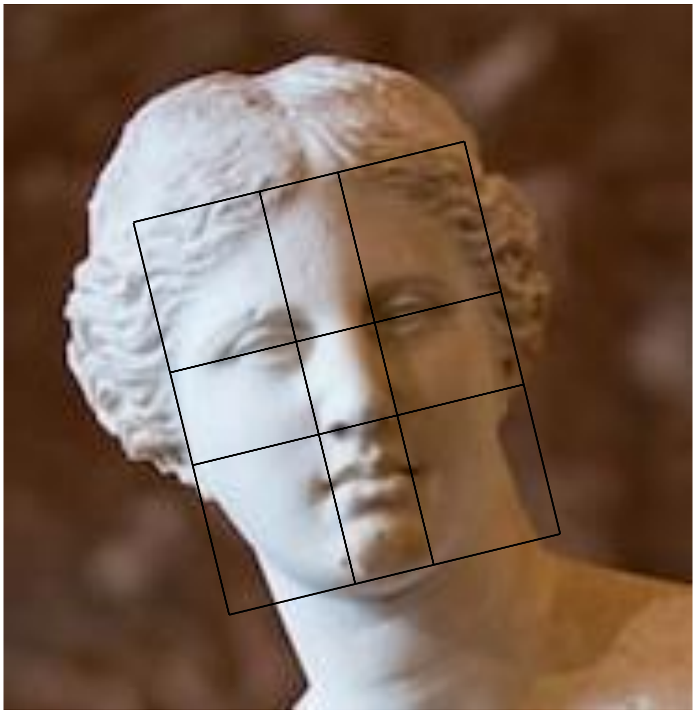


FIGURE 2. Detail of the head of the Venus de Milo. The face in the vertical proportions of the golden ratio: hairy scalp, eye line, tip of the nose, tip of the chin, and horizontally (symmetry also prevails here): cheekbones, inner corners of the eyes, nostrils.

The idea that the golden ratio has an aesthetic meaning comes from Zeising (1854), and Fechner, in his experiment in 1876, targets the psychological significance of the golden section. In his experiment, he did not study human body parts, but simple plane figures: rectangles that carry the proportions of their sides in their appearance as well. Fechner examined and summarized which of the rectangles was found to be the "best" (the most appealing) to the observers eyes (empirical verification) [1].

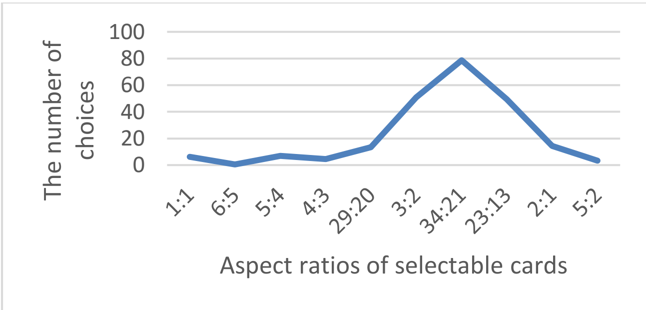


FIGURE 3. Fechner's conclusion: when the participants chose the most pleasing rectangle from ten cards, 34.5% of the test subjects preferred the card with the golden ratio. The second most favorable choice was only 22%.

In Fechner's experiment a number of errors can be observed: the ratios of the rectangles he chose did not evenly divide the intervals (Fig. 5.) and included "degenerate" rectangles as well (e.g. square and strongly elongated strip-like rectangle). Observers were allowed to line up the cards, in which case the cards in the middle were presumably preferred, and Fechner did not take into account the minimum threshold above which two ratios could be sensitively distinguished (i.e., very close ratios appeared in his experiment).

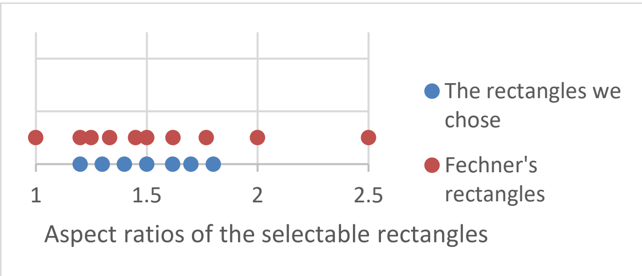


Figure 4. The ratios of the selectable rectangles are from 1:1 to 1:2.5. The ratios of the rectangles we use are uniform over the interval, only the ratio 1.618 is slightly different.

We performed a modified Fechner's experiment on April 1, 2020, on a web interface in the form of a questionnaire with 7 rectangles (Fig. 5). The description of the questionnaire was as follows.

III. BEAUTY CONTEST OF RECTANGLES

The connection between proportionality and beauty could already be found in antiquity. Is there a beauty accepted by everyone, or can it be different for everybody? Is it objective or subjective? In order to clarify this question, please fill out the short questionnaire below and take part in the jury of the rectangle beauty contest! What do you think is the most beautiful, most proportional rectangle? (1th place) Which should be the second and which should be third ranked? Your gender? Your age?"

(The rectangles were arranged as shown in the upper left part of Fig. 6.)

The questionnaire was filled out by 375 people in 9 days. Since 1.-3. ranking had to be given (a rectangle could only be chosen once), the choices were weighted, and the data were plotted taking this into account (Figure 6).

Instead of a clear jump of Fechner's graph, an interval showed the maximum values (1.6-1.8). The average obtained for the values of the summarized graph was 1.606.

Of course, a real beauty contest is much more exciting than the beauty contest of rectangles, where someone is declared the beauty of a given community (in the categories of women, men and children) according to a multi-member jury and, in many cases, the audience's opinion, or even the beauty of the world. The jury does not decide based on measurements but based on visual inspection of the candidate's body shape, general appearance, and movement. More recently, education and intelligence, style and communication skills are taken into account in the decision.

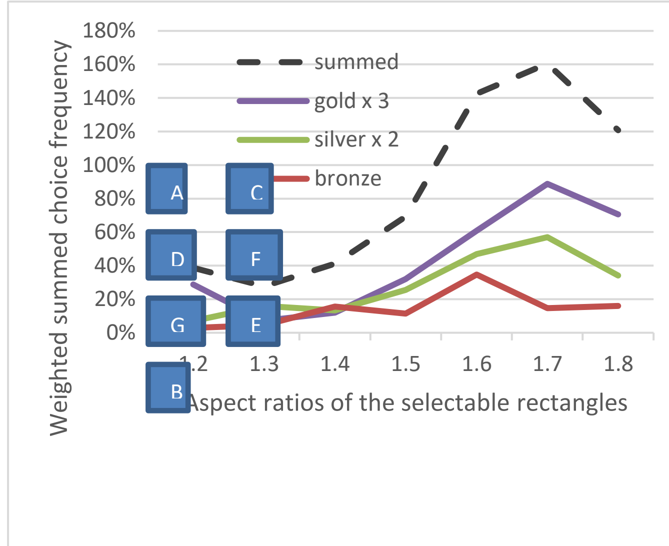


FIGURE 5. The majority of respondents preferred ratios of 1.6-1.8 over the others. The dashed line shows the weighted summed choice frequency.

Beauties declared by several people, even if it based on the opinion of a community, serve well in paralleling the golden ratio and beauty. The easiest way is to create a virtual grid with an editing software, where the right and left (and similarly upper and lower) lines divide the sides of the frame in a proportion ϕ. This grid is fitted to the subject's frontal face image using a graphic editor program, so that the sides of the rectangle fit the external facial features, the lower chin line and the beginning of the hairy scalp on the top of the forehead. Winners of beauty contests have the proportions of the Venus de Milo (Fig. 3).

Another question is that if we do not think we are beautiful enough, thanks to modern medicine, we can change the characteristic proportions of our face with plastic surgery. But what proportions should we choose in order to be satisfied with our appearance? Plastic surgeon Julian De Silva called on science to help: based on a computer program, he classified the ladies considered beautiful according to his patients' opinions. The program looked for the golden ratio based on the location of the details on the face (see L.B. Alberti's definition above). In the photos of the selected persons, the software classified in percentage terms how well the proportions follow the golden ratio. Based on the selected beauties, Bella Hadid came closest to the divine ratio, with an index of 94.35% (Fig. 7). The obtained results can be checked simply by applying our golden ratio grid to the Silva's top 10 placed: the face objects of the first place follow the golden ratios almost perfectly, while the 10th place Cara Delevingne shows a significant difference due to his higher forehead.

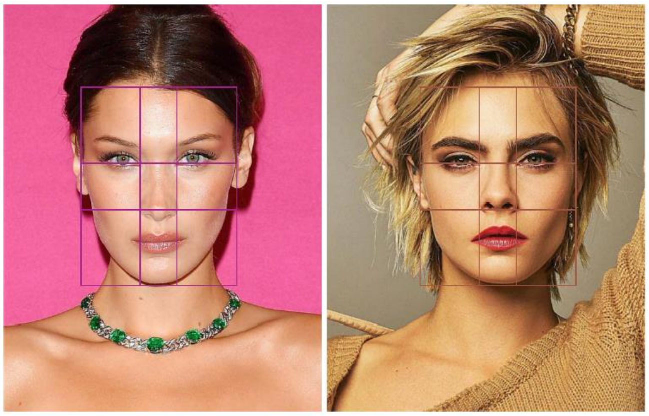


Figure 7. Plastic surgeon Dr. Julian De Silva, using a computer program that uses the rules of the golden ratio, found Bella Hadid to be the most beautiful woman with a 94.35% perfection index, while Cara Delevingne came in 10th place (89.99% perfection index).

IV. CONCLUSIONS

The golden ratio and the mystical statements surrounding it make an objective approach to the golden ratio quite difficult. It would be an exaggeration to say that a standard of nature has been found, but there is no doubt that the golden ratio and the Fibonacci sequence can be discovered in many places in the simple structures of nature. It is thanks to human's curiosity about nature that he discovered and then applied this ratio in the arts under the name of dynamic asymmetry, but it is certainly far from the "law of ONE" that would describes everything. If the golden ratio is considered as a one-dimensional self-similarity ratio, fractals can be considered as its three-dimensional manifestations, in which we can also find the fundamental organizing principle of nature. The presence of fractals was discovered by science progressively: it took a long way from the neglected mathematical "monster" of the Weierstrass [6] to its applications in fractal art, until today it has become the key to the description of many natural phenomena (see e.g. chaos theory).

Application of the golden ratio in classrom and study groups is an exciting and diverse task, but according to its exactness, it can only be considered in high school. It can be a good service in carrying out measurements, in calculating the average and standard deviation. (Below you can read some measurement tasks that students can complete in school sessions or even at home.)

We expect from science the highest possible degree of objectivity and the lowest possible degree of subjectivity. The graph shown in Figure 6 is the result of subjective choices, while the average calculated based on the data is already objective. The average was very close to ϕ, but as a conclusion of the study, we cannot declare that our sense of beauty is always determined by the golden ratio, even in case of a large sample size. However, to carry out new, similar experiments, as in the case of other open questions, the examination of subjective examples can be highly stimulating. We can take advantage of this even when we are not calculating the mean and standard deviation of the dimensions of school benches in the classroom, but rather calculating the mean and standard deviation of dimensions of human body parts in a target group (see below in the task), and then comparing our own body dimensions with the mean and standard deviation, because the effectiveness of the learning process is known to be positively influenced by emotions. The golden ratio appears in physics mainly in chaos theory and quantum physics. The discussion of these topics in high school requires a higher level of mathematical background, but certain tasks and examples that show coincidences with ϕ can be exciting (see the tasks below). Tasks like this seem more exciting to students, but care must be taken not to get lost in the maze of mysticism. Nor is the last aspect that we can emphasize the place of physics in society and in our basic thinking, if we talk to our students about science and the beauty of science.

V. TASKS

1. Cut rectangles out of paper with one side 5 cm and the other side of 6, 6.5, 7, 7.5, 8.09≈8.1, 8.5, 9 cm! Mark these rectangles with the letters A, B, C, D, E, F, G! Put these in an envelope, then hand the envelope to your partner and ask to participate in a game, i.e. to carefully observe the rectangles and rank them with 1st, 2nd and 3rd place according to their beauty! Ask as many of your friends as possible to jury the rectangles! Based on the data, calculate the mean and standard deviation, and plot the obtained values on a graph!
2. Navel ratio

   Create a loop to one end of a rubber band so that your foot fit comfortably, and fit a straight ruler to the other end. The length of the rubber band should be 140 centimeters. Put your foot in the loop, straighten the tape (it is not necessary to stretch it) and mark it with a pen at 140cm·0.618≈86.5 cm from the ground!
   a. Control experiment: stretch the tape to different extents and measure the ratio of the marked sections (average of 10 measurements, with standard deviation).
   b. Step into the loop, one of your partners will pull the tape up to the top of your head so that the ruler touches it horizontally! Place a pen to your navel, perpendicular to your body, and then have another partner measure the difference between the sign of the pen and the tape marking (made previously with a felt-tip pen) with a ruler. If the line is e.g. 2 cm above the pen, then d=-2cm. (Average, with standard deviation).

3. **Rectangle plasty**: Slide two paper squares over each other so that you get a rectangle you like best, then fasten them with a paper clip or adhesive tape!

Measure the length of the sides of your rectangle and calculate the ratio of the sides! Calculate the mean and standard deviation of the individually obtained values in the class!

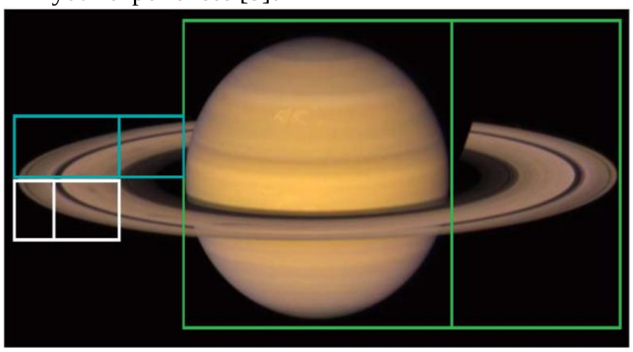
4. Calculate the difference in percentage between ϕ and the ratio of the density of the Earth to that of the Moon!

5. What do you think makes a physical formula beautiful? Find the one you like best!

6. Calculate the ratio of the orbital period of the Earth and Venus! By what percentage does it approximate ϕ?

7. Calculate the ratio of the sides of a plastic bank card!

8. Calculate the ratios shown in the picture! What are your experiences [5]?



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