REDIMAT
Journal of Research in Mathematics Education

Hipatia Press
www.hipatiapress.com

Journal of Research in Mathematics Education
Volume 13, Issue 3, 24th October 2024, Pages 268 – 292

http://dx.doi.org/10.17583/redimat.15066

Mathematical Modelling from a Semiotic-Cognitive Approach

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Abstract

Mathematical modelling has acquired relevance in different fields at an international level, both in education and research. This article states that, throughout the construction of the theoretical corpus of this mathematical process and competency – among others – two big issues have occurred: one of terminological nature since the definitions surrounding modelling have varied, and other of representational nature since different representations have been proposed to explain modelling as a cyclical process. These two issues occur mainly due to the diversity of positions on how modelling is understood and how this process is tried to be explained. To address the terminological issue, a position was adopted on the terminology surrounding modelling in Mathematics Education, based on the main theoretical developments of ICTMA Community. To address the representational issue, a modelling cycle from a semiotic-cognitive approach that represents this process in a non-set way is proposed, that is, without a strict separation between «real world» and «mathematical world». In this way, a proposal aligned with a system of theoretical principles (terminology and structure), a methodology (modelling cycle), and sketches of research questions (for future theoretical and empirical developments) is presented to address both issues.

Keywords
Modelling cycle, representations in modelling, terminology in modelling

To cite this article: Ledezma, C. (2024). Mathematical Modelling from a Semiotic-Cognitive Approach. *Journal of Research in Mathematics Education, 13*(3), pp. 268-292. http://dx.doi.org/10.17583/redimat.15066
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http://dx.doi.org/10.17583/redimat.15066

Modelización Matemática desde un Enfoque Semiótico-Cognitivo

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Resumen

La modelización matemática ha adquirido una relevancia en distintos ámbitos a nivel internacional, tanto en el plano educativo como investigativo. En este artículo se plantea que, a lo largo de la construcción del corpus teórico de este proceso y competencia matemáticos, se han sucedido – entre otros – dos grandes problemas: uno de carácter terminológico, pues las definiciones en torno a la modelización han ido variando, y otro de carácter representacional, pues se han propuesto distintas representaciones para explicar la modelización como un proceso cíclico. Estos dos problemas se deben, principalmente, a la diversidad de posturas sobre qué se entiende por modelización y cómo se intenta explicar este proceso. Para abordar el problema terminológico, se adopta una postura sobre la terminología en torno a la modelización en Didáctica de la Matemática, basada en los principales desarrollos teóricos de la Comunidad ICTMA. Para abordar el problema representacional, se propone un ciclo de modelización desde un enfoque semiótico-cognitivo que representa este proceso de manera no-conjuntista, es decir, sin una separación estricta entre el «mundo real» y el «mundo matemático». De este modo, se presenta una propuesta alineada con un sistema de principios teóricos (terminología y estructura), una metodología (ciclo de modelización), y esbozos de preguntas de investigación (para futuros desarrollos teóricos y empíricos) para atender ambos problemas.

Palabras clave

Ciclo de modelización, representaciones en modelización, terminología en modelización

Cómo citar este artículo: Ledezma, C. (2024). Modelización Matemática desde un Enfoque Semiótico-Cognitivo. *Journal of Research in Mathematics Education, 13*(3), pp. 268-292. http://dx.doi.org/10.17583/redimat.15066
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There is international consensus on the importance of modelling for the teaching and learning of mathematics, not only because of the benefits brought by the implementation of this process in the classroom (Blum, 2011), but also because its development as a competency is indispensable for educating individuals capable of linking their mathematical knowledge to current needs and demands (Maass et al., 2022). Evidence of that is, among other aspects, its curricular inclusion at an international level at different educational levels (Lingefjärd, 2006) and in international assessments, as in the case of PISA, in the domain of mathematical literacy (Turner, 2007).

In research terms, modelling has attracted the interest of many researchers in Mathematics Education in a more noticeable way than other forms of mathematical applications, due to an apparent reconciliation between the intra- and extra-mathematical that modelling provoked (Pollak, 2003). More specifically, since 2005, a substantial growth in scientific production around modelling became evident (Kutluca & Kaya, 2023), materialised in the publication of special issues of journals dedicated to this topic, such as *ZDM – Mathematics Education* (vol. 38, nos. 2 and 3, from 2006; vol. 50, nos. 1 and 2, from 2018), *Mathematics Education Research Journal* (vol. 22, no. 2, from 2010), *Avances de Investigación en Educación Matemática* (vol. 17, from 2021), *Educational Studies in Mathematics* (vol. 109, no. 2, from 2022), among others.

However, throughout the construction of its theoretical corpus – among others – two big issues have occurred. One is of terminological nature since the definitions surrounding modelling have varied; the other is of representational nature since different representations have been proposed to explain modelling as a process in the form of cycles. These two issues – which are not stated here as conflict situations but as change-and-evolution situations – occur mainly due to the diversity of theoretical positions around modelling and the use and consideration of this mathematical process and competency in different areas (Borromeo Ferri, 2013; Frejd & Vos, 2024). Thus, the terminological and representational issues are not stated in an independent way, but as interrelated.

From a broader perspective, one of the demands of Mathematics Education raised by Font and Godino (2011) is that its theoretical constructs are capable of describing, as clearly as possible, the mathematical activity occurring in a certain moment of a teaching and learning process. Therefore, the approach to the terminological and representational problems would address this demand, specifically in mathematical modelling. Having said that, the relevance of this article lies in the fact that it presents a proposal aligned with a system of theoretical principles (terminology and structure), a methodology (modelling cycle), and sketches of research questions (for future theoretical and empirical developments), in terms of Radford (2008), to address the terminological and representational problems along with the demand raised by Font and Godino (2011). To this end, the literature of research in Mathematics Education, especially from the last 50 years, is considered to address the terminological issue, and a modelling cycle, from a semiotic-cognitive approach, is proposed to address the representational issue.

Some Terminological Clarifications

In this article, the first issue stated is of terminological nature. This issue consists of throughout the construction of the theoretical corpus of modelling, the terminology surrounding this mathematical process and competency has varied due not only to the advances of research in Mathematics Education, but also to the use and consideration of modelling in different areas (Frejd & Vos, 2024). In 1983, the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) was formed, which, to this day, is the maximum reference on this matter. The discussions of the ICTMA Community allowed an evolution in the focus of educational curricula on the integration of mathematical modelling and applications, especially since the late 1980s. In turn, these discussions have given rise to reflections on *why* these topics should be taught, *what* of them should be taught, and *how* they should be taught in the classroom, like those made by the German mathematician Werner Blum and the Danish mathematician Mogens Niss (see Blum, 1991; Blum & Niss, 1989, 1991; Niss, 1989).

To address the terminological issue, a position is adopted on the terminology surrounding modelling in Mathematics Education, based on the main theoretical developments of ICTMA Community which, since its inception, has maintained a pluralistic attitude towards different modelling approaches (cf., Berry et al., 1984, 1987; Blum et al., 1989; among others). Having said that, adopting a terminological position does not imply imposing a hegemonic position, as this would lead to one of the two extreme positions proposed by Bikner-Ahsbahs and Prediger (2010) regarding ignoring other theories, but rather, it implies making explicit the position from which one speaks when using certain terms and definitions about modelling (see further discussion in Font et al., 2013). In other words, adopting a clearly defined terminological position would contribute to the theoretical proposal being capable of describing, as clearly as possible, the mathematical activity occurring in a modelling process, in response to one of the demands raised by Font and Godino (2011).

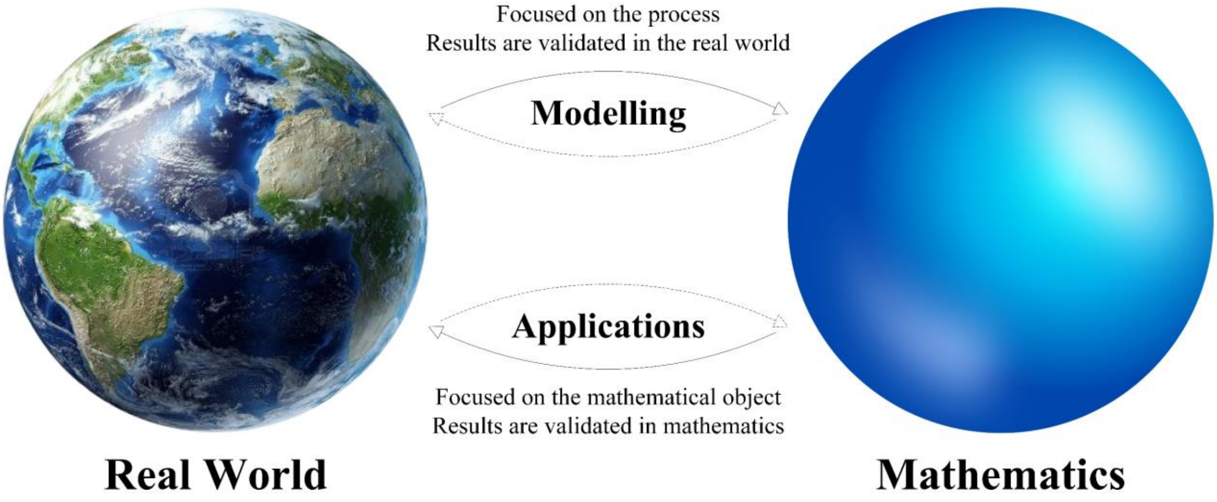
Thus, this second section [1] presents a clarification of the terms that form a fundamental part of modelling, and which serve as part of the theoretical principles (in terms of Radford, 2008) that support the proposal presented in the third section.

Mathematical Modelling and Applications

A first clarification is the differentiation between the terms *mathematical modelling* and *applications*, due to the colloquial consideration of both as synonyms. Blum (2002) states that both *mathematical modelling* and *applications* denote all types of relationships between the «real world» and «mathematical world», which Niss and colleagues (2007) complement by stating that *modelling* is the transition from the «real world» to «mathematical world», focusing on the mathematisation of reality, while *applications* are the transition in the reverse direction, focusing on the mathematical object involved. Although both are bidirectional processes, in the form «real world» ↔ «mathematical world», they differ in the starting point and the context in which they validate their results (Ledezma, 2024). Figure 1 represents both processes based on the differentiation established above.

Figure 1

Representation of the Processes of Mathematical Modelling and Applications



Source. Ledezma (2024, p. 323)

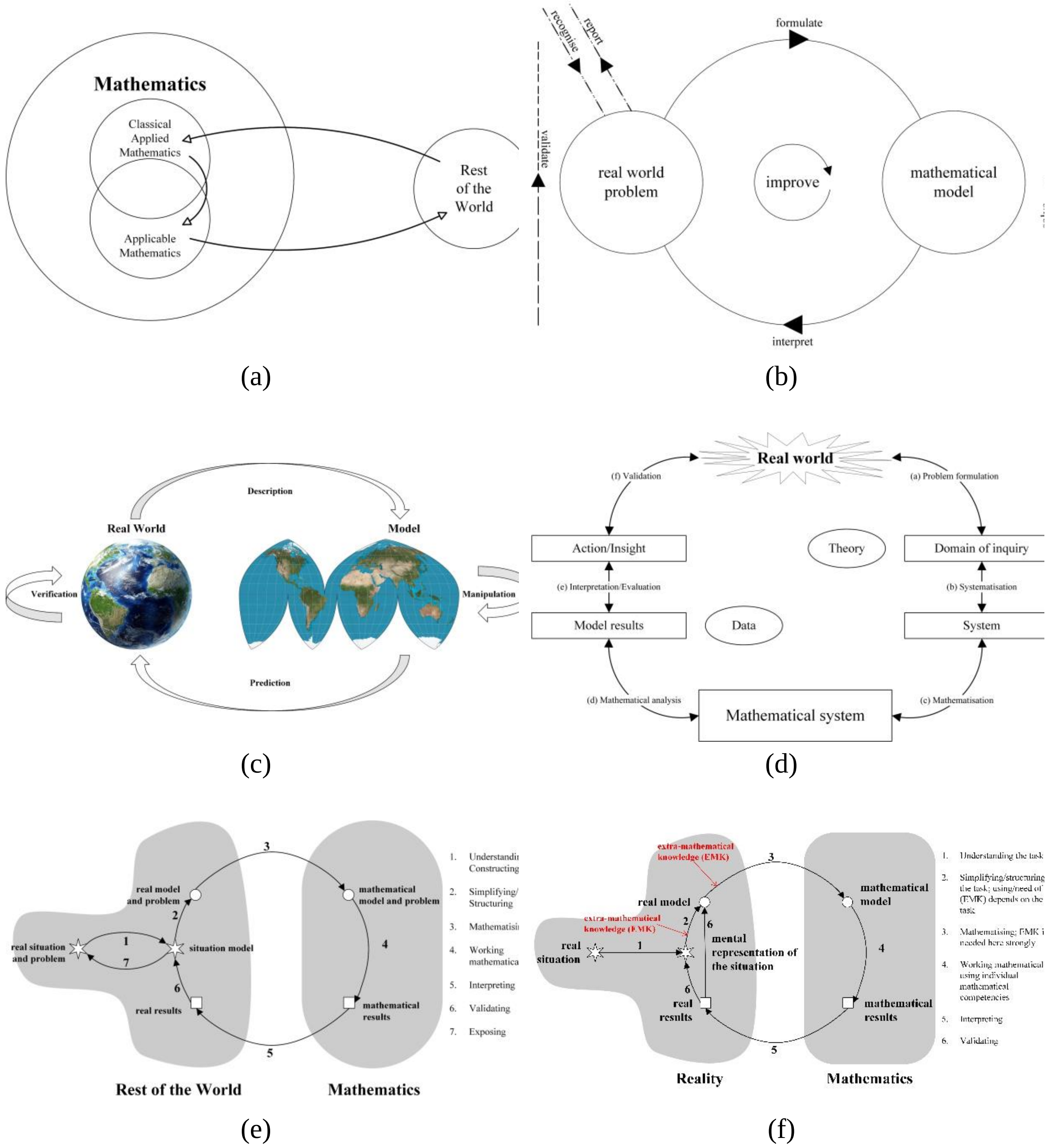
A second clarification requires considering that there is no single definition for *modelling*, this is because of the coexistence of different theoretical positions around this term. Nevertheless, a generic definition is that provided by Pollak (2007), who states that *modelling* is the process that begins with an extra-mathematical situation that is posed as a problem that is attempted (or expected) to be understood mathematically until an image that will allow the solver to obtain some answers is formed. This position leads to the clarifications presented in the following subsection.

Modelling Process and Cycle

A third clarification is the differentiation between the terms *modelling process* and *cycle*. While *modelling* is the mathematical activity in which a *modelling process* is carried out (i.e., the transition from the «real world» to «mathematical world», focusing on the mathematisation of reality), the *modelling cycle* is an idealised representation of the *modelling process* (Ledezma, 2024). In the specialised literature, different *cycles* have been proposed to explain the *modelling process*, which represent the approaches to its understanding and the type of tasks that its structure aims to describe (Borromeo Ferri, 2006). Figure 2 presents some of these *modelling cycles*.

Figure 2

Different Modelling Cycles: (a) Cycle of Applied Mathematics, (b) Cycle to Tackle a Real problem Mathematically, (c) Models-and-Modelling cycle, (d) Graphic Model of a Modelling Process, (e) Seven-Steps Cycle, and (f) Cycle from a Cognitive Perspective



Source. Adapted from (a) Pollak (1979, p. 323), (b) Burkhardt (1981, p. 3), (c) Lesh & Doerr (2003, p. 17), (d) Blomhøj (2004, p. 148), (e) Blum & Leiß (2007, p. 225), (f) Borromeo Ferri (2018, p. 15).

Despite having different structures, *modelling cycles* tend to converge in similar phases to explain the *modelling process*: identification of the essential characteristics of a problem in the real world, simplification of the problem to develop a workable model, elaboration of justifiable assumptions to accommodate missing information; translation of the real situation into a mathematical model (mathematisation), generation of an initial solution from the mathematical model, interpretation of the resulting solutions in the initial context of the problem, validation of a potential solution, and revision of the process until establishing an acceptable solution (Geiger et al., 2018). In a retrospective vision, the Dutch mathematician Hans Freudenthal's (1968 and so on) contributions allowed it to delve deeper into the mathematisation aspects of the *modelling process*, while the Austrian American mathematician Henry O. Pollak's (1968 and so on) contributions provided a circular structure to the *modelling cycles* (Kaiser et al., 2015). As shown in the previous description, a fundamental step in the *modelling process* is the *mathematisation* of the problem-situation, which leads to the clarifications presented in the following subsection.

Mathematisation and Mathematical Model

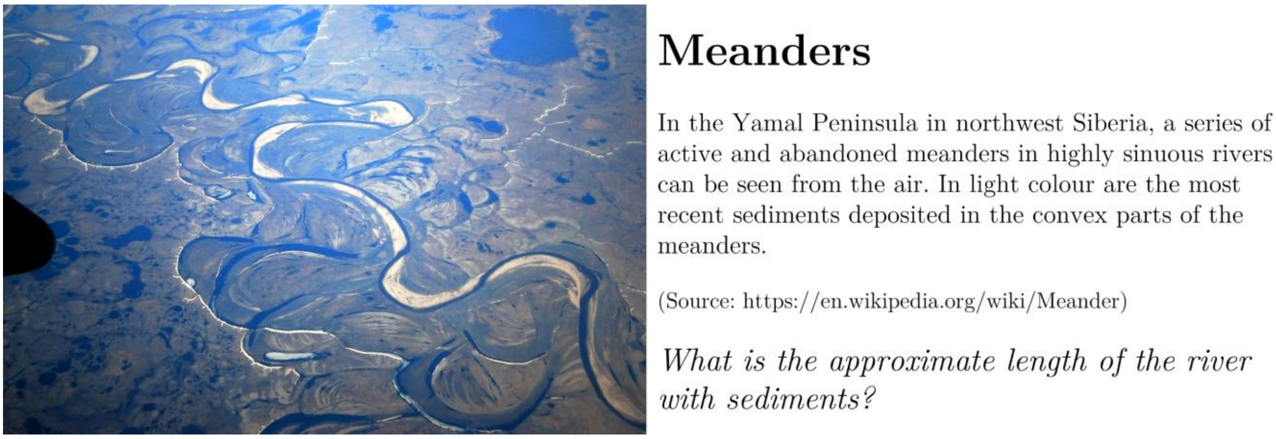
A fourth clarification is on the term *mathematisation*, which was refined during the end of the twentieth century and, broadly speaking, is understood as the translation of an extra-mathematical situation into mathematical language. Treffers (1987) raises the ideas of two types of *mathematisation*: one is *horizontal*, referred to making a problem-situation accessible for mathematical treatment (in the most formal sense of the word), and the other is *vertical*, referred to a more or less formal mathematical processing. Later, Freudenthal (1991) refined this distinction, referring to *horizontal mathematisation* as that which leads from the real world to the world of symbols and, in the latter, "symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, [and] reflectingly" (p. 42), referring to *vertical mathematisation*. It must be stressed that every *modelling process* implies the development of both types of *mathematisation*; however, *mathematisation* per se does not necessarily imply the development of a *modelling process* (Ledezma, 2024).

Mathematisation involves two types of representations: an input and output (Ledezma et al., 2023) known as *models*. At the beginning of the twentieth century, the term *model* was fundamentally associated with three-dimensional material objects, and its definition was evolving (see Friedman & Krauthausen, 2022). Therefore, a fifth clarification is about one of these two types of representations, that is, the term *mathematical model* (output representation) which, as with *modelling*, requires considering that there is no single definition for it, also because of the coexistence of different theoretical positions around this term. Nevertheless, a generic definition is that provided by Niss and colleagues (2007), who state that a *mathematical model* consists of a triad between an extra-mathematical domain (D), a mathematical domain (M), and a rule mapping from D to M (f), of the form (D, M, f) (see Figure 1). More specifically, a *mathematical model* consists of mathematical objects (functions, vectors, equations, etc.) defined as essential to explain the relationships between the problem-situation from the real world (*real model*) and between the objects that represent these relationships (Wess et al., 2021). Therefore, the construction and/or use of a *mathematical model* is the core of the *modelling process* (Ledezma, 2024). The other type of representation involved in *mathematisation*, that is, the *real model* (input representation), will be clarified in the next section.

Modelling Problem

A sixth clarification is the characterisation of a *modelling problem*, which corresponds to a type of *applied mathematical problem* (in terms of Blum & Niss, 1991) with certain characteristics (see Borromeo Ferri, 2018). To exemplify this characterisation, the *Meanders Problem* is used (see Figure 3).

Figure 3
Meanders Problem



Source. Author's elaboration.

The problem in Figure 3 is characterised by being *open*, that is, the situation presented is not limited to a specific answer and/or procedure; also, the situation is *complex*, since the solver must try to understand the context in which it is posed and look for the relevant data for its solving. The wording of this problem describes a *realistic* situation, as real-world elements are used (the geography of Yamal Peninsula), and it is also *authentic*, since the described situation is plausible to have occurred in the past, to be occurring in the present, or to occur in the near future (in terms of Palm, 2007). A *modelling problem* is essentially a *mathematical problem* that cannot be solved by applying known algorithms or routine procedures (Schoenfeld, 1994), but which does require strategies for its solving (Lesh & Doerr, 2003). Finally, a *modelling problem* must be *solvable through the modelling process*, which implies that all the phases of a *modelling cycle* are used for its solving. *Modelling problem* solving requires a competency work, which leads to the clarifications presented in the following subsection.

Modelling (Sub)Competency(ies)

As mentioned before, *modelling* is the mathematical activity in which a *modelling process* is carried out, which is why *modelling* is considered as a process; however, *modelling* can also be considered as a mathematical *competency*. Therefore, a seventh clarification is about the *competency* view of *modelling* which, as with the previous terms, requires considering that there is no single definition for it, because of the different approaches to this topic (see Kaiser & Brand, 2015).

During the first decade of the twenty-first century, from the development of the KOM Project (Danish acronym for 'Competencies and Learning of Mathematics') proposals were derived on how to define and characterise the mathematical competence and, among others, the *modelling competency* (see Niss, 2003). At a more advanced stage of this project, Blomhøj and Højgaard (2007) define the *modelling competency* as "someone's insightful readiness to carry through all parts of a mathematical modelling process in a certain context" (p. 48), based on the *modelling cycle* proposed in previous studies (see Figure 2d). Likewise, these authors proposed two approaches for the development of this *competency*: a holistic approach (encompassing the development of the complete *modelling cycle*) and an atomistic approach (focusing especially on *mathematisation* and working with the *mathematical model*). Thus, one can talk about the existence of *modelling competency* and *sub-competencies* (Ledezma, 2024).

Currently, assuming the existence of other positions (such as curricular ones), the modelling research community has converged on the approach of two major definitions of *modelling competencies*. On one hand, a general definition has been agreed upon (similar to the holistic approach of Blomhøj & Højgaard, 2007), proposed by Niss and Højgaard (2019), which is called *Mathematical modelling competency – analysing and constructing models of extra-mathematical contexts and situations*, defined as follows:

> This competency focuses on mathematical models and modelling, i.e., on mathematics being put to use to deal with extra-mathematical questions, contexts and situations. Being able to construct such mathematical models, as well as to critically analyse and evaluate existing or proposed models, whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account, are the core of this competency. It involves relating to and navigating within and across the key processes of the 'modelling cycle' in its various manifestations. (p. 16)

On the other hand, particular definitions have been agreed upon (similar to the atomistic approach of Blomhøj & Højgaard, 2007) for the different *modelling competencies*. In this position, these *competencies*, which correspond to the transitions of the modelling cycle, are understood through *sub-competencies* for each one (Kaiser & Brand, 2015). Table 1 presents the proposal by Greefrath and Vorhölter (2016), based on that by Kaiser (2007), to define *modelling sub-competencies*.

Table 1
Mathematical Modelling Sub-Competencies

Sub-competencies	**Indicators**
Constructing	They construct their own mental model from a given problem and thus formulate an understanding of their problem.
Simplifying	They identify relevant and irrelevant information from a real problem.
Mathematising	They translate specific, simplified real situations into mathematical models (for example, terms, equations, figures, diagrams, functions, etc.).
Interpreting	They relate results obtained from manipulation within the model to the real situation and thus obtain real results.

Sub-competencies	**Indicators**
Validating	They judge the real results obtained in terms of plausibility.
Exposing	They relate the results obtained in the situational model to the real situation, and thus obtain an answer to the problem.

Note. Working mathematically (working with mathematical methods in the mathematical model and getting mathematical solutions) does not appear in this table, since it is not considered as a specific sub-competency to the modelling process. *Source*. Adapted from Greefrath & Vorhölter (2016, p. 19).

Modelling Perspectives

A last clarification is about *modelling perspectives*, understood as the way in which the *modelling process* is defined, addressed, and characterised (Ledezma, 2024). Kaiser-Messmer (1986) distinguished two predominant currents of discussion until the 1980s: on one hand, a *scientific-humanist perspective*, focused on the interactions between the intra- and extra-mathematical, led by the work and reflections of Freudenthal and, on the other hand, a *pragmatic perspective*, focused on the use of mathematics to solve practical problems, led by the work and reflections of Pollak. Years later, Kaiser and Sriraman (2006) recognised other *perspectives* that began to be developed, as presented in Table 2.

Table 2

Classification of Modelling Perspectives According to Kaiser and Sriraman (2006)

Perspectives	**Objectives**	**Background**
Realistic or applied	Pragmatic-utilitarian objectives, such as solving real-world problems, understanding the real world, and promoting modelling competencies.	Pragmatic perspective (Anglo-Saxon pragmatism and applied mathematics).
Contextual	Psychological objectives and subject-related objectives, such as solving word problems.	Information processing approaches leading to systems approaches (American debate on problem solving as well as everyday school practice and psychological lab experiments).
Educational: (a) didactical (b) conceptual	Pedagogical and subject-related objectives, such as (a) structuring and promoting learning processes and (b) introducing and developing concepts.	Integrative perspective and further developments of the scientific-humanist perspective (didactical and learning theories).
Socio-critical	Pedagogical objectives, such as critical understanding of the surrounding world.	Emancipatory perspective (socio-critical approaches in political sociology).
Epistemological or theoretical	Theory-oriented objectives, such as promoting theoretical development.	Scientific-humanist perspective (Roman epistemology).

Perspectives	Objectives	Background
Cognitive	Research objectives: analysing and understanding the cognitive processes that take place during the modelling process. Psychological objectives: promoting mathematical thinking processes using models as mental or physical images, or by emphasising modelling as a mental process (abstraction or generalisation).	Cognitive psychology.

Source. Adapted from Kaiser & Sriraman (2006, p. 304).

In the classification of Table 2, Kaiser and Sriraman (2006) recognise the *cognitive perspective* as a *meta-perspective* which, at that time, was considered the most recent for the analysis of the *modelling process* (see Borromeo Ferri, 2006). Finally, these authors point out that this classification had not to be understood in exhaustive terms, since its intention was to demonstrate, on one hand, the current state of advances in *modelling* research and, on the other hand, that these advances come from the evolution of already existing traditions.

In their literature review, Abassian and colleagues (2020) notably refined the proposal by Kaiser and Sriraman (2006), based on the two main goals that are proposed for *modelling*: to facilitate the learning of mathematics using contextual situations and to explore extra-mathematical scenarios using mathematics as a tool to learn how to model. Thus, these authors proposed the classification presented in Table 3.

Table 3

Classification of Modelling Perspectives According to Abassian and Colleagues (2020)

Perspectives	Goals and Research Focus	Characteristics
Realistic	Aims to develop modelling skills to model and understand authentic real-world scenarios. Focuses on modelling competencies.	The MM explains the given real-world scenario. The MC is a multistep process that begins in the real world, is mathematised, and ends in the real world. The MP is an authentic and messy task.
Educational	Aims to develop modelling skills to model and understand mathematics. Focuses on mathematics in modelling and on modelling in the curriculum.	The MM has a relationship to the given real-world scenario. The MC is a multistep process that begins in the real world, is mathematised, and ends in the real world.

Perspectives	Goals and Research Focus	Characteristics
		The MP is an authentic task that can be simplified to reveal specific mathematical aspects.
Models and Modelling	Aims to develop a deep understanding of mathematics through a modelling context. Focuses on the use of model-eliciting activities to teach mathematics.	The MM is a conceptual system that maps the structural characteristics of a relevant system. The MC begins in the real world, develops a model, and goes back to the real world as many times as needed. The MP is a model-eliciting activity.
Socio-Critical	Aims to develop modelling skills to make decisions in society. Focuses on students' use of mathematics to understand society critically.	The MM is a representation of a relevant scenario. The MC considers all the aspects of the modeller's participation in the exploration of a real-world problem. The MP is a task in a societal context.
Epistemological	Aims to develop formal mathematical reasoning. Focuses on teaching and learning specific mathematical concepts.	The MM is the result of an activity based on situations and mathematical concepts. The MC are four-stage activities to develop formal mathematical reasoning. The MP does not have requirements set.

Note. MM = Mathematical model, MC = Modelling cycle, MP = Modelling problem. *Source*. Adapted from Abassian et al. (2020, p. 56).

Unlike the classification in Table 2, Abassian and colleagues (2020) included the *cognitive meta-perspective* within the *realistic* and partially *educational perspectives*. Finally, the literature review conducted by Preciado and colleagues (2023) is highlighted, who privileged the geographical attribute in the distribution of research on *modelling* and considered the publication media with the greatest international impact as sources of their review (publications until 2020). Thus, the following results stand out:

- The six countries with the most scientific production in *modelling* are (in descending order) Germany, United States, Australia, Brazil, Japan, and United Kingdom. Of these countries, the first three produced almost half of the total reviewed publications (249 of 500).
- The most worked *modelling perspective* is the *educational* one.
- The most researched topics are (in descending order) specific mathematical contents and modelling competencies.

Taking into consideration that the review by Preciado and colleagues (2023) is one of the most recent, certain trends can be determined at an international level. Firstly, the evident predominance of the *educational perspective* (and its *realistic* variant) in research on *modelling*; secondly, the articulated nature of the *models and modelling perspective*, mainly worked on the United States, with its national curricular proposal; thirdly, the existence of two minor *modelling* approaches, namely, the *socio-critical* and *epistemological perspectives* (Frejd & Vos, 2024); fourthly, regarding the *socio-critical perspective*, a localist nature is evident, being strongly focused on Brazil and, regarding the *epistemological perspective*, whose most notable manifestation is evident in the works developed from the Anthropological Theory of the Didactic, its approach is quite questionable in terms of its conception of what *modelling* is and how it is characterised.

A Semiotic-Cognitive Approach to Modelling

In this article, the second issue stated is of representational nature. This issue consists of throughout the construction of the theoretical corpus of modelling, the way of representing this process has been as cycles with set characteristics (similar to a Venn diagram), where a strict separation between «real world» and «mathematical world» is suggested (see Figure 2). More specifically, the descriptions of the modelling cycles use a natural-language register, accompanied by their graphical representation with a white region surrounding both worlds, thus raising some question about its meaning. In other words, two explicit worlds («real world» and «mathematical world») and a third implicit world (the white region) are created.

To address the representational issue, a modelling cycle that represents this process in a non-set way is proposed, that is, without a strict separation between «real world» and «mathematical world». This proposal is derived from the reflections on an analysis model for the modelling process developed in two previous studies, whose common denominator was the questioning of said representational structures for this process. Having said that, proposing a modelling cycle does not imply imposing a hegemonic representation, but rather, it implies presenting a new representation for the modelling process based on the theoretical principles underlying the terminology adopted in this study. Furthermore, this new representation for the modelling process includes a methodology (in terms of Radford, 2008), supported by the adopted terminological position, which would contribute to describing, as clearly as possible, the mathematical activity occurring in a modelling process, in response to one of the demands raised by Font and Godino (2011).

Thus, this third section [2] presents the proposal of a modelling cycle with its corresponding theoretical principles and methodology, based on the reflections of two previous studies and the terminology presented in the previous section.

Background of Previous Studies

In previous studies, modelling – as a relevant process of mathematical activity – was analysed from the perspective of other theoretical frameworks of Mathematics Education. In a first study, Ledezma and colleagues (2023) developed a theoretical articulation between the Modelling Cycle from a Cognitive Perspective (proposed by Borromeo Ferri, 2018, see Figure 2f) and the tools for the analysis of mathematical activity proposed by the Onto-Semiotic Approach (OSA, Godino et al., 2007). As a result of this study, a model was proposed for the analysis of the mathematical activity underlying the modelling process which coordinated the structure of phases and transitions of the cycle to explain the modelling activity with the tools provided by the OSA to analyse the mathematical activity.

In a second study, Ledezma and colleagues (2024) expanded the articulation of the first study, adding the analysis of intra- and extra-mathematical connections that can be established when solving a modelling problem, based on the proposal of the Extended Theory of Mathematical Connections (see Rodríguez-Nieto et al., 2023). As a result of this study, a more refined model was proposed for the analysis of the mathematical activity underlying the modelling process, which coordinated the structure and tools of the first study with the identification of intra- and extra-mathematical connections established by a solver.

As a result of the studies described above, the *Semiotic-Cognitive Analysis Model for the Modelling Process* was proposed, which allows to analyse the mathematical activity underlying the modelling process. To this end, this model allows to identify the following onto-semiotic aspects:

- The mathematical practices performed by an individual to solve a modelling problem.
- The mathematical objects intervening in such practices.
- The semiotic functions established among such objects.
- The mathematical processes occurring in the modelling activity.
- The intra- and extra-mathematical connections established by a solver.

Furthermore, this model has a structure of phases and transitions to describe the modelling process from a cognitive perspective (see a more detailed description of this analysis model in Ledezma, 2024).

Nevertheless, in the analyses of mathematical activity performed in both studies, a strict separation between «real world» and «mathematical world» was not appreciated as such, which is why the need arose to propose a new representation for the modelling process, as detailed in the following subsection.

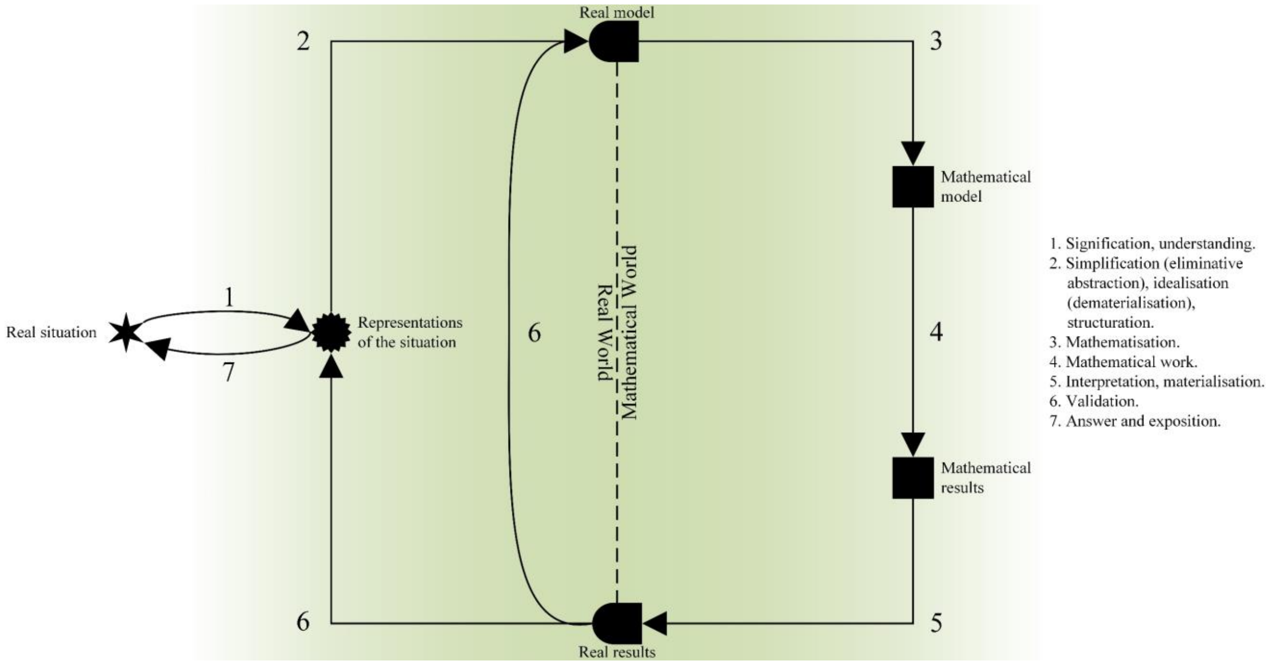
Semiotic-Cognitive Mathematical Modelling Cycle

Faced with the situation of addressing the representational issue in modelling, two main positions can be assumed (see Font, 2003). On one hand, an empiricist position, in which mathematical thinking is a certain way of thinking about the «world of things» with which they have mutual dependence. On the other hand, a Platonist position, in which «mathematics» is part of a different world from the «world of things», where symbols and other mathematical representations are the ostensive part of a series of mathematical objects that have an idealised and independent existence of individuals in a «mathematical world». Historically, the development of research on modelling has tended to assume the existence of two worlds (cf. Kaiser-Messmer, 1993; Niss, 1987; among others), in which the objects of the «mathematical world» allow explaining (and solving) a situation from the «real world».

The proposal presented below assumes this same Platonist position as a theoretical-philosophical principle (in terms of Radford, 2008), maintaining the tradition of the most important developments of ICTMA Community and being aligned with the terminological position adopted in this study. Thus, Figure 4 presents the *Semiotic-Cognitive Mathematical Modelling Cycle* (SCMMC) which, in line with the cycles presented in Figure 2, assumes the existence of two worlds: the real and the mathematical world. Nevertheless, this cycle does not include the white region in its representation to establish a separation between two (or three) worlds.

Figure 4

Semiotic-Cognitive Mathematical Modelling Cycle



Source. Ledezma (2024, p. 596).

The SCMMC takes as a basis a structure of six phases and seven transitions to explain the modelling process from the Blum/Kaiser-Messmer's proposal and its subsequent theoretical developments (see Figure 2e and Figure 2f). Likewise, this cycle refines the transitions based on the analyses of the mathematical activity underlying the modelling process developed in the studies mentioned before (Ledezma et al., 2023, 2024), supporting them in the modelling sub-competencies (see Table 1) and refining the terminology of some of them. In this way, the SCMMC describes the modelling process as follows:

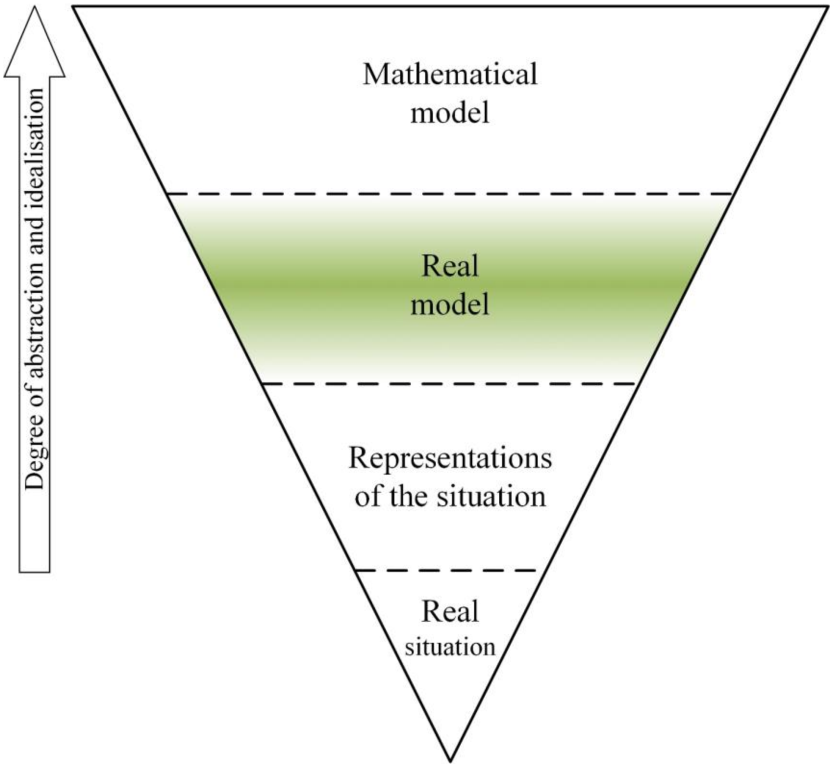
> The individual begins with a *real situation* (RS), which corresponds to a situation with a realistic and authentic context, which can use different types of representations to be posed, and which is already problematised. Then, the individual generates his own *representations of the situation* (RoS) based on his understanding of the RS, taking into consideration his previous experiences, what the RS requests to be solved, and how it could be solved. To build a *real model* (RM), the individual must simplify (make eliminative abstractions) and idealise (dematerialise) the RS, taking into consideration the conditions and characteristics of the RS context, and structure it in a representation on which later to be able to work in mathematical terms. The *mathematical model* (MM) takes into consideration the mathematical objects involved in the modelling activity that allow explaining the RS through the mathematisation of the RM. From the mathematical work with the MM,

mathematical results (MR) emerge, which must be interpreted and materialised in the RS context to have *real results* (RR). Finally, the validation of the RR will occur through a comparison of the RR↔RoS↔RM triad, which will lead to the formulation and exposition of a plausible answer.

Representations in the SCMMC

One of the didactic criticisms made to the Platonist position is about the little importance it attributes to ostensive representations (see Font & Peraire, 2001); however, this issue is addressed by the SCMMC. Unlike the Blum/Kaiser-Messmer's proposal and its subsequent theoretical developments (see Figure 2e and Figure 2f), the representation of the SCMMC in Figure 4 does not separate the «real» and «mathematical» worlds as two sets, but rather establishes a boundary between them. Therefore, it is intended to contribute to the refinement of the characterisation of the representations used in the transitions of the RS→RoS→RM→MM phases, as shown in Figure 5.

Figure 5
Representations in the SCMMC



Source. Ledezma (2024, p. 598).

The *real situation* is the starting point of modelling activity, which can use different types of representations to be posed, which include experiencing the situation, working with a material representation, a statement with text and a picture, with only a picture, or with only text. There different representations are horizontal in nature, since they are located in the same «real world», in addition to being vicarious, that is, each one can act on behalf of the others. For example, if a modelling problem consists of calculating the height of a mountain, there are different options to pose the *real situation*, such as visiting the mountain (experiencing the situation), working with a scale model such as a mock-up (material representation), or with a photograph of the mountain scenery (statement with text and/or picture).

Now, understanding the *real situation* is a very complex process that requires the articulation of many cognitive elements to give it meaning and, in this way, generate *representations of the situation*. Nevertheless, there *representations* (mental images, sketches, etc.) are already somewhat more general and abstract with respect to the *real situation*, but particular with respect to the *real model*, which is why they are understood as a necessary intermediate phase between the *real situation* and the *real model*.

The construction of the *real model* captures the essential elements of the *real situation* (through simplification, eliminative abstraction) and idealises (dematerialises) them to enable its subsequent work with a *mathematical model*. The structuring of the *real model* must be made in a convenient representation for such mathematical purposes. In other words, although the *real model* contains elements of the *real situation*, it is already a much more simplified and idealised representation (vertical representation), so it cannot be completely located in one world or another, but rather on the boundary between both worlds. Something similar happens with *real results* that, although they are derived from the interpretation and materialisation of *mathematical results* in the *real situation* context, still contain mathematical elements. For example, continuing with the problem of calculating the height of a mountain, the *real model* would simplify some irregularities in the relief of the mountain and idealise it as a triangle or a set of polygons that make it possible to later calculate its height, structuring all this in some drawing made by hand or with graphing software.

Finally, the *mathematical model* is the (or the set of) mathematical object(s) that allows the *real situation* (located in the «real world») to be explained in the language and system of the «mathematical world». This last representation is completely simplified and idealised regarding the *real situation* (vertical representation), and it is the result of mathematising the *real model*. For example, continuing with the problem of calculating the height of a mountain, the *mathematical model* would correspond to the mathematical objects that allow calculating said height.

Example of Solving a Modelling Problem

Up to this point in the article, the theoretical principles that support the SCMMC proposal have been described as thoroughly as possible; therefore, it is now necessary to describe its methodology (in terms of Radford, 2008). Thus, to exemplify how the SCMMC in Figure 4 works and, in turn, to address the use of different representations in this cycle, this subsection presents a possible solving to the *Meanders Problem* (see Figure 3) from the perspective of a solver individual.

The *real situation* corresponds to the statement of the *Meanders Problem* (with text and picture), in which the approximate length of a fragment of river with sediments is asked. Since a modelling problem may imply different ways for its solution (Lesh & Doerr, 2003), the *representations of the situation* may vary depending on said solution and the extra-mathematical considerations made by the solver. For example, by considering the original picture of the problem (see Figure 6a), a curved line could be drawn over the river with sediments, thus focusing the attention on it (see Figure 6b), and some additional representation could be searched for in a maps application as Google Earth® (see Figure 6c). Since the different *representations of the situation* involve some extra-mathematical representations, the solver assumes the context of the *real situation* as a source to collect additional information to solve the problem.

Figure 6

(a) Real Situation → (b) Representations of the Situation

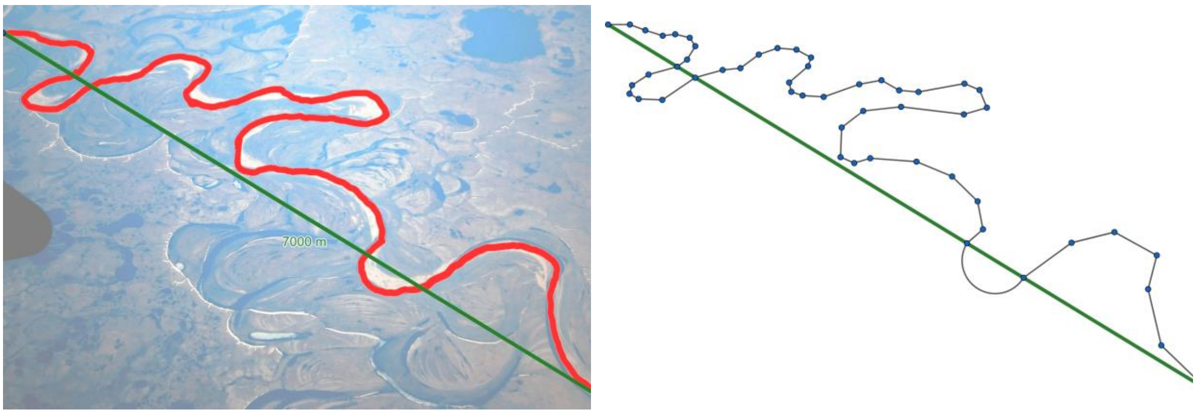
(a) (b) (c)

Source. (a) Extracted from "Oxbow Lake, Yamal Peninsula, Russia" (2006); (b) Author's modification on (a); (c) Screenshot from Google Earth®

The different *representations of the situation* are already somewhat more general and abstract with respect to the *real situation*, but particular with respect to the *real model*, which is why they are needed for its construction. By considering the similarities between the original picture of the problem (Figure 6a) and the satellite photo (Figure 6c), the solver could use the technical data of the latter as a reference (height = 4600 metres; slope = 45º; map scale 3 cm = 700 m). Furthermore, a linear (Figure 7a, green-coloured) and a real length (Figure 7a, red-coloured) could be considered for the river with sediments, simplifying the management of lengths. Thus, by considering the scale of the satellite photo, a linear length of 7000 metres could be obtained, from which constructing a *real model* using a graphing software as GeoGebra®. To construct a *real model*, the river could be idealised as a set of line segments surrounding the linear length of the river (Figure 7b).

Figure 7

(a) Representations of the Situation → (b) Real Model



(a) (b)

Source. (a) Author's modification on Figure 6b; (b) Author's elaboration with GeoGebra®

The solver has captured the essential elements of the *real situation* (through simplification), idealised (dematerialised) them, and structured them in a *real model*, that is, in a representation that allows its subsequent work with a *mathematical model*. Since the solver could have constructed the *real model* using a graphing software (Figure 7b), two *mathematical models* could be used: one that implies the addition of the lengths of all the line segments around the longest segment (by knowing the lengths of each segment with the software tools), and other that implies the restructuring of the line segments as semi-circumferences of different measurement around the longest segment (by performing new manipulations with the software, which would require additional idealisations of the *real model*). The choice of the *mathematical model* will depend on the epistemic norms that govern the modelling process, which will determine the degree of accuracy of the answer, the mathematical procedures to follow, the consistency of the results, etc. (Ledezma et al., 2023).

Based on the work with some of the *mathematical models*, the *mathematical results* would be approximately 13300 metres in length for the sum of the line segments / semi-circumferences around the longest segment. Since these *mathematical results* must be interpreted and materialised in the context of the *real situation*, the *real results* correspond to the approximate length of a fragment of river with sediments, which would be 13300 metres (or 13.3 kilometres). The validation of these *real results* could imply their comparison with other satellite photos in Google Earth® from the same zone in which the *real situation* is contextualised or the search for additional information from the internet.

Final Remarks

This article stated two of the issues that occurred throughout the construction of the theoretical corpus of modelling, namely, one of terminological nature and the other of representational nature. To address the terminological issue, a position was adopted on the terminology surrounding modelling in Mathematics Education, based on the main theoretical developments of ICTMA Community. To address the representational issue, a modelling cycle that represents this process in a non-set way is proposed, that is, without a strict separation between «real world» and «mathematical world». Addressing these two issues is not a trivial question, since it is considered that they contribute to one of the demands of Mathematics Education raised by Font and Godino (2011), namely, that its theoretical constructs are capable of describing, as clearly as possible, the mathematical activity occurring in a certain moment of a teaching and learning process, in this case, contextualised in the mathematical activity of modelling.

As mentioned in the previous section, in the studies conducted by Ledezma and colleagues (2023, 2024), the modelling process was analysed from the perspective of other theoretical frameworks of Mathematics Education. As a result of these studies, the *Semiotic-Cognitive Analysis Model for the Mathematical Modelling Process* was proposed. This tool, along with the SCMMC presented in the previous section, in a complementary way, meet the characteristics to be placed in the educational modelling perspective. This is justified by the fact that, on one hand, the SCMMC provides a structure of phases and transitions that allows explaining the modelling process in order to identify the development of sub-competencies in this process and, on the other hand, the *Semiotic-Cognitive Analysis Model* provides the tools that allow describing the mathematical activity underlying the modelling process in order to identify the development of mathematical knowledge in this process.

Finally, some sketches of research questions (in terms of Radford, 2008) are proposed for future theoretical and empirical developments. The first line consists of testing both the SCMMC and the *Semiotic-Cognitive Analysis Model* in empirical studies. This is justified by the need to validate both tools beyond the theoretical studies conducted so far (Ledezma, 2024; Ledezma et al., 2023, 2024), but also in different implementations contexts and educational levels. The second line consists of articulating the *Semiotic-Cognitive Analysis Model* with other relevant processes of mathematical activity, such as argumentation and representation, the latter with the objective of refining the proposal of representations in Figure 5 and thus enrich the SCMMC regarding its capability of explaining the development of the modelling process.

Acknowledgments

This research was conducted within Project no. 72200458 funded by ANID/PFCHA (Chile); and Grant PID2021-127104NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". Special thanks to Prof. Dr Vicenç Font Moll (University of Barcelona, Spain) for the theoretical reflections on the SCMMC.

Notes

[1.] This section is a synthesis of the conceptual framework prepared by the author in his doctoral thesis.
[2.] This section is a synthesis of the theoretical proposal developed by the author in his doctoral thesis.

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