Revista de Matemática: Teoría y Aplicaciones **31**(1): 57-97
CIMPA - UCR ISSN: 1409-2433(Print), 2215-3373(Online)
DOI: https://doi.org/10.15517/rmta.v31i1.55440

VACATING A PARKING LOT UNDER UNCERTAINTY

DESOCUPAR UN ESTACIONAMIENTO BAJO INCERTIDUMBRE

José Antonio Martínez Díaz[1] Damián Emilio Gibaja Romero[2]
Juan José Reyes Salgado[3] José Luis Martínez Flores[4]

Received: 20/Jun/2023; Accepted: 08/01/2024

---

[1] Universidad Popular Autónoma del Estado de Puebla, Departamento de matemáticas, Puebla, Mexico. E-Mail: joseantonio.martinez03@upaep.edu.mx
[2] Universidad Popular Autónoma del Estado de Puebla, Departamento de matemáticas, Puebla, Mexico. E-Mail: damianemilio.gibaja@upaep.mx
[3] Universidad Popular Autónoma del Estado de Puebla, Departamento de matemáticas, Puebla, Mexico. E-Mail: juanjose.reyes@upaep.mx
[4] Universidad Popular Autónoma del Estado de Puebla, Departamento de matemáticas, Puebla, Mexico. E-Mail: joseluis.martinez01@upaep.mx

Abstract

First-mile problems have become a major problem for the automobile industry since moving cars from production plants to selling destinations is characterized by using vehicle carriers with limited space. Although supply chain processes have automatized last-mile operations to improve productivity and increase benefits, first-mile analysis has been widely ignored. For example, in the automotive industry, cars are stored in parking lots until they are demanded, which negatively impacts delivery times and increases transportation costs. The previous issues impact the first-mile logistics of the automobile industry to the detriment of copying with delivery times and increasing the operation cost. In this paper, we deal with the previous issues by modeling the movement of cars from the parking lot to the car carrier as an optimal control problem. Considering that not all cars should leave the parking lot, we search for conditions that guarantee the existence of a unique optimal path when the cars' requisition is uncertain. Theoretical results provide a closed-form solution that indicates the optimal path to fill the car carrier in a time window. Such solutions allow us to study the impact of exogenous parameters (such as the parking lot size, the starting point, and marginal costs) on the behavior and features of the optimal path.

Keywords: calculus of variations, optimization, uncertainty, transshipment, optimal paths.

Resumen

Los problemas de la primera milla se han convertido en un problema importante para la industria automotriz, ya que el traslado de los automóviles desde las plantas de producción a los destinos de venta se caracteriza por el uso de portavehículos con espacio limitado. Aunque los procesos de la cadena de suministro han automatizado las operaciones de última milla para mejorar la productividad y aumentar los beneficios, el análisis de la primera milla ha sido ampliamente ignorado. Por ejemplo, en la industria automotriz, los automóviles se almacenan en estacionamientos hasta que se demandan, lo que impacta negativamente en los tiempos de entrega y aumenta los costos de transporte. Lo anterior impacta la logística de primera milla de la industria automotriz en detrimento de copiar tiempos de entrega y aumentar el costo de operación. En este artículo, abordamos las cuestiones anteriores modelando el movimiento de los automóviles desde el estacionamiento hasta el portavehículos como un problema de control óptimo. Considerando que no todos los autos deben salir del estacionamiento, buscamos condiciones que garanticen la existencia de un único camino óptimo cuando la requisición de los autos es incierta. Los resultados teóricos proporcionan una solución de forma cerrada que indica la ruta óptima para llenar el portavehículos en una ventana de tiempo. Estas soluciones nos permiten estudiar el impacto de parámetros exógenos (como el tamaño del estacionamiento, el punto de partida y los costos marginales) sobre el comportamiento y las características de la ruta óptima.

Palabras clave: cálculo de variaciones; optimización; incertidumbre; transporte; caminos óptimos.

Mathematics Subject Classification: Primary: 49Q22, 35Q49, 90B06.

1 Introduction

Transportation processes are unavoidable in logistics since they are required in the entire production procedure. From manufacturing to customers and in the presence of returns concerning reverse logistics, transportation plays a key role in logistics since it constitutes 40-50% of logistics costs and 4-10% percent of the products' selling price. Hence, it is clear that transportation decisions directly impact the logistics costs and other areas in the company [30]. Given transportation is constrained to cope with delivery times [18], it is a dynamic process that requires planning efforts to guarantee the effectiveness and efficiency of logistics processes [10]. So, forward-thinking organizations recognize the need to optimize first-mile operations since they represent an opportunity to improve the companies's competitiveness. Particularly, the first and last-mile delivery market is forecasted to reach 288.38 billion by 2030, with a compound annual growth rate of 6.12%. This growth is driven by the interest in e-commerce, declining shipping costs, improved ground delivery vehicles, automated warehouses, and new supply chain platforms [16].

Nowadays, first-mile transportation logistics plays a major role in fulfilling delivery times since it is the first step to transporting products from the production plant to a warehouse or distribution center. However, first-mile logistics is complex because it is a dynamic pickup process under which disruptions may happen due to cargo features such as volume and uncertain clients' requisitions. So, optimizing such problems requires implementing mathematical models related to static and dynamic vehicle routing problems with packing constraints [16]. Moreover, it is necessary to consider specific time windows and personalized shipments to cope with clients' expectations [7].

In the context of the Automotive Industry (AI), the dynamics of demand and supply for cars are intertwined with uncertainties that pose a significant challenge for AI's logistics operations [11]. Dell proposes a model grounded in Possibility Theory to simulate users' parking choice behavior and assess the impact of various parking policies on the movement of cars, which subsequently influences transportation from production plants to end customers. Notably, clearing parking lots, where vehicles are stored, introduces delays due to spatial constraints, limiting the flow of cars. In simpler terms, there isn't enough room to efficiently transfer cars from their parking positions to carriers [5]. Bahrami's work explores the potential of autonomous vehicles in optimizing land space utilization and minimizing parking space requirements, affecting the logistics of moving other vehicles [23]. Consequently, this process delays delivery times and increases transportation costs. Furthermore, the decision to transport a car from the parking lot depends on specific customer requisitions [34].

In this paper, we delve into the intricate problem of filling a car carrier while relocating cars from the production plant's parking lot to a car carrier with limited capacity by considering uncertainty. It is worth distinguishing our approach from [23], which primarily focuses on moving cars from a parking lot to a mother ship within a restricted time frame with cost minimization as the core objective, but without addressing the underlying uncertainty. Our modeling strategy adopts principles from optimal control theory, aiming to minimize the cost of clearing the parking lot within a fixed time frame. For simplicity, we assume the parking lot is a two-dimensional rectangle, with each point corresponding to a parking slot. Consequently, we tackle a dynamic optimization problem to identify the optimal paths that minimize the cost of loading the car carrier. In this framework, the state vector represents the car's position, and the control variable denotes its velocity.

Not all cars need to be moved due to customer requisitions, so moving a car is modeled as a random variable following a Bernoulli distribution. Consequently, we seek a movement path that minimizes the expected cost of loading the car carrier. By incorporating a polynomial cost function, we establish the existence of a unique movement path that dictates which cars should be relocated within the specified time frame. These movement paths are influenced by exogenous factors that affect the process of cars leaving the parking lot.

Our key contributions revolve around the critical impact of clearing parking lots in the initial logistics phase, ultimately affecting transportation costs. Reducing these costs poses a substantial challenge for businesses, as transportation involves various stakeholders with diverse objectives [12], affecting different layers of the supply chain, including public policies, tariffs, transport modes, energy consumption, and distance [33]. Consequently, optimization models play a pivotal role in enhancing supply chain efficiency and coordination [31]. Efficient transportation logistics bridge the gap between raw materials, finished products, and consumers, further underlining their significance [32].

By considering the capacity of the car carrier, we can optimize the cost associated with filling the car carrier. Numerous studies have proposed models and algorithms to address analogous inventory management problems. For instance, Nagasawa et al. developed a multi-item inventory model that accounts for truck capacity, associated costs, and receiving inspection costs [26]. Aghajani and Kalantar presented a methodology to model the interaction between parking lots and distribution system operators in the energy and reserve market while considering uncertainties related to load and wind power [1].

The supply chain is profoundly influenced by various factors, with the selection of optimal paths for moving goods from one point to another playing a pivotal role. This is why we employ diverse techniques such as operations research, heuristics, and optimal control, with a particular emphasis on the latter due to its potential to stabilize variables robustly [35]; [22] and optimize logistics planning. Other studies aim to maximize service levels while minimizing costs within distribution networks. Notable examples include [25], which employs a multi-objective optimization approach, and [15], which utilizes genetic algorithms.

Economic variables are inherently intertwined with logistics, and the dynamic nature of these systems introduces uncertainty due to demand fluctuations and market changes. Mathematical models are useful tools for analyzing uncertainty problems, as demonstrated by [28], which introduces a methodology for eliciting logistic regression parameters with a single covariate. In these cases, a binomial distribution is assumed. Optimizing the time required to clear a parking lot to load a car carrier can be achieved through a cost function [14], which considers factors such as the distance between the user and the parking lot, the distance between the parking lot and service areas, the availability of parking spaces, and the cost of parking for a given duration [21].

The calculus of variations serves as a valuable tool for analyzing dynamic phenomena, offering a framework for solving problems related to dynamic systems, as commonly encountered in physics and engineering. By optimizing functionals over time, we can determine the optimal trajectory for a system while considering various constraints and variables [19]. For instance, [4] highlights its applicability in enhancing nanostructures, [20] introduces a framework for optimizing energy utilization through battery management in a cooperative environment using calculus of variations, and [27] explores its use in optimizing functional parameters of compacted modified soils for geotechnical applications. Additionally, [13] addresses uncertainty in clearing parking lots and optimizes a polynomial cost function to determine the cars that should be relocated.

The Bernoulli distribution, commonly employed in our study, quantifies the number of successful events within a given time frame based on discrete events. It postulates that for any random event, there are only two possible outcomes: success or failure, with each experiment being independent. Our work extends the scope of [23] by considering the uncertainty surrounding the decision of whether to move specific cars.

The paper is organized as follows. Section 2 describes the optimal control model we use to analyze the vacation of a parking lot when clients' requisition list is uncertain. Section 3 presents the optimal paths, while Section 4 discusses the features of the previous paths. In Section 5, we provide some numerical examples to illustrate the behavior of the optimal path. Additional numerical examples are presented in the appendices.

2 Model

2.1 Basic elements

In this paper, we study the filling of a car carrier $\mathcal{M}$ in a time T given a requisition list $\mathcal{L}$; i.e., such a list has cars requested by customers of the parking lot. So, our model is closely related to the one of [23] in the sense that cars move from a parking lot $\mathcal{P}$ to $\mathcal{M}$. However, not all cars belong to the requisition list. The length of $\mathcal{L}$ equals the capacity of $\mathcal{M}$, which we assume is finite.

To simplify the analysis, the parking lot is the rectangular surface $\mathcal{P} = [0, a] \times [0, b]$, where $a, b \in \mathbb{R}_+$. A parking slot is a point $p = (x, y) \in \mathcal{P}$. Without loss of generality, the carrier is located in the point (a, b), which we assume without loss of generality.

Let $\mathcal{S} = \mathcal{P} \setminus \{(a, b)\}$ be the set of all the parking slots that are not occupied by the car carrier. Following the model of [23], each parking lot $p \in \mathcal{S}$ is occupied by a car; so, whenever there is no confusion, p also refers to a car. We write $p \in \mathcal{L}$ if the customers request the car p of the parking lot.

Notice that cars move from position p to (a, b) for each $p \in \mathcal{S}$. Moreover, there are no empty spaces in the parking lot, like streets, to move a car from its position to $\mathcal{M}$. Consequently, cars' movement requires freeing space by moving other cars. We use m to represent the number of movements to take a car from its position to its carrier. So, m is a function from $\mathcal{S}$ to $\mathbb{R}_+$. Given that $\mathcal{P}$ is a rectangle, cars move in vertical or horizontal directions. The total number of horizontal movements is denoted by m_X, while m_Y represents the total number of movements in the vertical direction.

Assumption 1. *For all $p \in \mathcal{P}$, we consider that $m(p) = m_X(p) + m_Y(p)$.*

Concerning the requisition list, freeing up space for movement involves the participation of several drivers because cars are not stored as $\mathcal{L}$ requests. In general, not all drivers know if a car should be carried out by $\mathcal{M}$ or not. Thus, belonging to $\mathcal{L}$ is a random event that follows a binomial distribution $f : [0, 1] \to \mathbb{R}$.

It is worth mentioning that moving a car implies using fuel and different drivers to free up space. Thus, filling $\mathcal{M}$ induces costs such as the drivers' salary and fuel expenditure. Notice that freeing space depends on p and the maneuver's complexity to move p to $\mathcal{M}$. So, $m(p)$ determines the cost of moving cars from the parking lot to the car carrier. For example, many cars need more drivers and fuel to free space [8]. Let C_M be the cost of moving cars from $\mathcal{P}$ to $\mathcal{M}$. Given that not all cars should be moved to $\mathcal{M}$, let C_S be the cost of rearranging the cars in the parking lot.

2.2 The problem of filling $\mathcal{M}$

In this paper, we study the problem of filling a car carrier when a requisition is not common knowledge. In other words, there is uncertainty concerning the cost of moving cars.

Remembering that belonging to $\mathcal{L}$ follows a binomial distribution, let $Pr[R]$ be the probability of belonging to the requisition list. At the same time, $Pr[NR]$ is the probability of not being requested. So, we have that $Pr[NR] = 1 - Pr[R]$. Also, it is worth mentioning that $Pr[NR]$ refers to the event where cars are moved to free up space, but they should be rearranged in the parking lot. Moreover, $Pr[R]$ implies moving a car p from $\mathcal{P}$ to $\mathcal{M}$. Hence, the expected cost function is the following:

$$E[C(m(t))] \quad = \quad Pr[R]C_M(m(t)) + Pr[NR]C_S(m(t)).$$

Notice that filling the car carrier $\mathcal{M}$, in the period $[0, T]$, plays a crucial role in guaranteeing the effectiveness of the first mile logistics [6]. Aside from copying with the filling time, the previous discussion points out that such an activity induces costs that the company should minimize. So, we have to solve the following minimization problem.

$$\min \int_0^T [Pr[R]C_M(m(t)) + Pr[NR]C_S(m(t))]\, dt.$$

It is important to note that movements depend on the car's location and velocity. Thus, we can rewrite the movement functions as follows:

$$m_x(t) = x(t) + \dot{x}(t) \quad \text{and} \quad m_y(t) = y(t) + \dot{y}(t). \tag{1}$$

Consequently, the cost minimization problem can be written as the following calculus of variations problem:

$$\begin{aligned} &\min \int_0^T [Pr[R]C_M(x + \dot{x} + y + \dot{y}) + Pr[NR]C_S(x + \dot{x} + y + \dot{y})]\, dt, \\ &\text{subject to} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x(T) \\ y(T) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \end{aligned} \tag{2}$$

3 Optimal paths

The optimization problem of equation 2 searches for minimizing the expected cost function $E[C(m)]$, which is not necessarily a linear function. Previously, we found the optimal movement paths that minimize $E[C(m)]$. The function is detailed in the following sections.

3.1 The expected cost function

The expected cost function weights the cost of leaving the parking lot with the cost of remaining on it, summarized by the probability events R and NR, respectively. Since it is the First, the expected cost function can be rewritten as follows:

$$E[C(m(t))] \quad = \quad Pr[R](C_M(m(t)) - C_S(m(t))) + C_S(m(t)). \tag{3}$$

Notice that C_M depends on the total number of movements that drivers perform to fill the car carrier. We simplify the analysis by assuming that such functions are linear concerning horizontal and vertical movements.

Assumption 2. *The total cost of moving a car from $\mathcal{P}$ to $\mathcal{M}$ is given by*

$$C_M(m) = C_x(m_x) + C_y(m_y), \tag{4}$$

where C_x and C_y indicate the cost of performing vertical and horizontal movements, respectively.

As usual, we consider that horizontal and vertical movement follows a polynomial behavior [24], [3]. Mathematically, we consider that

$$C_x(m_x) = \frac{m_x^r}{r(r-1)}, \qquad \text{and} \qquad C_y(m_y) = \frac{m_y^s}{s(s-1)},$$

where s and r are positive constants.

It is important to emphasize that all cars $p \notin \mathcal{L}$ need to be moved to free up space, but at the same time, drivers should arrange them in the parking since they remain in $\mathcal{P}$. So, function C_S summarizes the cost of freeing up space and rearranging those cars that should stay in the parking lot. in this sense, we have that

$$C_S(m) = C_x^2(m_x) + C_y^2(m_y). \tag{5}$$

By the assumptions in equations 1, 4, and 5, the total cost function is rewritten as follows:

$$C(m) = C_x(m_x) + C_y(m_y) + C_x^2(m_x) + C_y^2(m_y). \tag{6}$$

Now, by using equation 3, the expected cost function is given by

$$E[C(m)] \quad = \quad \sum_{k\in\{x,y\}} Pr[R]\left[C_k(m_k) - C_k^2(m_k)\right] + C_k^2(m_k).$$

The previous expression allows us to rewrite the dynamic optimization problem 2 regarding the state variables x and y as:

$$\min \int_0^T \left[\sum_{k\in\{x,y\}} Pr[R]\left[C_k(k+\dot{k}) - C_k^2(k+\dot{k})\right] + C_k^2(k+\dot{k})\right] dt. \tag{7}$$

3.2 General solution

The dynamic optimization problem in equation 2 searches for those optimal paths that minimize the expected cost function of moving cars from their slot to the point (a, b) in the period $[0, T]$. Given that movements depend on the location and the movement's velocity, the optimization problem can be analyzed through the Calculus of Variations theory lens.

Note that cost functions C_x and C_y are twice differentiable, and their derivatives are continuous. So, we can find the optimal movements through the first-order criterion. In other words, we need to solve Euler's equation. First, we consider that

$$\begin{aligned} D_x &= P_r(R)\left(\frac{m_x^r}{r(r-1)}\right) + 2\frac{m_x^{2r}}{r^2(r-1)^2} - P_r(R)\left(2\frac{m_x^{2r}}{r^2(r-1)^2}\right), && (8)\\ D_y &= P_r(R)\left(\frac{m_y^s}{s(s-1)}\right) + 2\frac{m_y^{2s}}{s^2(s-1)^2} - P_r(R)\left(2\frac{m_y^{2s}}{s^2(s-1)^2}\right). && (9) \end{aligned}$$

By equation 7, the optimization problem is equivalent to minimizing two functions regarding the state variables: one in terms of x and one in terms of y. So, Euler's equations establish the following system of equations:

$$\begin{aligned} \frac{\partial D_1}{\partial x} - \frac{d}{dt}\left(\frac{\partial D_1}{\partial \dot{x}}\right) &= 0,\\ \frac{\partial D_2}{\partial y} - \frac{d}{dt}\left(\frac{\partial D_2}{\partial \dot{y}}\right) &= 0. \end{aligned}$$

Euler's equation induces a linear system of differential equations that can be expressed in terms of the probability of being requested and the parameters r, s. Specifically, we have that

$$\begin{bmatrix} \dot{m}_x \\ \dot{m}_y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ x \\ y \end{bmatrix}, \tag{10}$$

where:

$$v = \frac{Pr(R)}{r-1} + \frac{1}{2(r-1)} - \frac{Pr(R)}{2(r-1)}, \tag{11}$$

and

$$w = \frac{Pr(R)}{s-1} + \frac{1}{2(s-1)} - \frac{Pr(R)}{2(s-1)}. \tag{12}$$

Thus, the optimal movement paths are the solutions of the system 10. The following theorem establishes our main result, which summarizes the optimal paths of moving cars from $\mathcal{P}$ to the car carrier $\mathcal{C}$.

Theorem 3.1. *Given system 10, the general solution of problem 2 is*

$$\begin{pmatrix} \dot{m}_x \\ \dot{m}_y \\ \dot{x} \\ \dot{y} \end{pmatrix} = K_1 e^{vt} \begin{pmatrix} v+1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + K_2 e^{wt} \begin{pmatrix} 0 \\ w+1 \\ 0 \\ 1 \end{pmatrix} + \ldots$$
$$K_3 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + K_4 t e^{-t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{13}$$

where $K_i \in \mathbb{R}$ *for all* $i = 1, \ldots, 4$.

Proof. The equation in 10 is a system of differential equations with constant coefficients. Hence, we can find its solutions by proposing a path $ke^{\lambda t}V$, where $k, \lambda \in \mathbb{R}$ and $v \in \mathbb{R}^4$. Thus, we first compute the eigenvalues of the coefficients' matrix:

$$A = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

These are the roots of the characteristic polynomial:

$$pol(\lambda) = det \begin{bmatrix} v-\lambda & 0 & 0 & 0 \\ 0 & w-\lambda & 0 & 0 \\ 1 & 0 & -1-\lambda & 0 \\ 0 & 1 & 0 & -1-\lambda \end{bmatrix} = 0.$$

In other words, the eigenvalues of matrix A are the solutions of the following equation:

$$(\lambda+1)(\lambda+1)(\lambda-w)(\lambda-v) = 0.$$

Consequently, the matrix of coefficients A has the following eigenvalues: $\lambda_1 = v$, $\lambda_2 = w$, $\lambda_3 = -1$ and $\lambda_4 = -1$. In other words, we have three different eigenvalues, one with multiplicity equal to two. Now, we compute the eigenvectors associated with each eigenvalue. In other words, we need to solve the following linear system:

$$\begin{pmatrix} v-\lambda & 0 & 0 & 0 \\ 0 & w-\lambda & 0 & 0 \\ 1 & 0 & -1-\lambda & 0 \\ 0 & 1 & 0 & -1-\lambda \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

for each eigenvalue $\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Concerning the eigenvalue λ_1, we can rewrite system 14) in the following way

$$\begin{aligned} 0(e_1) &= 0, \\ (w-v)e_2 &= 0, \\ e_1 + (-1-v)e_3 &= 0, \\ e_2 + (-1-v)e_4 &= 0. \end{aligned}$$

By the previous system of equations, we have that $e_2 = e_4 = 0$, while e_1 and e_3 are different to zero. Then, the eigenvector associated to λ_1 is $\hat{e}_{\lambda_1} = (e_1(1+v), 0, e_1, 0)$, where $e_1 \in \mathbb{R}$. As usual, we set the free term as $e_1 = 1$, i.e., we have that

$$\hat{e}_{\lambda_1} = \begin{bmatrix} v+1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now, by substituting the eigenvalue λ_2 into the system 14, the corresponding eigenvector is found by solving the following equations system

$$\begin{aligned} (v-w)e_1 &= 0, \\ (0)e_2 &= 0, \\ e_1 + (-1-w)e_3 &= 0, \\ e_2 + (-1-w)e_4 &= 0. \end{aligned}$$

Note that $e_1 = e_3 = 0$, while e_2 and e_4 are different to zero. Hence, we can write the eigenvector associated to λ_2 in terms of e_2, i.e., we have that $\hat{e}_{\lambda_2} = (0, e_2(1+w), 0, e_2)$, where $e_2 \in \mathbb{R}$. As before, we set e_2 equal to 1, which implies that

$$\hat{e}_{\lambda_2} = \begin{bmatrix} 0 \\ w+1 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, we repeat the previous procedure to find the eigenvector associated with λ_3, which has a multiplicity equal to two. In other words, such an eigenvalue has two associated eigenvectors, non-trivial solutions to the following equations' system.

$$\begin{aligned} (v(-1))e_1 &= 0 \\ (w-(-1))e_2 &= 0 \\ e_1 + (-1-(-1))e_3 &= 0 \\ e_2 + (-1-(-1))e_4 &= 0. \end{aligned}$$

The first two equations show that $e_1 = e_2 = 0$. We substitute the previous solutions into equations three and four and get that e_3 and e_4 are different from zero. Since they are independent of each other, the two eigenvectors of λ_3 are $\hat{e}_3 = (0, 0, e_3, 0)$ and $\hat{e}_4 = (0, 0, 0, e_4)$, where $e_3, e_4 \in \mathbb{R}$. By considering that $e_3 = e_4 = 1$, the eigenvectors that we use for $\lambda_3 = -1$ are the following

$$\hat{e}_{\lambda_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \hat{e}_{\lambda_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{15}$$

Given the eigenvalues λ_1, λ_2, λ_3 and λ_4, and their eigenvectors $\hat{e}_{\lambda_1}$, $\hat{e}_{\lambda_2}$, $\hat{e}_{\lambda_3}$ and $\hat{e}_{\lambda_4}$, the general solution of the system is the one in equation 13. $\square$

3.3 The initial value problem

Remembering the calculus of variations problem of equation 2 states initial and ending conditions. At $t = 0$, we have that the slot of departure is (x_0, y_0), and the slot of destination is $(x(T), y(T)) = (a, b)$ since the car carrier departs from the parking lot at $t = T$. By Theorem 3.1, there is a unique solution concerning the dynamic optimization of the cost of moving cars from $\mathcal{P}$ to C.

Specifically, equation 13 illustrates the optimal path (x^*, y^*) and the optimal number of movements (m_x^*, m_y^*) that follow a car requested by the car carrier. Given the initial and ending conditions, we use the previous paths to find the values of constants K_1, K_2, K_3, and K_4.

By considering $t = 0$ and $t = T$, we get the following system of four equations

$$\begin{aligned} K_1 + K_3 &= x_0, && (16)\\ K_2 &= y_0, && (17)\\ K_1 e^{vT} + K_3 e^{-T} &= a, && (18)\\ K_2 e^{wT} + K_4 T e^{-T} &= b. && (19) \end{aligned}$$

The previous equations system can be split into two systems with different unknown variables. Equations 16 and 18 allows us to compute the value of K_1 and K_3. Moreover, we can get the K_2 and K_4 from equations 17 and 19. By doing some algebra, we get that

$$\begin{aligned} K_1 &= x_0 - \frac{ae^{-vT} - x_0}{e^{-T(1+v)} - 1},\\ K_2 &= y_0,\\ K_3 &= \frac{ae^{-vT} - x_0}{e^{-T(1+v)} - 1},\\ K_4 &= \frac{be^{-wT} - y_0}{Te^{-T(1+w)}}. \end{aligned} \tag{20}$$

We also normalize the period to simplify the visualization of the optimal paths. In other words, we consider that time belongs to $[0, 1]$ by considering the transformation t/T. Consequently, we can rewrite the optimal movement path as follows:

$$\begin{aligned} x^*(t) &= \left(x_0 - \frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right) e^{\frac{vt}{T}} + \left(\frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right) e^{\frac{-t}{T}},\\ y^*(t) &= y_0 e^{\frac{wt}{T}} + \left(\frac{be^{-w} - y_0}{Te^{-(1+w)}}\right) te^{\frac{-t}{T}}. \end{aligned} \tag{21}$$

Hence, the optimal paths in expression (21) imply that the total number of movements that we require to move cars from the parking lot to the car carrier is the following:

$$m^*(t) = \left(x_0 - \frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right)(v+1)e^{\frac{vt}{T}} + \left(\frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right)e^{\frac{-t}{T}} + \ldots$$
$$\ldots + (w+1)y_0 e^{\frac{wt}{T}} + \left(\frac{be^{-w} - y_0}{Te^{-(1+w)}}\right)te^{\frac{-t}{T}}.$$

4 Model solution analysis

It is worth emphasizing the fact that assumptions 1 and 2 imply the existence of a unique solution (x^*, y^*) that points out the path that cars should follow to leave the parking lot. By equation 20, we observe that optimal paths depend on the exogenous parameters v, w, x_0, y_0, a, b, and T. We generally observe that paths' behavior depends on the relationship between the previous parameters.

First, we investigate the behavior of the horizontal position x^* concerning time.

Proposition 4.1. *Consider* $\Delta_x = (ae^{-v} - x_0)/(e^{-(1+v)} - 1)$*. The optimal path* x^* *is increasing for all* $t \in [0, \overline{\Delta_x}]$*, where*

$$\overline{\Delta}_x = \frac{T}{1+v} ln\left(\frac{\Delta_x}{v(x_0 - \Delta_x)}\right).$$

Proof. To find a domain's condition where x^* is increasing, note that we can rewrite it as follows:

$$x^* = (x_0 - \Delta_x)e^{vt/T} + \Delta_x e^{-t/T}.$$

The horizontal position increases as time passes by whenever the derivative concerning time is positive. In other words, x^* is monotonically increasing when

$$0 < \frac{dx^*}{dt} = \frac{v}{T}(x_0 - \Delta_x)e^{vt/T} - \frac{\Delta_x}{T}e^{-t/T}.$$

By doing some algebra, we find that $dx^*/dt > 0$ if and only if

$$e^{(1+v)t/T} > \frac{\Delta_x}{v(x_0 - \Delta_x)}.$$

By applying the logarithmic function on both sides of the previous inequality, we get that

$$\frac{(1+v)}{T}t > ln\left(\frac{\Delta_x}{v(x_0 - \Delta_x)}\right).$$

Therefore, x^* is monotonically increasing when

$$t > \frac{T}{1+v}\ln\frac{\Delta_x}{v(x_0 - \Delta_x)} = \overline{\Delta_x}.$$

□

Now, we search for a domain where the vertical position y^* monotonically increases.

Proposition 4.2. *Consider* $\Delta_y = (be^{-w} - y_0)/(Te^{-(1+w)})$. *If* $\ln y_0/b < -w$, *then the optimal path* y^* *is increasing for all* $t \in [0, T]$.

Proof. First, note that it is possible to rewrite y^* as follows

$$y^* = y_0 e^{wt/T} + \Delta_y t e^{-t/T}.$$

Then, y^* is monotonically increasing if

$$\begin{aligned} 0 &< \frac{dy^*}{dt} \\ &= \frac{wy_0}{T} e^{wt/T} - \frac{t\Delta_y}{T} e^{-t/T} + \Delta_y e^{-t/T} \\ &< \frac{wy_0}{T} e^{wt/T} + \Delta_y e^{-t/T}. \end{aligned}$$

From the previous expression, we get that

$$\begin{aligned} -\Delta_y e^{-t/T} &< \frac{wy_0}{T} e^{wt/T}, \\ -\frac{\Delta_y}{wy_0} &< e^{(w+1)t/T}. \end{aligned}$$

Since $e^{(w+1)t/T}$ is greater than zero for all $t \in [0, T]$, the previous inequality implies that y^* is monotonically increasing if and only if Δ_y is greater than zero. By the definition of Δ_y, the previous condition is guaranteed when

$$be^{-w} - y_0 > 0.$$

Therefore, we conclude that y^* is monotonically increasing when

$$\ln \frac{y_0}{b} < -w.$$

□

Finally, it is worth recalling that the parameters a and b are associated with the size of the parking lot $\mathcal{P}$, which may impact the optimal paths in a significant way. Intuitively, the movement position should be nearer to the parking lot exit (represented by the point (a, b)) when the parking lot size increases. At each time t, the following proposition shows the existence of a positive relationship between x^* and a. Also, the relationship between y^* and b is positive.

Proposition 4.3. *Let (x^*, y^*) be the vector of optimal paths as in expression (21). We have that:*

1. *The relationship between x^* and a is positive.*
2. *The relationship between y^* and b is positive.*

Proof. By expression (21), the optimal path on the horizontal axis is

$$x^*(t) = \left(x_0 - \frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right) e^{\frac{vt}{T}} + \left(\frac{ae^{-v} - x_0}{e^{-(1+v)} - 1}\right) e^{\frac{-t}{T}}.$$

At a time t, its derivative with respect to a is

$$\begin{aligned} \frac{\partial x^*}{\partial a} &= -\frac{e^{-v}}{e^{-(1+v)} - 1} e^{vt/T} + \frac{e^{-v}}{e^{-(1+v)} - 1} \\ &= \frac{e^{-v}}{e^{-(1+v)} - 1} \left(e^{-t/T} - e^{vt/T}\right). \end{aligned}$$

In the previous expression, note that

$$e^{-(1+v)} - 1 < 0, \quad e^{-v} > 0, \quad \text{and} \quad e^{-t/T} - e^{vt/T} < 0.$$

Hence, we conclude that

$$\frac{\partial x^*}{\partial a} > 0.$$

Now, we take the derivative of y^* with respect to b. By expression (21), we have that

$$y^*(t) = y_0 e^{\frac{wt}{T}} + \left(\frac{be^{-w} - y_0}{Te^{-(1+w)}}\right) te^{\frac{-t}{T}}.$$

Then, the derivative of y^* with respect to b is

$$\frac{\partial y^*}{\partial b} = \frac{e^{-w}}{Te^{-(1+w)}} te^{\frac{-t}{T}}.$$

Since e^{-w}, $te^{\frac{-t}{T}}$ and $Te^{-(1+w)}$ are positive, we conclude that

$$\frac{\partial y^*}{\partial b} > 0.$$

□

5 Numerical examples

The previous section points out the importance of the relationship between the exogenous parameters concerning the behavior of the optimal paths x^* and y^*. Propositions 4.1 and 4.2 show the non-existence of a general relationship that implies a monotonic behavior in the period $[0, T]$. Hence, this section provides numerical examples to illustrate the behavior of optimal solutions.

As is usual in the literature, we consider that the cost function is quadratic; that is to say, we assume that $r = s = 2$. Also, the time is normalized to $T = 1$ while the parking lot is $\mathcal{P} = [0, 40] \times [0, 40]$. First, we vary the initial conditions of the problem of equation 2. Table 1 shows 10 possible initial slots that point out the position of the first car to be moved to the car carrier.

Table 1: Initial conditions for simulation.

x_0	y_0
34	30
8	3
25	29
5	4
24	33
33	4
30	37
19	3
10	30
2	17

We use MATLAB R2022a software to illustrate the optimal solutions as we change the exogenous parameters. MATLAB is a powerful software for analyzing and solving optimal control problems [17, 2, 29, 9]. It provides a toolbox called YALMIP and LMI, which can be used to model and solve optimization problems in control systems. MATLAB also offers numerical methods, such as Euler's method and second-order and fourth-order Runge-Kutta methods, for solving ordinary differential equations. Additionally, MATLAB can be used to implement Pontryagin's maximum principle for solving optimal control problems. Appendix A summarizes the code we use to illustrate the optimal paths for moving cars from the parking lot to the car carrier and the characteristics of the computer.

Figure 1 illustrates the path requested cars follow from the parking lot to the car carrier. It is worth emphasizing that initial conditions in Table 1 change the behavior of the optimal paths x^* and y^*. If y^* depends on x^*, paths are monotonically increasing as the initial conditions are close to the X and Y axis. In

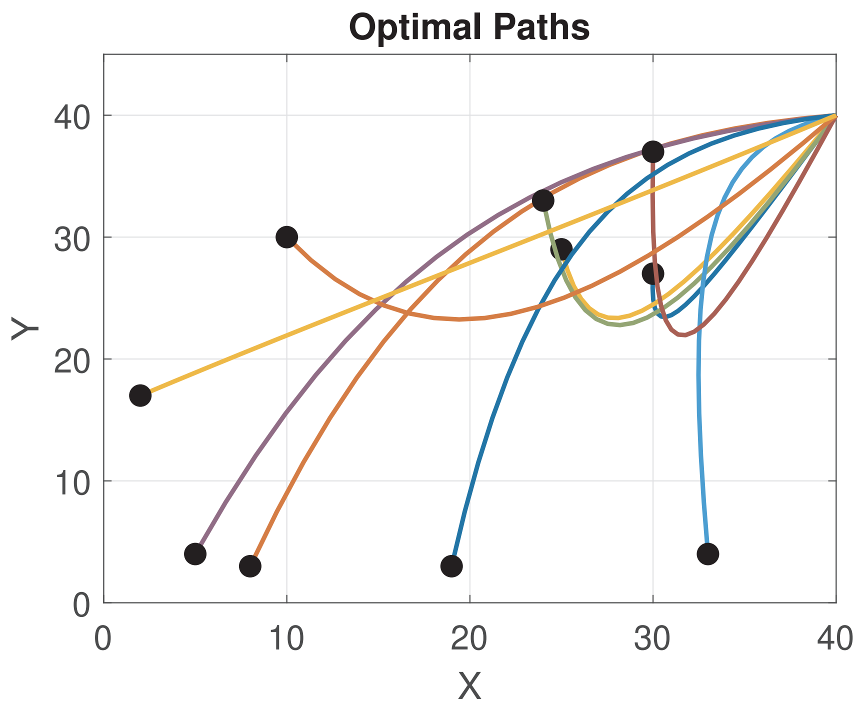

Figure 1: Optimal Movements

contrast, paths follow a $U-$shaped behavior when the initial condition (x_0, y_0) is located in the interior of the parking lot $\mathcal{P}$.

The data must be analyzed with variations to understand the model's performance change. The changes proposed are variations in the terms r, s, and P_x using $t = 0$ with increments of 0.01 until the time T that is normalized.

Case 1.

The following tabulation is given at a certain instant of time, in this case at $t = 0.5$, and the next conditions $x_0 = 30$, $y_0 = 37$, $r = 2$, $s = 2$ with the variation in the probability term.

Table 2 shows the relative change in the Y axis and explains the type of graphics in Figure 2.

Table 2: Case 1.

P_x	X	Y
0.1	33.2571	28.8195
0.2	33.0005	27.3421
0.3	32.7464	25.7570
0.4	32.4947	24.0578
0.5	32.2455	22.2378
0.6	31.9989	20.2900
0.7	31.7549	18.2069
0.8	31.5135	15.9808
0.9	31.2747	13.6033
1.0	31.0387	11.0659

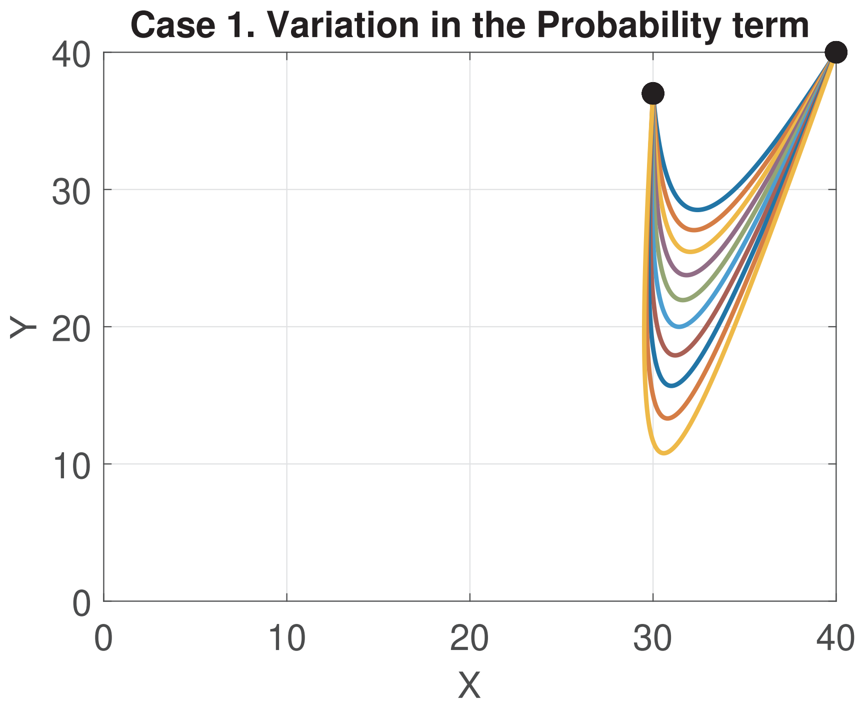

Figure 2: Optimal Movements changing P_x

Case 2

The following tabulation is given at a certain instant of time, in this case at $t = 0.5$ and the next conditions $x_0 = 30$, $y_0 = 37$, $P_x = 0.5$, $r = 2$, with the variation in the s term.

Table 3: Case 2.

s	X	Y
3	32.2455	33.2258
4	32.2455	35.7364
5	32.2455	36.8193
6	32.2455	37.4186
7	32.2455	37.7982
8	32.2455	38.0598
9	32.2455	38.2509
10	32.2455	38.3966
11	32.2455	38.5114
12	32.2455	38.6040

Table 3 shows a steady behavior in X and a growth in Y is presented explaining the type of graphics in Figure 3.

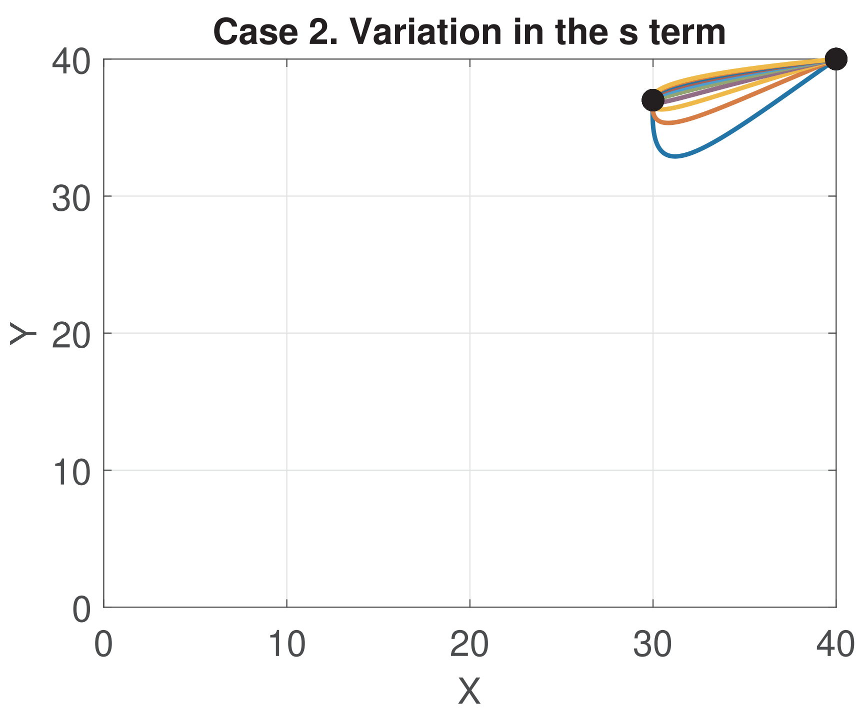

Figure 3: Optimal Movements changing s value

Case 3.

The following tabulation is given at a certain instant of time, in this case at $t = 0.5$ and the next conditions $x_0 = 30$, $y_0 = 37$, $P_x = 0.5$, $r = 2$, with the variation in the r term.

Table 4: Case 3.

r	X	Y
3	34.1735	22.2378
4	34.8447	22.2378
5	35.1852	22.2378
6	35.3910	22.2378
7	35.5288	22.2378
8	35.6275	22.2378
9	35.7017	22.2378
10	35.7594	22.2378
11	35.8057	22.2378
12	35.8436	22.2378

Table 4 denotes an increasing behavior in X and remains constant in Y explaining the type of graphics in Figure 4

Figure 2 shows that the number of movements reduces as the probability of allocating the car in the parking lot increases. In other words, uncertainty is a decisive variable where the time to locate the car to be moved plays a major role in coping with the clients' requisitions. Figure 3, shows that as term s increases, the movements in axis Y increase too, with a problem in finding an inflection point as the values in x_0 and y_0 get bigger, finally Figure 4 gives an almost stationary state as r value increases with very small changes in the remain variables.

The curves shown in Figures 2, 3 and 4 simulate the optimal paths due to changes in the exogenous parameters of the model, which modifies the trajectories and, therefore the arrival time at the parking lot exit. This contributes to defining a logistics strategy to reduce uncertainty and thus carry out the transfer of vehicles to the mother ship in less time.

It is important to recall that the previous numerical examples fix the parking lot's size together with the lot where the movement of cars initializes. However, Proposition 4.3 emphasizes that previous parameters positively impact the optimal movement path. Si, Appendix B provides additional examples where it is possible to visualize the impact of the parking lot size. From Figure 11 to 16, in Appendix B, the behavior of the optimal paths is very similar to those presented in Figures 2, 3 and 4. So, the further the car is from the exit point (a, b), we observe very slight curved movements. On the contrary, the closer it is, the movements are more erratic.

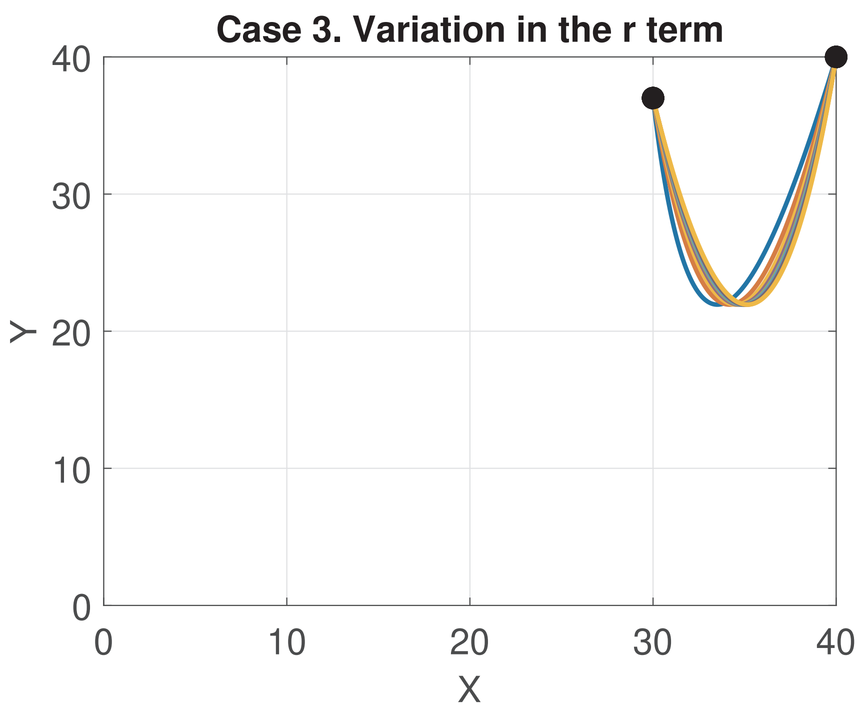

Figure 4: Optimal Movements changing r value

6 Conclusions

The present paper presents a theoretical analysis concerning first-mile logistics within the automotive industry by modeling the vacating of a parking lot in a time window as an optimal control problem. We provide conditions under which a unique path exists to minimize the cost of vacating the parking lot. Such conditions allow us to find closed-form solutions for the movements that minimize the cost of filling a car carrier when the cars to move are uncertain. The closed-form solutions are affected by exogenous parameters related to the parking lot size and marginal cost. So, we also present graphical representations of the optimal paths by setting the values of the exogenous parameters.

The sensitivity analysis shows that the number of movements reduces as the probability of allocating the car in the parking lot increases. So, uncertainty is decisive in coping with clients' requisitions since it modifies the optimal movement path. In other words, the movement of cars is nearer to the parking lot when the probability of not being listed in the clients' requisition increases. Intuitively, the parking lot prefers to vacate its closest part to the car carrier when most cars are not requested.

Concerning marginal cost parameters, both parameters have a positive relationship with the location of movement. So, the cost of vacating the parking lot is minimal when the cars are near the point where the car carrier is located. As before, larger marginal costs drive movement to the frontiers of the parking lot to reduce the total cost.

The main limitation of our study is the lack of data to compare our theoretical results with the empirical experience. In future works, we pretend to calibrate the parameters of our model by collecting data from parking that needs to be vacated. This last exercise is out of the scope of our research, given the permits it requires, aside from coordinating agents of different companies. However, to provide some insights about the model implications with collected data, Appendix C analyzes the variation between data provided at different parking lot sizes with the probability of appearing or not appearing on the list of cars to pick up through the ANOVA methodology. As expected, the previous variables generate a significant impact on the optimal, which means that the null hypothesis is rejected.

Finally, future works are related to two open questions that this paper does not address. The first one is related to the capacity of the car carrier, which can be modeled as an isoperimetric problem. The second one is the calibration of the model by considering real data.

Acknowledgement

JAMD acknowledges support from Consejo Nacional de Humanidades, Ciencias y Tecnologías (CONAHCyT), Universidad Popular Autónoma del Estado de Puebla (UPAEP), and Asociación Sindical del Personal Académico de la BUAP (ASPABUAP).

Appendix A

The characteristics of the computer are listed below:

- Processor Intel(R) Core(TM) i7-6600U
- CPU @ 2.60GHz at 2.80 GHz
- RAM installed 16.0 GB, 64 bits, x64

The Matlab code used to simulate:

```
clc;
close all;
a=40;
b=40;
T=1;
s=2;
r=2;
Px=0.5;
v=Px.*(1./(r-1))+(1-Px).*(1./(2.*(r-1)));
w=Px.*(1./(s-1))+(1-Px).*(1./(2.*(s-1)));
t=0:0.05:T;
n=10;
X1=[];
Y1=[];
dx=[];
dy=[];
x01=[];
y01=[];
for i=1:n
    x0=randi(40);
    y0=randi(40);
    k3=(a.*exp(-v)-x0)./(exp(-(1+v))-1);
    k1=x0-k3;
```

```
k4=(b*exp(-w)-y0)./(T.*exp(-(1+w)));
k2=y0;
X=(v+1)*k1.*exp(v.*t./T)+k3.*exp(-t./T);
Y=(w+1)k2.*exp(w.*t./T)+k4.*t.*exp(-t./T);

dX=(exp(-t./T).*(x0 - a*exp((Px - 1)/(2.*r - 2) -
Px./(r - 1))))/(T.*(exp((Px - 1)/(2.*r - 2) -
Px./(r - 1) - 1) - 1))-(exp(-(t.*((Px - 1)./
(2.*r - 2)-Px./(r - 1)))./T).*
(x0 + (x0 - a.*exp((Px - 1)./(2.*r - 2) -
Px./(r - 1)))./(exp((Px - 1)./ (2.*r - 2) -
Px./(r - 1) - 1) - 1)).*((Px - 1)./(2.*r - 2) -
Px./(r - 1)))./T;

dY=(t.*exp(Px./(s - 1) - (Px - 1)./(2.*s - 2)
+ 1).*exp(-t/T).*(y0 - b.*exp((Px - 1)./(2*s - 2)
- Px./(s - 1))))/T^2 - (y0.*exp(-(t.*((Px - 1)./
(2*s - 2) - Px./(s - 1)))./T).*((Px - 1)./
(2.*s - 2) - Px./(s - 1)))./T - (exp(Px./(s - 1)
-(Px - 1)./(2.*s - 2) + 1).*exp(-t./T).*(y0 - b.*
exp((Px - 1)./(2.*s - 2) - Px./(s - 1))))./T;

X1=[X1 X];
Y1=[Y1 Y];
dx=[dx dX];
dy=[dy dY];
x01=[x01 x0];
y01=[y01 y0];
A=[X1' Y1' dx' dy'];
mx=(v+1).*X;
my=(w+1).*Y;
mt=mx+my;
figure (1)
plot(X,Y,'-o', 'MarkerIndices',1:40:length(Y),
'MarkerEdgeColor',
'black','MarkerFaceColor','black',
'MarkerSize',8,LineWidth=2)
xlabel('X')
ylabel('Y')
ax = gca;
ax.FontSize = 15;
grid on
```

```
    title('X vs Y','FontSize',16)
    hold on
    figure (2)
    plot(dX, dY, '-o', 'MarkerIndices', 1:40:length(Y)
    ,'MarkerEdgeColor',
    'black','MarkerFaceColor','black', 'MarkerSize',
    8,LineWidth=2)
    xlabel('dX')
    ylabel('dY')
    ax = gca;
    ax.FontSize = 15;
    title('dX vs dY','FontSize',16);
    hold on
    figure (3)
    plot (t, mx, '-o', 'MarkerIndices',1:40:length(Y),
    'MarkerEdgeColor',
    'black','MarkerFaceColor','black', 'MarkerSize',
    8,LineWidth=2)
    xlabel('t')
    ylabel('mx')
    ax = gca;
    ax.FontSize = 15;grid on
    title('X Axe Movements','FontSize',16)
    hold on
    figure (4)
    plot (t,my, '-o', 'MarkerIndices',1:40:length(Y),
    'MarkerEdgeColor',
    'black','MarkerFaceColor','black', 'MarkerSize',
    8,LineWidth=2)
    xlabel('t')
    ylabel('my')
    ax = gca;
    ax.FontSize = 15;
    grid on
    title('Y Axe Movements','FontSize',16)
    hold on
end
```

Appendix B

These simulations were computed with two different initial points and show the difference between each sector in the graphics. The derivative plots were also included. The condition is the same that was used in the first simulation proposed with variations in the terms r, s, P_x and the size of the $P = [100, 100]$.

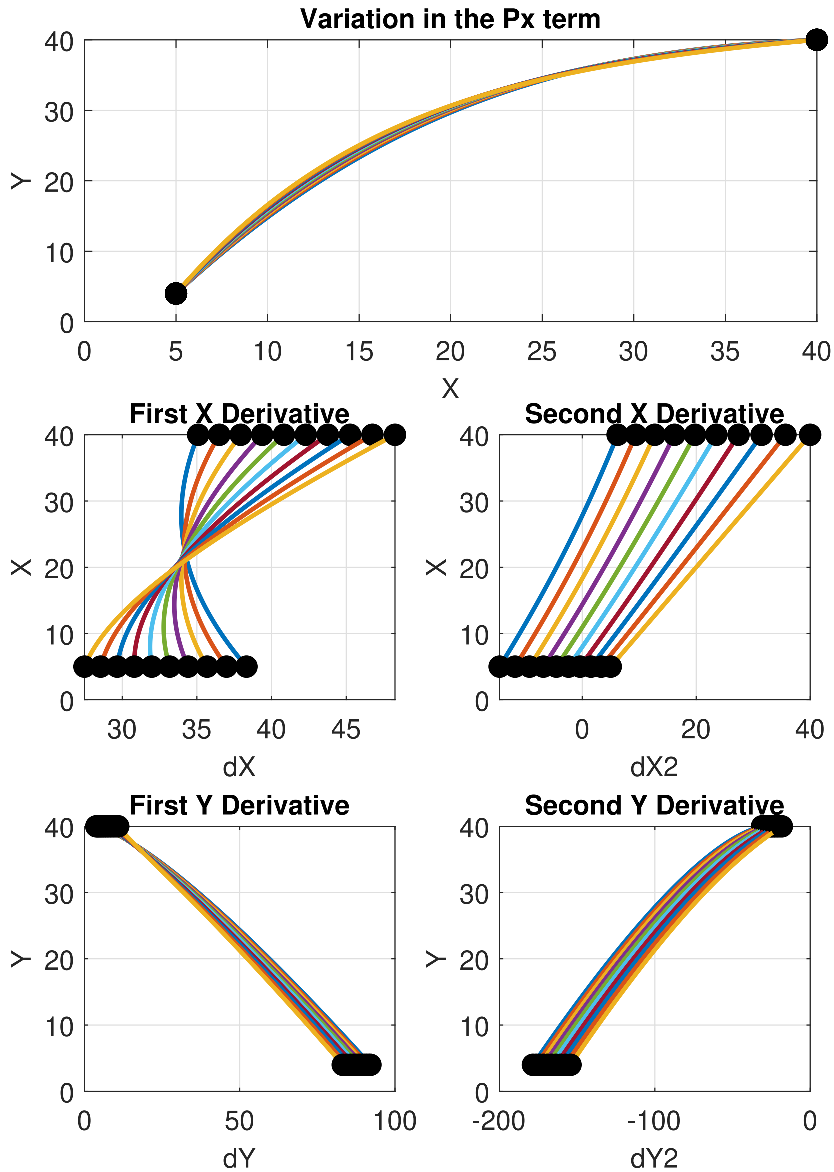

Figure 5: Variations at Point (5, 4) in Case 1

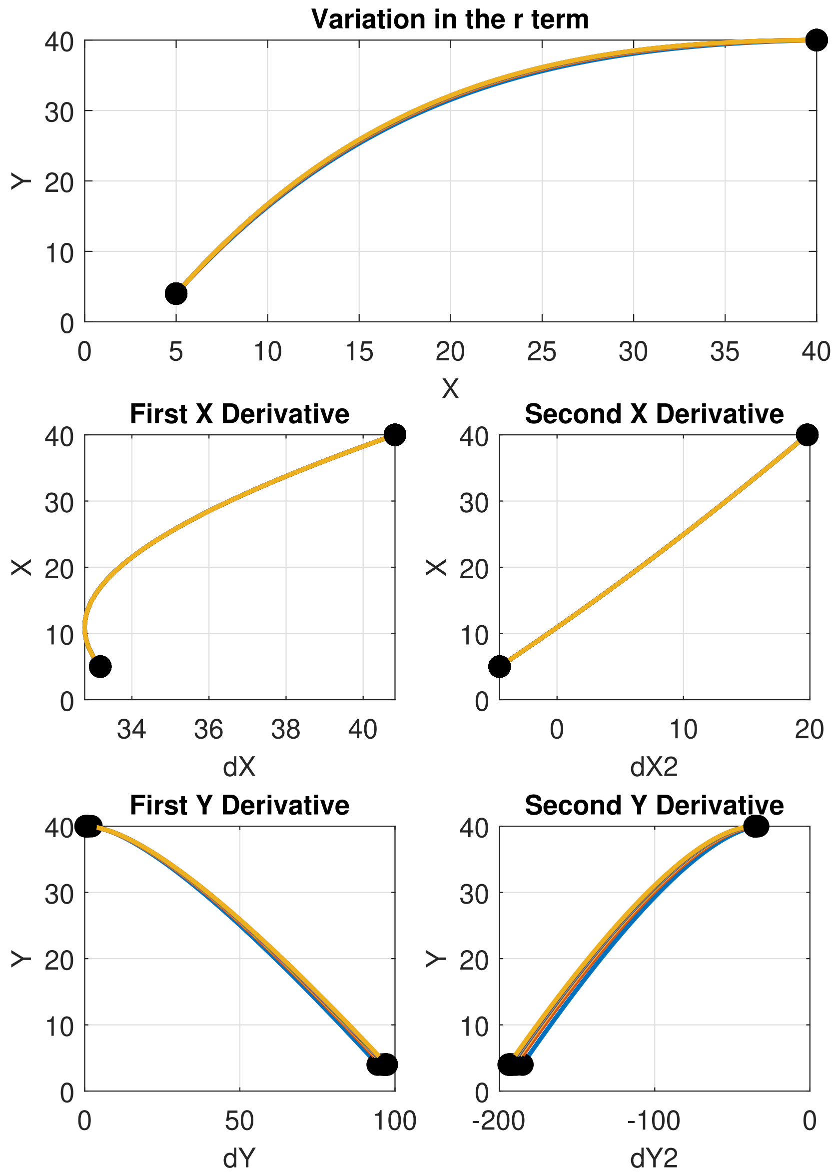

Figure 6: Variations at Point (5, 4) in Case 2

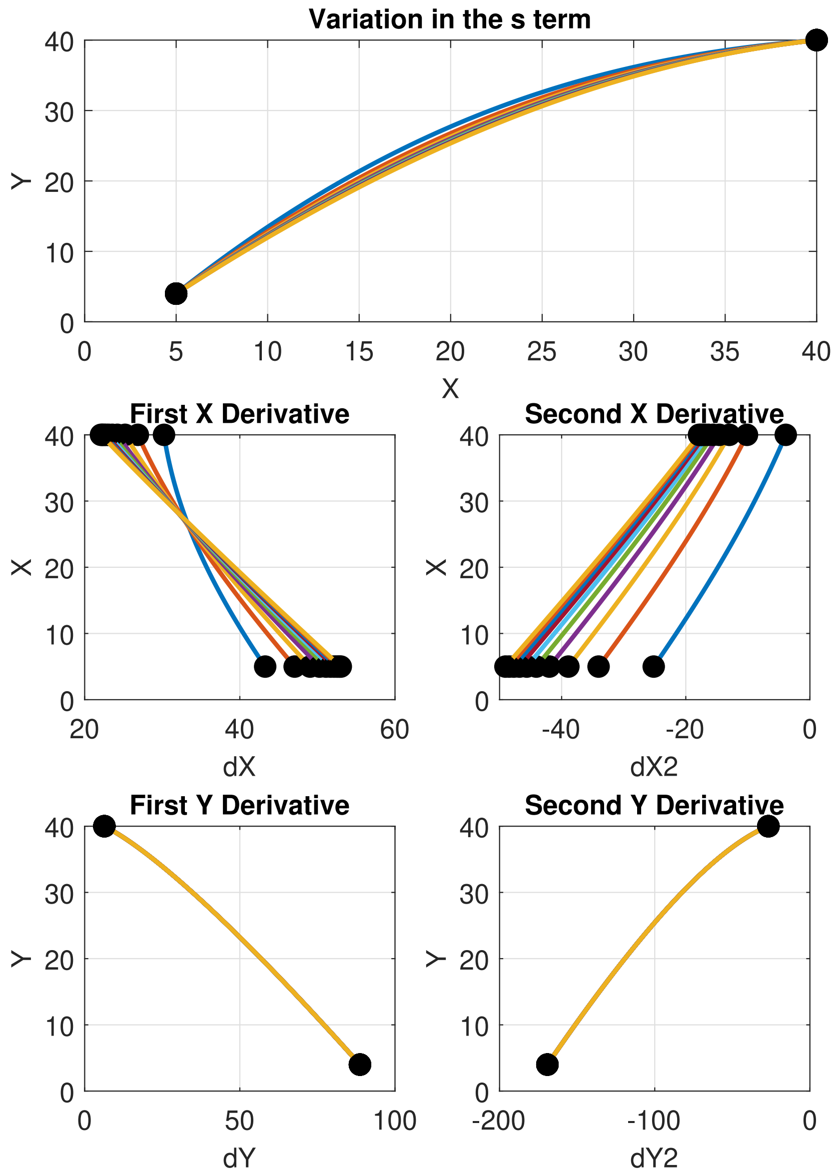

Figure 7: Variations at Point (5, 4) in Case 3

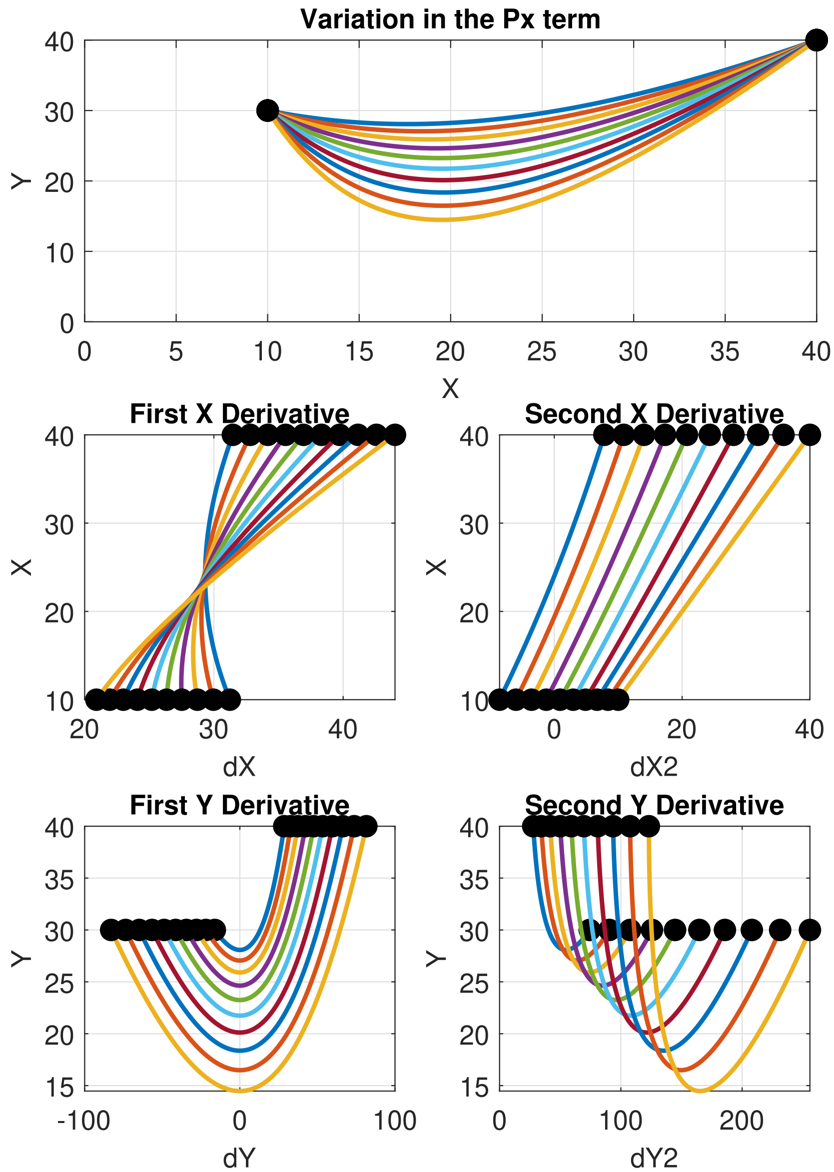

Figure 8: Variations at Point (10, 30) in Case 1

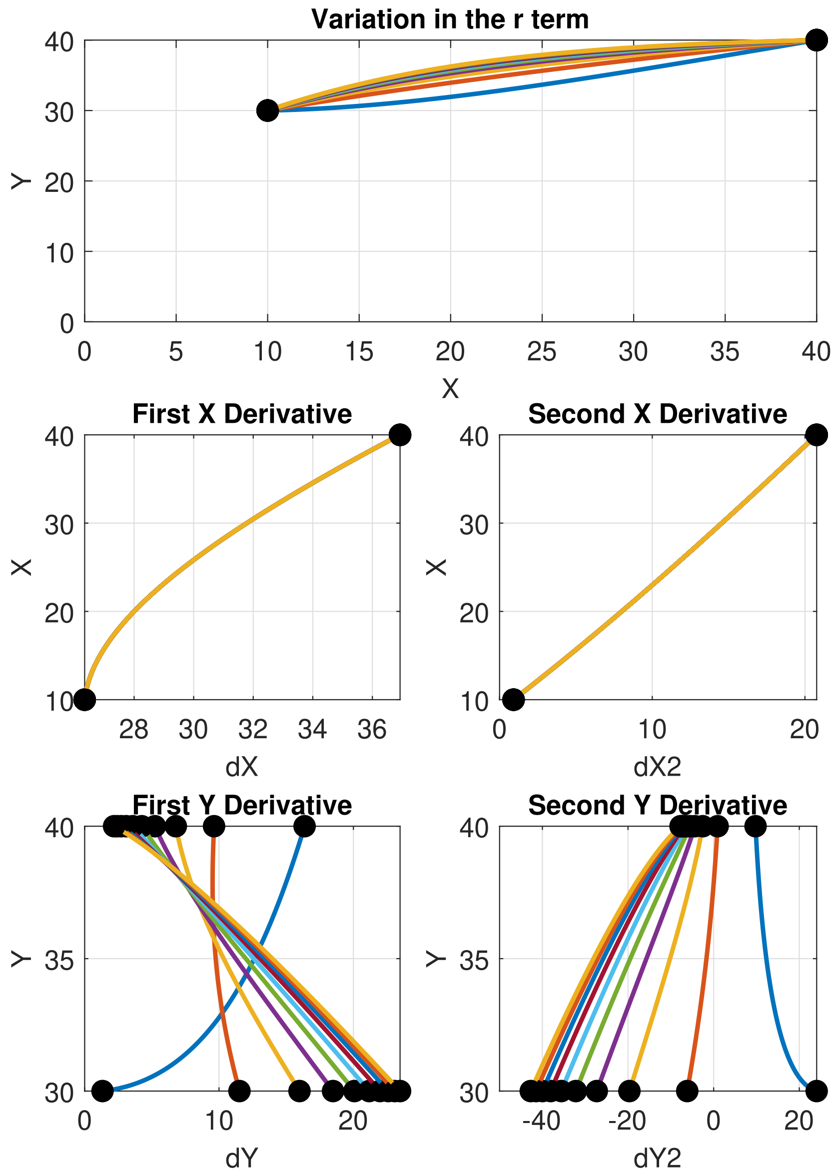

Figure 9: Variations at Point (10, 30) in Case 2

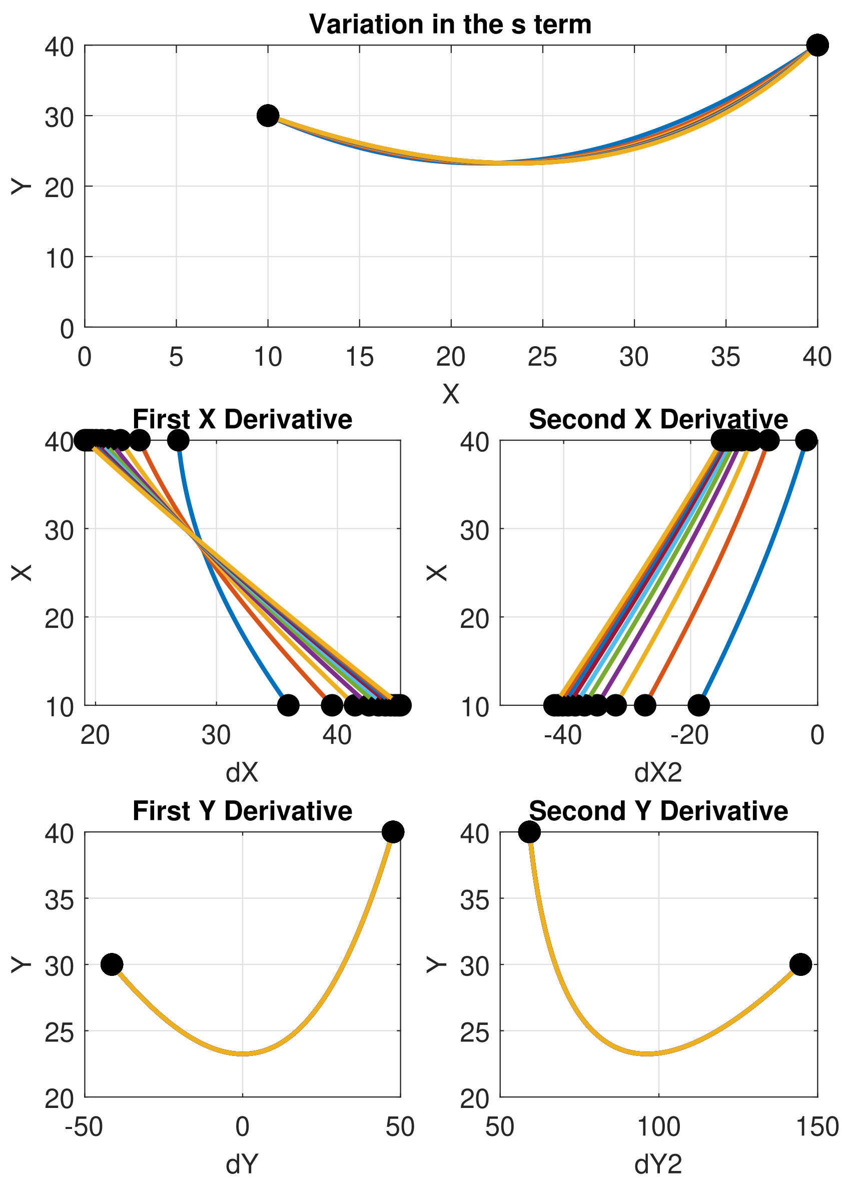

Figure 10: Variations at Point (10, 30) in Case 3

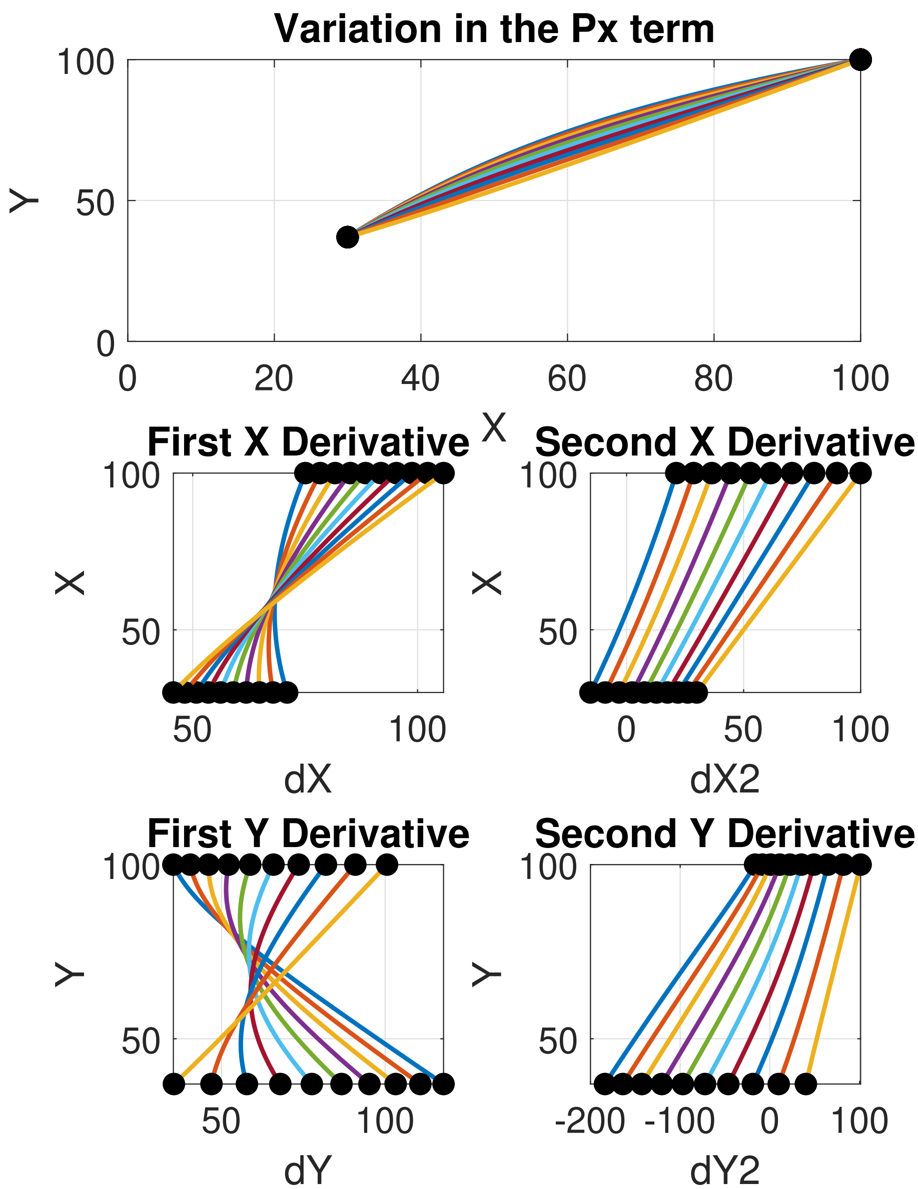

Figure 11: Variations at Point (30, 37) in Case 1 with $\mathcal{P} = [0, 100] \times [0, 100]$

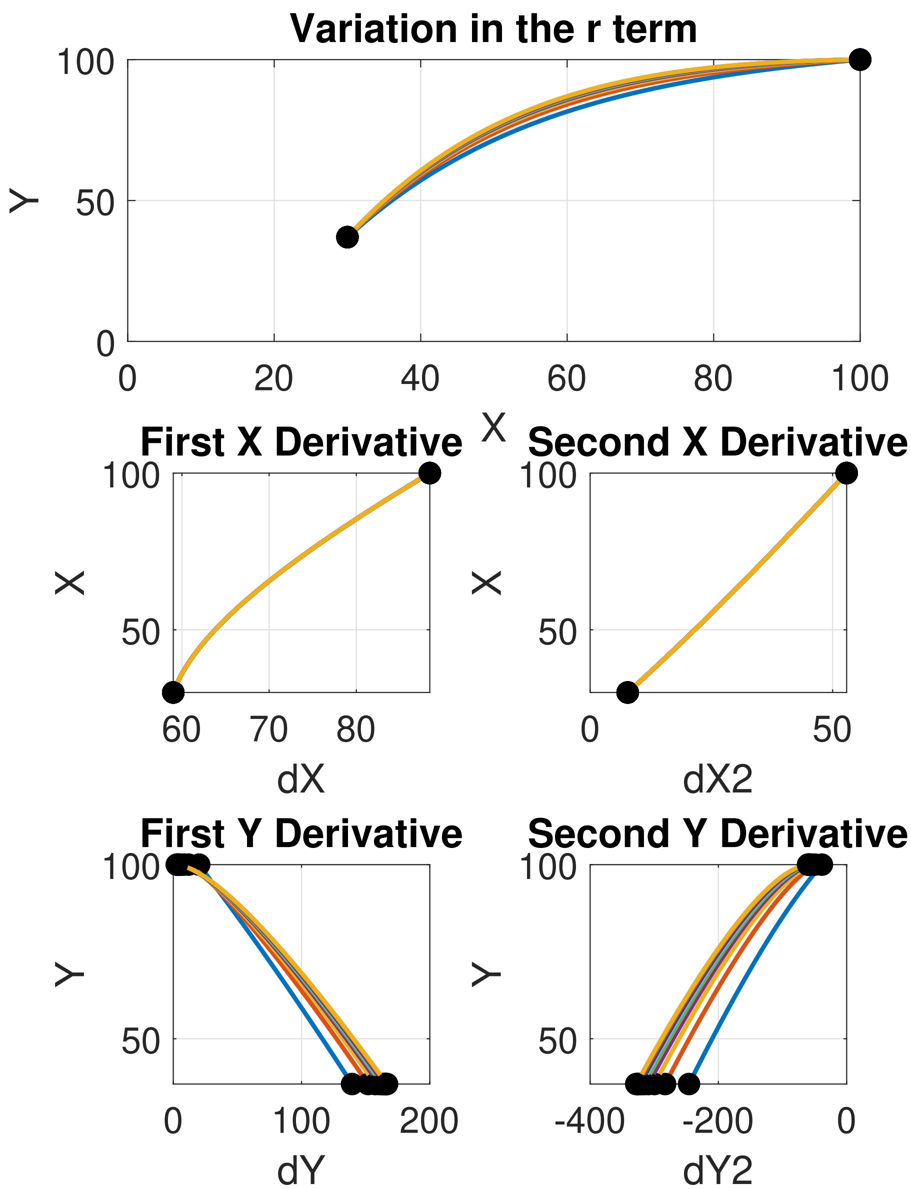

Figure 12: Variations at Point (30, 37) in Case 2 with $\mathcal{P} = [0, 100] \times [0, 100]$

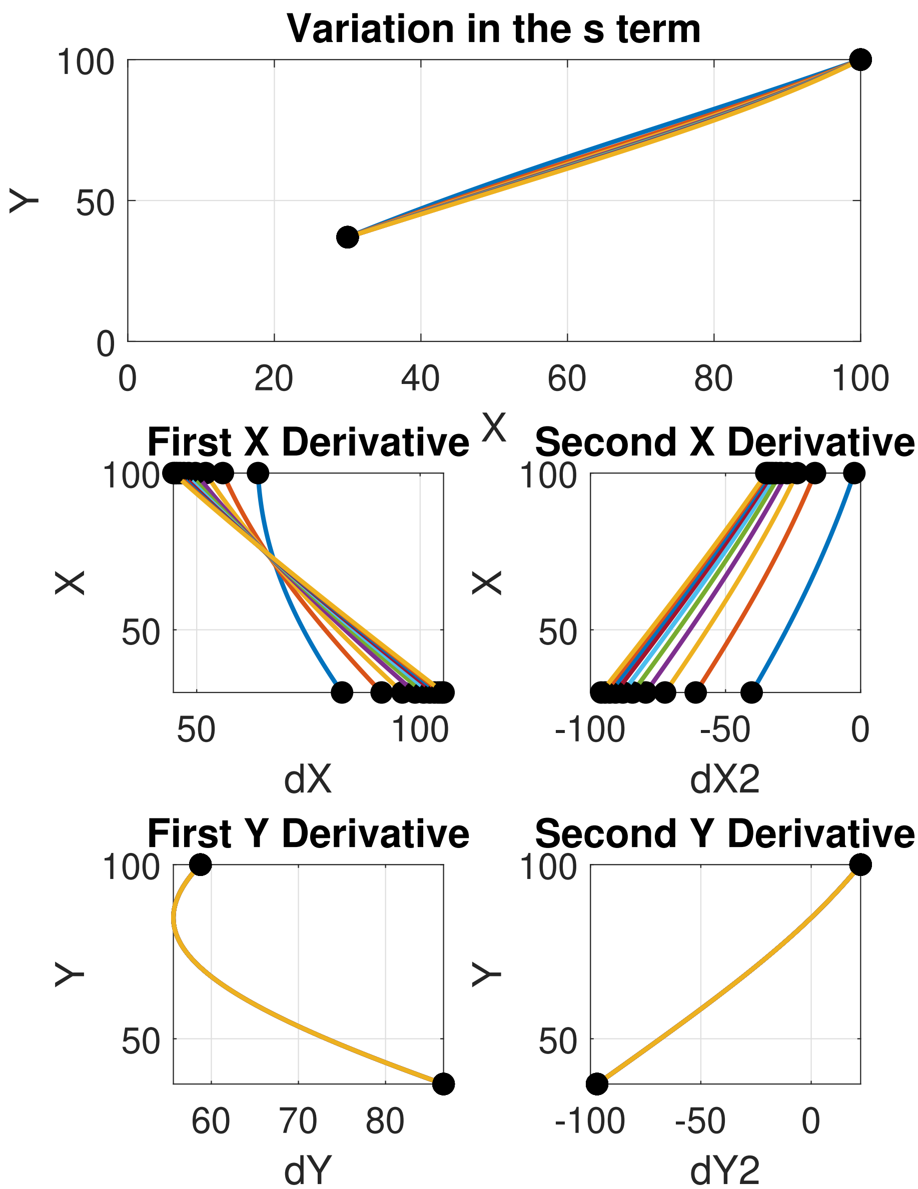

Figure 13: Variations at Point (30, 37) in Case 3 with $\mathcal{P} = [0, 100] \times [0, 100]$

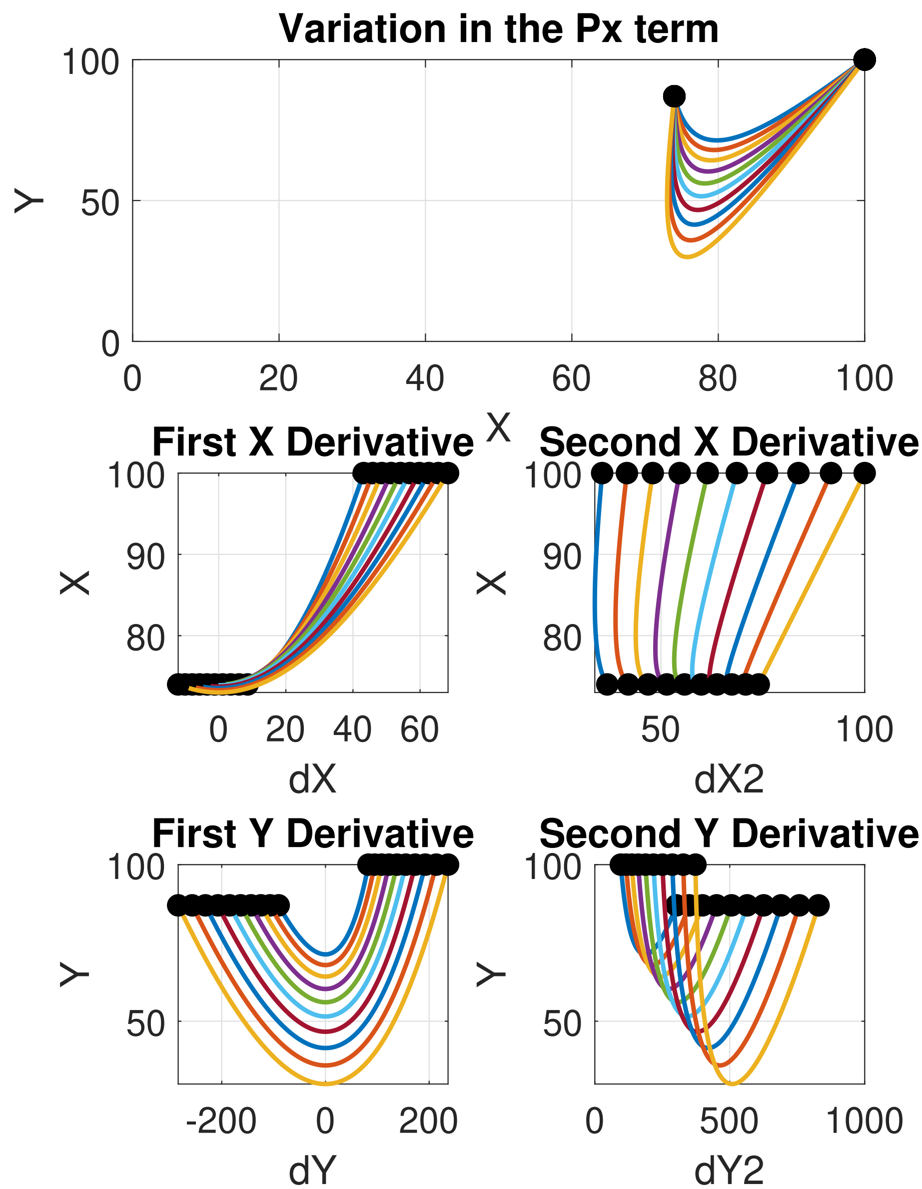

Figure 14: Variations at Point (74, 87) in Case 3 with $\mathcal{P} = [0, 100] \times [0, 100]$

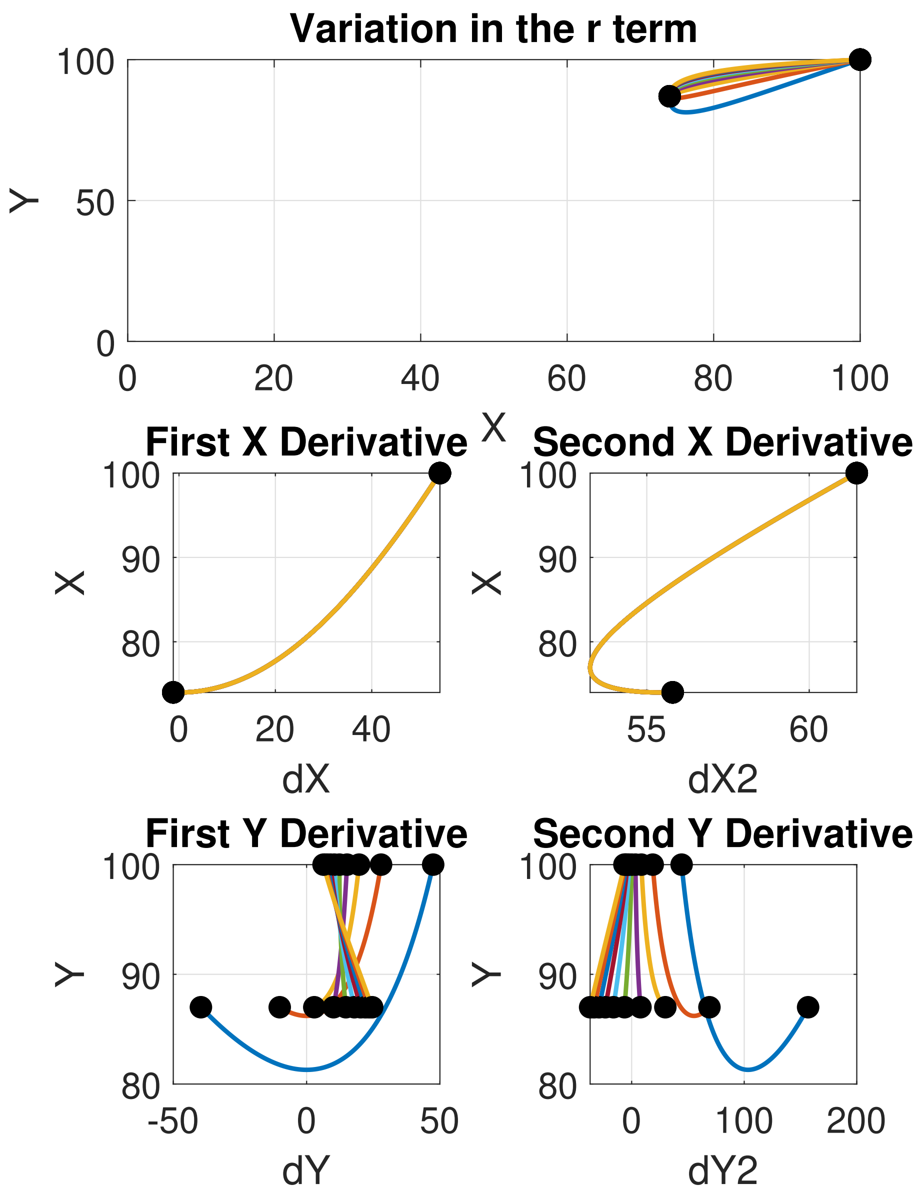

Figure 15: Variations at Point (74, 87) in Case 3 with $\mathcal{P} = [0, 100] \times [0, 100]$

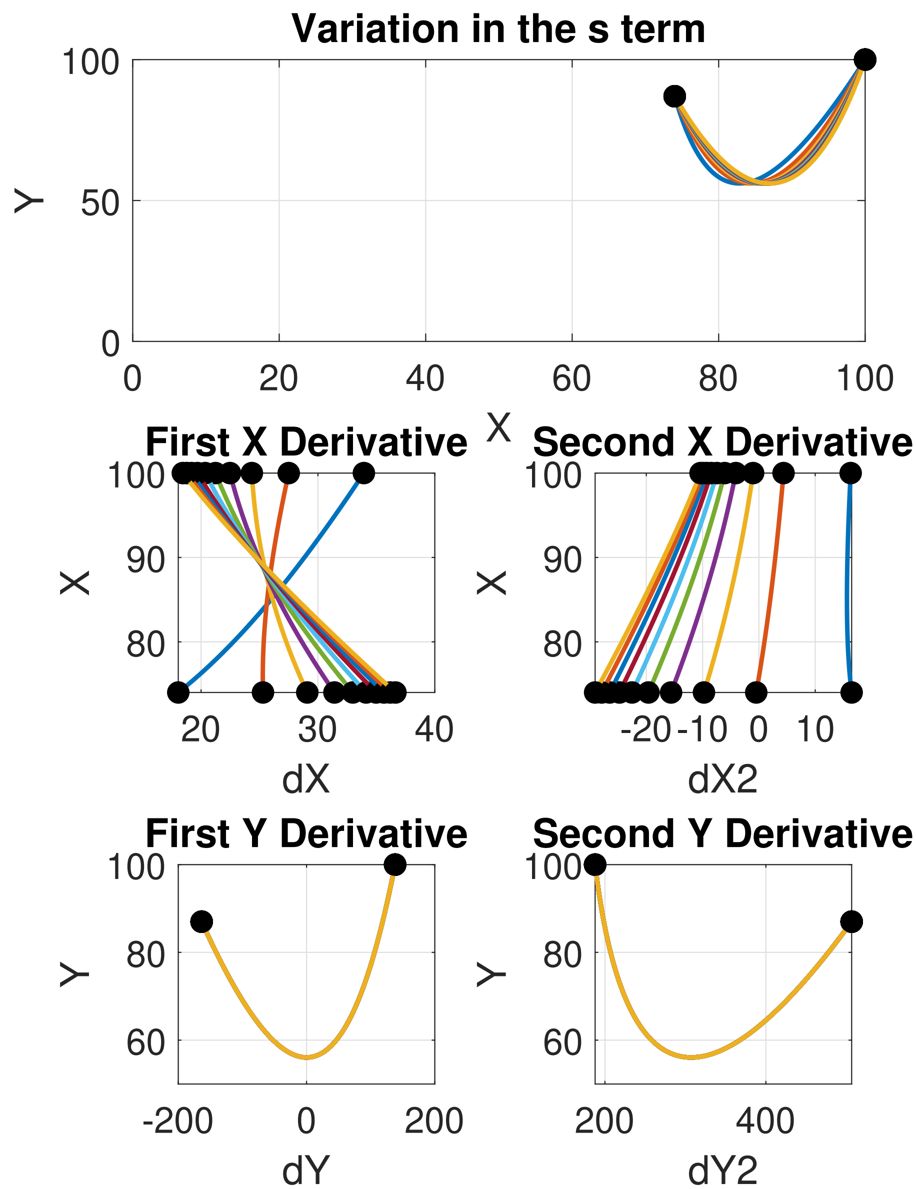

Figure 16: Variations at Point (74, 87) in Case 3 with $\mathcal{P} = [0, 100] \times [0, 100]$

Appendix C

In this appendix, we analyze the variations that may arise in our results when the size of the parking changes and the initial location is not the origin $(0, 0)$. We use the ANOVA methodology to determine if there is a significant difference by comparing three groups. The first one considers data generated by a parking lot $[0, 40] \times [0, 40]$ at initial location $(30, 37)$, the second group with a parking size $[0, 100] \times [0, 100]$ at initial location $(30, 37)$, and the third group considers a parking lot $[0, 100] \times [0, 100]$ at starting location $(90, 97)$. the ANOVA's results point out the variability between groups for all probabilities from 0 to 1 at all simulated times $t \in [1, T]$, with time increments equal to 0.01.

Table 5 summarizes the ANOVA's results. We observe that the F-value is greater than the f critical value, while the p-value equals zero. Hence, the null hypothesis is rejected; that is to say, there is a significant difference between groups when the parking lot size and initial location vary.

Table 5: ANOVA.

Source of variation	SS	df	MS	F	P-value	F critic
Sample	1267090.38	2	633545.19	6535.99	0	2.99
Columns	10081479.74	3	3360493.24	34668.68	0	2.60
Interaction	1500779.61	6	250129.93	2580.47	0	2.09
Whithin	1173648.645	12108	96.93			
TOTAL	14022998.39	12119				

References

[1] S. Aghajani, M. Kalantar, *Operational scheduling of electric vehicles parking lot integrated with renewable generation based on bilevel programming approach*. Energy **139**(2017), 422–432. DOI: 10.1016/j.energy.2017.08.004

[2] A. Ahuja, D. Waghole, S. S. Ramdasi, *Optimal Control Strategy for PEM Electric Vehicle using Matlab Simulink*. J Electrical Electron Eng **2**(2023), no. 2, 97–108. DOI: 10.33140/JEEE.02.02.04

[3] E. Albert, P. Arenas, S. Genaim, G. Puebla, *A practical comparator of cost functions and its applications*. Science of Computer Programming **111**(2015), no. 3, 483–504. DOI: 10.1016/j.scico.2014.12.001

[4] N. A. Alshammari, *Mathematical modelling for joining boron nitride graphene with other bn nanostructures*. Advances in Mathematical Physics **2020**(2020). DOI: 10.1155/2020/2394037

[5] S. Bahrami, M. J. Roorda, *Autonomous vehicle relocation problem in a parking facility*. Transportmetrica A Transport Science **16**(2020), no. 3, 1604–1627. DOI: 10.1080/23249935.2020.1769226

[6] T. Bányai, *Real-time decision making in first mile and last mile logistics: How smart scheduling affects energy efficiency of hyperconnected supply chain solutions*. Energies **11**(2018), no. 7, 1–25. DOI: 10.3390/en11071833

[7] F. M. Bergmann, S. M. Wagner, M. Winkenbach, *Integrating first-mile pickup and last-mile delivery on shared vehicle routes for efficient urban e-commerce distribution*. Transportation Research Part B: Methodological **131**(2020), 26–62. DOI: 10.1016/j.trb.2019.09.013

[8] M. D. K. Breteler, R. G. Meulenbroek, S. C. Gielen, *An evaluation of the minimum-jerk and minimum torque-change principles at the path, trajectory, and movement-cost levels*. Motor Control **6**(2002), no. 1, 69–83. DOI: 10.1123/mcj.6.1.69

[9] C. Campos, C. J. Silva, D. F. Torres, *Numerical optimal control of HIV transmission in Octave/MATLAB*. Mathematical and computational applications **25**(2019), no. 1, 1. DOI: 10.3390/mca25010001

[10] D. J. Closs, Y. A. Bolumole, *Transportation's role in economic development and regional supply chain hubs*. Transportation Journal **54**(2015), no. 1, 33–54. DOI: 10.5325/transportationj.54.1.0033

[11] M. Dell'Orco, M. Ottomanelli, D. Sassanelli, *Modelling uncertainty in parking choice behaviour*. **82**(2003).

[12] J. Eliasson, *Efficient transport pricing–why, what, and when?* Communications in Transportation Research **1**(2021), 100006. DOI: 10.1016/j.commtr.2021.100006

[13] O. Fallah-Mehrjardi, M. H. Yaghmaee, A. Leon-Garcia, *Charge scheduling of electric vehicles in smart parking-lot under future demands uncertainty*. IEEE Transactions on Smart Grid **11**(2020), no. 6, 4949–4959. DOI: 10.1109/TSG.2020.3000850

[14] M. S. Farag, M. Mohie-Eldin, H. El Shenbary, *Smart Parking Guidance Using Optimal Cost Function*. Comput. Inf. Sci. **10**(2017), no. 1, 48–53. DOI: 10.5539/cis.v10n1p48

[15] R. Z. Farahani, M. Elahipanah, *A genetic algorithm to optimize the total cost and service level for just-in-time distribution in a supply chain*. International Journal of Production Economics **111**(2008), no. 2, 229–243. DOI: 10.1016/j.ijpe.2006.11.028

[16] I. Giménez-Palacios et al., *First-mile logistics parcel pickup: Vehicle routing with packing constraints under disruption*. Transportation Research Part E: Logistics and Transportation Review **164**(2022), 102812. DOI: 10.1016/j.tre.2022.102812

[17] A. Karbowski, *Matlab implementation of direct and indirect shooting methods to solve an optimal control problem with state constraints*. Journal of Automation Mobile Robotics and Intelligent Systems **15**(2021). DOI: 10.14313/JAMRIS/1-2021/6

[18] M. G. Kay, K. Karagul, Y. Şahin, G. Gunduz, *Minimizing Total Logistics Cost for Long-Haul Multi-Stop Truck Transportation*. Transportation Research Record **2676**(2022), no. 2, 367–378. DOI: 10.1177/03611981211041596

[19] L. Komzsik, *Applied calculus of variations for engineers*. CRC Press, 2019. DOI: 10.1201/9781315215129

[20] J. Leithon, S. Werner, V. Koivunen, *Energy optimization through cooperative storage management: A calculus of variations approach*. Renewable Energy **171**(2021), 1357–1370. DOI: 10.1016/j.renene.2021.02.093

[21] Y. Li et al., *A Game Theoretic Approach for Parking Spot Search with Limited Parking Lot Information*. 2020 IEEE 23rd International Conference on Intelligent Transportation Systems (ITSC). IEEE. 2020, 1–6. DOI: 10.1109/ITSC45102.2020.9294257

[22] C. Lima, S. Relvas, A. Barbosa-Póvoa, J. M. Morales, *Adjustable robust optimization for planning logistics operations in downstream oil networks*. Processes **7**(2019), no. 8, 507. DOI: 10.3390/pr7080507

[23] J. L. Martínez, D. E. Gibaja, *An application of calculus of variations to transshipment*. Advances in Differencial Equations and Control Processes **18**(2017), no. 1, 1–16. DOI: 10.17654/DE018010001

[24] E. Martínez-Budría, S. Jara-Díaz, F. J. Ramos-Real, *Adapting productivity theory to the quadratic cost function. An application to the Spanish electric sector*. Journal of Productivity Analysis **20**(2003), no. 2, 213–229. DOI: 10.1023/A:1025184306832

[25] E. Melachrinoudis, A. Messac, H. Min, *Consolidating a warehouse network: A physical programming approach*. International Journal of Production Economics **97**(2005), no. 1, 1–17. DOI: 10.1016/j.ijpe.2004.04.009

[26] K. Nagasawa, T. Irohara, Y. Matoba, S. Liu, *Joint replenishment problem in multi-item inventory control with carrier capacity and receiving inspection cost*. Operations and Supply Chain Management: An International Journal **6**(2014), no. 3, 111–116. DOI: 10.31387/oscm0150096

[27] K. C. Onyelowe, *Application of calculus of variation in the optimization of functional parameters of compacted modified soils: a simplified computational review*. Mathematical Problems in Engineering **2021**(2021). DOI: 10.1155/2021/6696392

[28] J. A. T. Quintero, J. C. C. Morales, *Elicitación de una distribución a priori para el modelo logístico*. Comunicaciones en Estadística **10**(2017), no. 2, 225–246. DOI: 10.15332/2422474x.3175

[29] A. K. Ravat, A. Dhawan, M. Tiwari, *LMI and YALMIP: Modeling and Optimization Toolbox in MATLAB*. Advances in VLSI, Communication, and Signal Processing: Select Proceedings of VCAS 2019. Springer. 2021, 507–515. DOI: 10.1007/978-981-15-6840-4_41

[30] T. T. Sang et al., *The Optimization of Transportation Costs in Logistics Enterprises during the Covid-19 Pandemic*. ARRUS Journal of Mathematics and Applied Science **1**(2021), no. 2, 62–71. DOI: 10.35877/mathscience567

[31] I. C. Sechel, F. Mariasiu, *Efficiency of governmental policy and programs to stimulate the use of low-emission and electric vehicles: The case of Romania*. Sustainability **14**(2021), no. 1, 45. DOI: 10.3390/su14010045

[32] T. Sudan, R. Taggar, *Recovering supply chain disruptions in post-COVID-19 pandemic through transport intelligence and logistics systems: India's experiences and policy options*. Frontiers in Future Transportation **2**(2021), 660116. DOI: 10.3389/ffutr.2021.660116

[33] T. D. Vo, M. D. Tran, *The impact of covid-19 pandemic on the global trade*. Int. J. Soc. Sci. Econ. Invent **7**(2021), 1–7. DOI: 10.23958/ijssei/vol07-i01/261

[34] K. Winter, O. Cats, K. Martens, B. van Arem, *Relocating shared automated vehicles under parking constraints: assessing the impact of different strategies for on-street parking*. Transportation **48**(2021), no. 4, 1931–1965. DOI: 10.1007/s11116-020-10116-w

[35] C.-S. Yu, H.-L. Li, *A robust optimization model for stochastic logistic problems*. International Journal of Production Economics **64**(2000), no. 1-3, 385–397. DOI: 10.1016/S0925-5273(99)00074-2