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New Flat Surfaces in $\mathbb{S}^3$

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Abstract

In this paper, we consider a method of constructing flat surfaces based on Ribaucour transformations in $\mathbb{S}^3$. *By applying the theory to the flat torus, we obtain a family of complete flat surfaces in* $\mathbb{S}^3$ *which is determined by several parameters. We provide explicit examples.*

Keywords . Flat surfaces, Ribaucour transformations, 3-sphere, flat torus, complete surface.

1. Introduction. Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [2]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given such surface, respectively, with constant Gaussian curvature and constant mean curvature. The first application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [3]-[5]. In [24], Tenenblat and Wang extended such transformations to surfaces in space forms. For more applications of this method, see [6]-[8], [10], [12], [22], [23] and [25].

Using Ribaucour transformations and applying the theory to the rotational flat surfaces in $\mathbb{H}^3$, in [9] the authors obtained families of new such surfaces.

The study of flat surfaces in $\mathbb{S}^3$ goes back to Bianchi's works in the 19th century, and it has a very rich global theory, as evidenced by the existence of a large classes of flat torus in $\mathbb{S}^3$, see [13], [19], [20] and [26]. Indeed, these flat torus constitute the only examples of compact surfaces of constant curvature in space forms that are not totally umbilical round spheres. Flat surfaces in $\mathbb{S}^3$ admit a more explicit treatment than other surfaces of constant curvature. Moreover, there are still important open problems regarding flat surfaces in $\mathbb{S}^3$, some of them unanswered for more than 40 years. For instance, it remains unknown if there exists an isometric embedding of $\mathbb{R}^2$ into $\mathbb{S}^3$. These facts show that the geometry of flat surfaces in $\mathbb{S}^3$ is a worth studying topic, although the number of contributions to the theory is not too large. Some important references of the theory are [11], [13]-[17], [19], [20] and [26].

In [18], the authors give a complete classification of helicoidal flat surfaces in $\mathbb{S}^3$ by means of asymptotic coordinates lines.

In [1], the authors characterized the flat surfaces in the unit 3-sphere that pass through a given regular curve of $\mathbb{S}^3$ with a prescribed tangent plane distribution along this curve.

In this paper, motivated by [9] we use the Ribaucour transformations to get a family of complete flat surfaces in $\mathbb{S}^3$ from a given such surface in $\mathbb{S}^3$. As an application of the theory, we obtain a family of complete flat surfaces in $\mathbb{S}^3$ associated to the flat torus. The obtained family depend on four parameters. One of these parameters is given by parameterization of the flat torus. The other parameters, appear from integrating the Ribaucour transformation. We show explicit examples of these surfaces. This work is organized as follows. In Section 1, we give a brief description of Ribaucour transformations in space forms. In Section 2, we give an additional condition for the transformed surface to be flat. In Section 3, we describe

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all flat surfaces of the sphere 3-space obtained by applying the Ribaucour transformation to the flat torus. We prove that such surfaces are complete and provide explicit examples.

2. Preliminary. This section contains the definitions and the basic theory of Ribaucour transformations for surfaces in $\mathbb{S}^3$ (for more details see [24]).

Let M be an orientable surface in $\mathbb{S}^3$ without umbilic points, with Gauss map denoted by N. Suppose that there exist 2 orthonormal principal vector fields e_1 and e_2 defined on M. We say that $\widetilde{M} \subset \mathbb{S}^3$ is associated to M by a Ribaucour transformation with respect to e_1 and e_2, if there exist a differentiable function h defined on M and a diffeomorphism $\psi : M \to \widetilde{M}$ such that:

(a) for all $p \in M$, $exp_p h(p)N(p) = exp_{\psi(p)} h(p)\widetilde{N}(\psi(p))$, where $\widetilde{N}$ is the Gauss map of $\widetilde{M}$ and exp is the exponential map of $\mathbb{S}^3$.

(b) The subset $\{exp_p h(p)N(p), p \in M\}$, is a two-dimensional submanifold of $\mathbb{S}^3$.

(c) $d\psi(e_i)$ $1 \leq i \leq 2$ are orthogonal principal directions of $\widetilde{M}$.

The following result gives a characterization of Ribaucour transformation(see [24] for a proof and more details).

Theorem 2.1. *Let M be an orientable surface of $\mathbb{S}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points. Assume $e_i = \frac{X_{,i}}{a_i}$, $1 \leq i \leq 2$ where $a_i = \sqrt{g_{ii}}$, are orthogonal principal directions, $-\lambda_i$ the corresponding principal curvatures, and N is a unit normal vector field to M. A surface $\widetilde{M}$ is locally associated to M by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subseteq U \to \mathbb{R}$ which satisfy*

$$\begin{aligned} \Omega_{i,j} &= \Omega_j \frac{a_{j,i}}{a_i}, \qquad for \ \ i \neq j, \\ \Omega_{,i} &= a_i \Omega_i, \\ W_{,i} &= -a_i \Omega_i \lambda_i. \end{aligned} \tag{2.1}$$

$W(W + \lambda_i \Omega) \neq 0$ and $\widetilde{X} : V \subseteq U \to \widetilde{M}$, is a parametrization of $\widetilde{M}$ given by

$$\widetilde{X} = \left(1 - \frac{2\Omega^2}{S}\right) X - \frac{2\Omega}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.2}$$

where

$$S = \sum_{i=1}^{2} (\Omega_i)^2 + W^2 + \Omega^2. \tag{2.3}$$

Moreover, the normal map of $\widetilde{X}$ is given by

$$\widetilde{N} = N + \frac{2W}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN + \Omega X\right), \tag{2.4}$$

and the principal curvatures and coefficients of the first fundamental form of $\widetilde{X}$, are given, respectively, by

$$\widetilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i}, \qquad \widetilde{g}_{ii} = \left(\frac{S - \Omega T_i}{S}\right)^2 g_{ii} \tag{2.5}$$

where Ω_i, Ω and W satisfy (2.1), S is given by (2.3), g_{ii}, $1 \leq i \leq 2$ are coefficients of the first fundamental form of X, and

$$T_1 = 2\left(\frac{\Omega_{1,1}}{a_1} + \frac{a_{1,2}}{a_1 a_2}\Omega_2 - W\lambda_1 + \Omega\right), \; T_2 = 2\left(\frac{\Omega_{2,2}}{a_2} + \frac{a_{2,1}}{a_1 a_2}\Omega_1 - W\lambda_2 + \Omega\right). \tag{2.6}$$

3. Ribaucour transformation for flat surfaces in $\mathbb{S}^3$. In this section we provides a sufficient condition for a Ribaucour transformation to transform a flat surface into another such surface.

Theorem 3.1. *Let M be a surface of $\mathbb{S}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points and let $\widetilde{M}$ be associated to M by a Ribaucour transformation, such that the normal lines intersect at a* *distance function h. Assume that $h = \frac{\Omega}{W}$ is not constant along the lines of curvature and the function Ω, Ω_i and W satisfy one of the additional relation*

$$\Omega_1^2 + \Omega_2^2 = c\big(\Omega^2 + W^2\big), \tag{3.1}$$

where $c > 0$, S is given by (2.3) and W, Ω_i, $1 \le i \le 2$ satisfies (2.1). Then $\widetilde{M}$ is a flat surface, if and only if, M is a flat surface.

Proof: See [24] Theorem 2.1, with $\alpha = \gamma = 1$ and $\beta = 0$. □

Remark 3.1. Let X be as in the previous theorem. Then the parameterization $\widetilde{X}$ of $\widetilde{M}$, locally associated to X by a Ribaucour transformation, given by (2.2), is defined on

$$V = \{(u_1, u_2) \in U;\ (\Omega T_1 - S)(\Omega T_2 - S) \neq 0\}.$$

4. Family of flat surfaces associated to the flat torus in $\mathbb{S}^3$. In this section, by applying Theorem 3.1 to the flat torus in $\mathbb{S}^3$, we obtain a four parameter family of complete flat surfaces in $\mathbb{S}^3$.

Theorem 4.1. *Consider the flat torus in $\mathbb{S}^3$ parametrized by*

$$X(u_1, u_2) = (r_1\cos(r_2u_1), r_1\sin(r_2u_1), r_2\cos(r_1u_2), r_2\sin(r_1u_2)), \quad (u_1, u_2) \in \mathbb{R}^2, \tag{4.1}$$

r_i, $1 \le i \le 2$ are positive constants satisfying $r_1^2 + r_2^2 = 1$, where the first fundamental form is $I = r_1^2r_2^2\big(du_1^2 + du_2^2\big)$. A parametrized surface $\widetilde{X}(u_1, u_2)$ is a flat surface locally associated to X by a Ribaucour transformation as in Theorem 2, if and only if, up to an isometry of $\mathbb{S}^3$, it is given by

$$\widetilde{X} = \frac{1}{S}\begin{bmatrix} \big(r_1S - 2r_1\Omega^2 - 2r_2\Omega W\big)\cos(r_2u_1) + 2f'\Omega\sin(r_2u_1), \\ \big(r_1S - 2r_1\Omega^2 - 2r_2\Omega W\big)\sin(r_2u_1) - 2f'\Omega\cos(r_2u_1), \\ \big(r_2S - 2r_2\Omega^2 + 2r_1\Omega W\big)\cos(r_1u_2) + 2g'\Omega\sin(r_1u_2), \\ \big(r_2S - 2r_2\Omega^2 + 2r_1\Omega W\big)\sin(r_1u_2) - 2g'\Omega\cos(r_1u_2) \end{bmatrix}. \tag{4.2}$$

$\widetilde{X}$ is defined on

$$V = \{(u_1, u_2) \in \mathbb{R}^2;\ (r_1^2g^2 - r_2^2f^2 - 2r_2^2fg)(r_2^2f^2 - r_1^2g^2 + 2r_1^2fg) \neq 0\},$$

where

$$\Omega = r_1r_2\big(f(u_1) + g(u_2)\big),\ \ W = r_2^2f(u_1) - r_1^2g(u_2),\ \ S = (1 + c)\big(\Omega^2 + W^2\big) \neq 0, \tag{4.3}$$

$c > 0$, and the functions f and g are given by

$$i)\ f(u_1) = \cosh(r_2\sqrt{c}\,u_1),\ \ g(u_2) = \frac{r_2}{r_1}\sinh(r_1\sqrt{c}\,u_2), \tag{4.4}$$

or

$$ii)\ f(u_1) = \sinh(r_2\sqrt{c}\,u_1),\ \ g(u_2) = \frac{r_2}{r_1}\cosh(r_1\sqrt{c}\,u_2), \tag{4.5}$$

or

$$iii)\ f(u_1) = a_1e^{\epsilon_1r_2\sqrt{c}\,u_1},\ \ g(u_2) = b_1e^{\epsilon_2r_1\sqrt{c}\,u_2},\ \epsilon_i^2 = 1,\ 1 \le i \le 2. \tag{4.6}$$

Moreover, the normal map of $\widetilde{X}$ is given by

$$\widetilde{N} = \frac{1}{S}\begin{bmatrix} \big(-r_2S - 2r_2W^2 + 2r_1\Omega W\big)\cos(r_2u_1) + 2f'\Omega\sin(r_2u_1), \\ \big(-r_2S - 2r_2W^2 + 2r_1\Omega W\big)\sin(r_2u_1) - 2f'\Omega\cos(r_2u_1), \\ \big(r_1S + 2r_1W^2 - 2r_2\Omega W\big)\cos(r_1u_2) + 2g'\Omega\sin(r_1u_2), \\ \big(r_1S + 2r_1W^2 - 2r_2\Omega W\big)\sin(r_1u_2) - 2g'\Omega\cos(r_1u_2) \end{bmatrix}. \tag{4.7}$$

Proof: Consider the first fundamental form of the flat torus $ds^2 = r_1^2r_2^2(du_1^2 + du_2^2)$ and the principal curvatures $-\lambda_i$ $1 \le i \le 2$ given by $\lambda_1 = \frac{-r_2}{r_1}$, $\lambda_2 = \frac{r_1}{r_2}$. Using (2.1), to obtain the Ribaucour transformations, we need to solve the following equations

$$\Omega_{i,j} = 0, \qquad \Omega_{,i} = r_1r_2\Omega_i, \qquad W_{,i} = -r_1r_2\Omega_i\lambda_i,\ \ 1 \le i \neq j \le 2. \tag{4.8}$$

Therefore we obtain

$$\Omega = r_1 r_2\big(f_1(u_1) + f_2(u_2)\big), \qquad W = -r_1 r_2\big(\lambda_1 f_1 + \lambda_2 f_2\big) + \bar{c}, \qquad \Omega_i = f_i', \ \ 1 \le i \neq j \le 2, \tag{4.9}$$

where $\bar{c}$ is a real constant. Using (3.1) the associated surface will be flat when

$$\Omega_1^2 + \Omega_2^2 = c(\Omega^2 + W^2).$$

Therefore, we obtain that $c > 0$ and the functions f_1 and f_2 satisfy

$$(f_1')^2 + (f_2')^2 = c(\Omega^2 + W^2). \tag{4.10}$$

Differentiating this last equation with respect to x_1 and x_2, using (4.8) and (4.9) we get

$$\begin{aligned} f_1'' &= c r_2^2 f_1 + c r_2^2 \bar{c}, \\ f_2'' &= c r_1^2 f_2 - c r_1^2 \bar{c}. \end{aligned}$$

Defining $f(u_1) = f_1(u_1) + \bar{c}$ and $g(u_2) = f_2(u_2) - \bar{c}$, we have

$$f'' - c r_2^2 f = 0, \quad g'' - c r_1^2 g = 0, \tag{4.11}$$

$$\Omega = r_1 r_2\big(f(u_1) + g(u_2)\big), \quad W = r_2^2 f(u_1) - r_1^2 g(u_2). \tag{4.12}$$

By Theorem 2.1, we have that $\widetilde{X}$ and $\widetilde{N}$ are given by (4.2) and (4.7). Using (2.6) and (4.12), we get

$$T_1 = \frac{2r_2(1+c)f}{r_1}, \quad T_2 = \frac{2r_1(1+c)g}{r_2}.$$

Thus, from Remark 3.1 , $\widetilde{X}$ is defined in

$$V = \{(u_1,u_2) \in \mathbb{R}^2;\ (r_1^2 g^2 - r_2^2 f^2 - 2r_2^2 fg)(r_2^2 f^2 - r_1^2 g^2 + 2r_1^2 fg) \neq 0\}.$$

From (4.11), we get

$$f(u_1) = a_1 \cosh(r_2\sqrt{c}u_1) + a_2 \sinh(r_2\sqrt{c}u_1), \tag{4.13}$$

$$g(u_2) = b_1 \cosh(r_1\sqrt{c}u_2) + b_2 \sinh(r_1\sqrt{c}u_2). \tag{4.14}$$

Substituting (4.13), (4.14) and (4.12) in (4.10), we have

$$\big(a_1^2 - a_2^2\big) r_2^2 = \big(b_2^2 - b_1^2\big) r_1^2. \tag{4.15}$$

Let $A_1 = a_1^2 - a_2^2$. If $A_1 > 0$, then from (4.15) $b_2^2 > b_1^2$. Hence, (4.13) and (4.14) can be rewritten as

$$f(u_1) = \sqrt{A_1}\cosh(r_2\sqrt{c}u_1 + A_2), \tag{4.16}$$

$$g(u_2) = \sqrt{A_1}\frac{r_2}{r_1}\sinh(r_1\sqrt{c}u_2 + B_2), \tag{4.17}$$

where

$$\cosh(A_2) = \frac{a_1}{\sqrt{a_1^2 - a_2^2}}, \quad \sinh(A_2) = \frac{a_2}{\sqrt{a_1^2 - a_2^2}},$$

$$\sinh(B_2) = \frac{b_2}{\sqrt{b_2^2 - b_1^2}}, \quad \cosh(B_2) = \frac{b_1}{\sqrt{b_2^2 - b_1^2}}.$$

The constants A_2 and B_2, without loss of generality, may be considered to be zero. One can verify that the surfaces with different values of A_2 and B_2 are congruent. In fact, using the notation $\widetilde{X}_{A_2B_2}$ for the surface $\widetilde{X}$ with fixed constants A_2 and B_2, we have

$$\widetilde{X}_{A_2B_2} = R_{\left(\frac{-A_2}{r_2\sqrt{c}}, \frac{-B_2}{r_1\sqrt{c}}\right)}\widetilde{X}_{00} \circ h,$$

where $h(u_1,u_2) = \left(u_1 + \dfrac{A_2}{r_2\sqrt{c}}, u_1 + \dfrac{B_2}{r_1\sqrt{c}}\right)$ with

$$R_{(\theta,\phi)}(x_1,x_2,x_3,x_4) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3\cos\phi - x_4\sin\phi, x_3\sin\phi + x_4\cos\phi).$$

Now substituting (4.16) with $A_2 = 0$, (4.17) with $B_2 = 0$ and (4.12) in (4.2) we obtain that $\widetilde{X}$ does not depend on A_1. Thus without loss of generality, we can consider $A_1 = 1$. Therefore we conclude that f and g are given by (4.4).

On the other hand, if $A_1 < 0$, then from (4.15) $b_2^2 < b_1^2$. Hence, (4.13) and (4.14) can be rewritten as

$$f(u_1) = \sqrt{-A_1}\sinh(r_2\sqrt{c}u_1 + A_2), \tag{4.18}$$

$$g(u_2) = \sqrt{-A_1}\frac{r_2}{r_1}\cosh(r_1\sqrt{c}u_2 + B_2). \tag{4.19}$$

Proceeding in a similar way to the previous case, we obtain that f and g are given by (4.5).

If $A_1 = 0$, then $a_2 = \epsilon_1 a_1$, and from (4.15) $b_2 = \epsilon_2 b_1$, with $\epsilon_i^2 = 1$, $1 \le i \le 2$. Thus, substituting this in (4.13) and (4.14), we obtain (4.6). □

Remark 4.1. As Ribaucour transformation preserves lines of curvature, then each flat surface associated to the flat torus as in Theorem 4.1, is parametrized by lines of curvature and from (2.5), the metric is given by

$$ds^2 = \psi_1^2 du_1^2 + \psi_2^2 du_2^2,$$

where

$$\psi_1 = \frac{(-r_2^2 f^2 + r_1^2 g^2 - 2r_2^2 fg)r_1 r_2}{r_1^2 g^2 + r_2^2 f^2}, \quad \psi_2 = \frac{(r_2^2 f^2 - r_1^2 g^2 - 2r_1^2 fg)r_1 r_2}{r_1^2 g^2 + r_2^2 f^2}. \tag{4.20}$$

Moreover, from (2.5), the principal curvatures of the $\widetilde{X}$ are given by

$$\widetilde{\lambda}_1 = \frac{-r_2\psi_2}{r_1\psi_1}, \quad \widetilde{\lambda}_2 = \frac{r_1\psi_1}{r_2\psi_2}. \tag{4.21}$$

Theorem 4.2. *All the flat surfaces associated to the flat torus $\widetilde{X}$, given by Theorem 4.1 are complete surfaces.*

Proof: For all divergent curves $\gamma(t) = (u_1(t), u_2(t))$, such that $\lim\limits_{t\to\infty}\left(u_1^2 + u_2^2\right) = \infty$, we have $l(\widetilde{X}\circ\gamma) = \infty$.

In fact, the functions f and g are given by (4.4) or (4.5) or (4.6) and the coefficients of the first fundamental form ψ_i, $1 \le i \le 2$, of $\widetilde{X}$ are given by (4.20). Therefore $\lim\limits_{|u_1|\to\infty}|\psi_i| = r_1 r_2$, $1 \le i \le 2$, uniformly in u_2 and $\lim\limits_{|u_2|\to\infty}|\psi_i| = r_1 r_2$, $1 \le i \le 2$, uniformly in u_1. Hence, there exist $k_1 > 0$ and $k_2 > 0$ such that $|\psi_i(u_1, u_2)| > \frac{r_1 r_2}{2}$, $1 \le i \le 2$, for all $(u_1, u_2) \in \mathbb{R}^2$ with $|u_1| > k_1$ and $|u_2| > k_2$.
Let

$$m_i = min\{|\psi_i(u_1, u_2)|;\ (u_1, u_2) \in [-k_1, k_1] \times [-k_2, k_2]\}.$$

Therefore $|\psi_i(u_1, u_2)| \ge m_i$ in $[-k_1, k_1] \times [-k_2, k_2]$. Now consider $m_0 = min\{m_1, m_2, \frac{r_1 r_2}{2}\}$, then $|\psi_i(u_1, u_2)| \ge m_0$ in $\mathbb{R}^2$. We conclude that $\widetilde{X}$ is a complete surface. □

In the following, we provide some examples.

Example 4.1. Consider stereographic projection $\pi : \mathbb{S}^3 \to \mathbb{R}^3$

$$\pi(x_1, x_2, x_3, x_4) = \frac{1}{1 - x_4}\Big(x_1, x_2, x_3\Big),$$

where $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

1. In Figure 4.1, we provide the surface parametrized by $\pi\circ\widetilde{X}$, where $\widetilde{X}$ given by (4.2) is locally associated to the flat torus in $\mathbb{S}^3$ by a Ribaucour transformation. In this case to obtain this surface we have considered

$$f(u_1) = \cosh\left(\frac{8u_1}{5}\right) \quad \textit{and} \quad g(u_2) = \frac{4}{3}\sinh\left(\frac{6u_2}{5}\right).$$

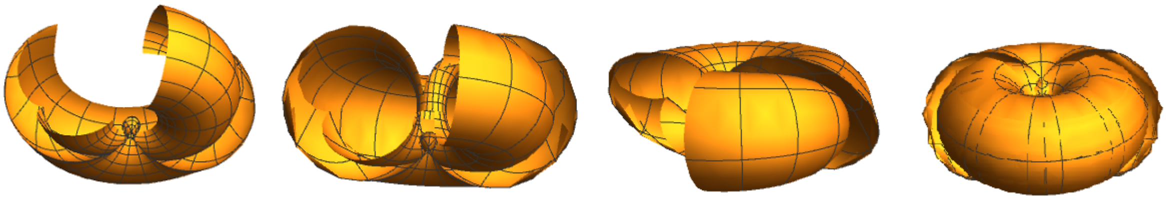

Figure 4.1: In the figure above we have $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $c = 4$.

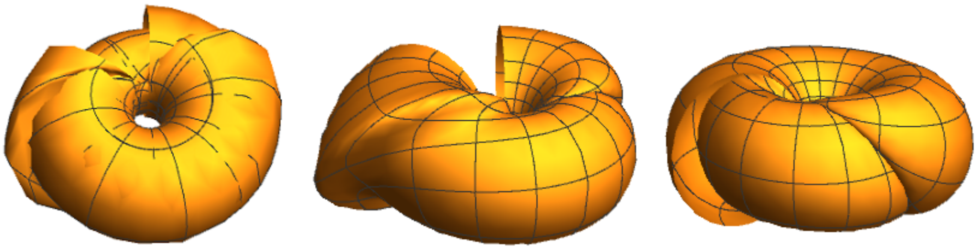

Figure 4.2: In the figure above we have $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $c = 4$.

2. In Figure 4.2, we provide the surface parametrized by $\pi \circ \widetilde{X}$, where $\widetilde{X}$ given by (4.2) is locally associated to the flat torus in $\mathbb{S}^3$ by a Ribaucour transformation. In this case to obtain this surface we have considered

$$f(u_1) = \sinh\left(\frac{8u_1}{5}\right) \quad \textit{and} \quad g(u_2) = \frac{4}{3}\cosh\left(\frac{6u_2}{5}\right).$$

3. In Figure 4.3, we provide the surface parametrized by $\pi \circ \widetilde{X}$, where $\widetilde{X}$ given by (4.2) is locally associated to the flat torus in $\mathbb{S}^3$ by a Ribaucour transformation. In this case to obtain this surface we have considered

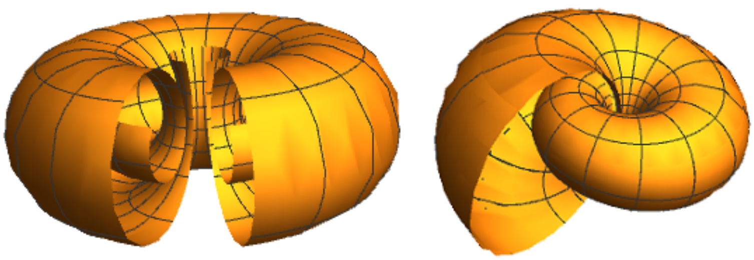

Figure 4.3: In the figure above we have $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $c = \frac{1}{1000}$.

$$f(u_1) = \cosh\left(\frac{2u_1}{25\sqrt{10}}\right) \quad \textit{and} \quad g(u_2) = \frac{4}{3}\sinh\left(\frac{3u_2}{50\sqrt{10}}\right),$$

in the first surface and

$$f(u_1) = \sinh\left(\frac{2u_1}{25\sqrt{10}}\right) \quad \textit{and} \quad g(u_2) = \frac{4}{3}\cosh\left(\frac{3u_2}{50\sqrt{10}}\right),$$

in the second one.

4. In Figure 4.4, we provide the surface parametrized by $\pi \circ \widetilde{X}$, where $\widetilde{X}$ given by (4.2) is locally associated to the flat torus in $\mathbb{S}^3$ by a Ribaucour transformation. In this case to obtain this surface we have considered

$$f(u_1) = e^{\frac{2u_1}{25\sqrt{10}}} \quad \textit{and} \quad g(u_2) = e^{\frac{3u_2}{50\sqrt{10}}}.$$

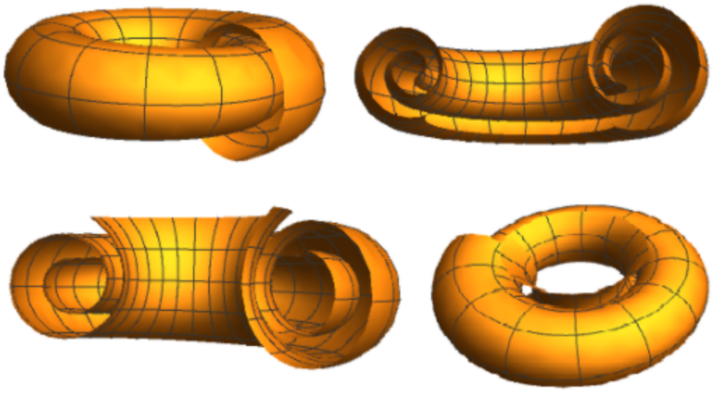

Figure 4.4: In the figure above we have $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $c = \frac{1}{1000}$.

5. In Figure 4.5, we provide the surface parametrized by $\pi \circ \widetilde{X}$, where $\widetilde{X}$ given by (4.2) is locally associated to the flat torus in $\mathbb{S}^3$ by a Ribaucour transformation. In this case to obtain this surface we have considered

$$f(u_1) = \cosh\left(\frac{2u_1}{65\sqrt{10}}\right) \quad and \quad g(u_2) = \frac{5}{12}\sinh\left(\frac{6u_2}{65\sqrt{10}}\right).$$

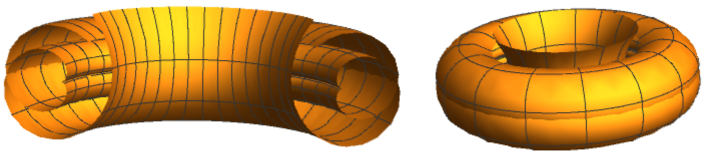

Figure 4.5: In the figure above we have $r_1 = \frac{12}{13}$, $r_2 = \frac{5}{13}$, $c = \frac{1}{1000}$.

5. Conclusions. From the results obtained in this work we can make the following conclusions: All flat surfaces in $\mathbb{S}^3$ locally associated to the flat torus by a Ribaucour transformation are complete surfaces. Moreover, these surfaces has no umbilic points. The geometry of flat surfaces in $\mathbb{S}^3$ is a worth studying topic, although the number of contributions to the theory is not too large.

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References

[1] Aledo JA, Gálvez JA, Mira P. A D´ Alembert Formula for Flat Surfaces in the 3-Sphere. Journal of Geometric Analysis. 2009; 19:211-232.
[2] Bianchi L. Lezioni di geometria Differenziale. Terza Edicione. Nicola Zanichelli Editore; 1927.
[3] Corro AMV, Ferreira WP, Tenenblat K. On Ribaucour transformations for hypersurfaces. Mat. Contemp. 1999; 17:137-160.
[4] Corro AMV, Ferreira WP, Tenenblat K. Ribaucour transformations for Constant mean curvature and linear Weingarten surfaces. Pacific Journal of Mathematics. 2003; 212(2):265-296.
[5] Corro AMV, Ferreira WP, Tenenblat K. Minimal surfaces obtained by Ribaucour transformations. Geometriae Dedicata, Netherlands. 2003; 96(1):117-150.
[6] Corro AMV, Tenenblat K. Ribaucour transformation revisited. Comum. Geom. 2004; 12(5):1055-1082.
[7] Corro AMV. Generalized Weingarten surfaces of Bryant type in Hiperbolic 3-space. Mat. Contemp. 2006; 30:71-89.
[8] Corro AMV, Martinez A, Milán F. Complete flat surfaces with two isolated singularities in hyperbolic 3-space. J. Math. Anal. Appl. 2010; 366:582-592.

[9] Corro AMV, Martinez A, Tenenblat K. Ribaucour transformations for flat surfaces in the hyperbolic 3-space. J. Math. Anal. Appl. 2014; 412:720-743.

[10] Cui NW, Tenenblat K. New minimal surfaces in the hyperbolic space. Science China Mathematics. 2017; 60:1679-1704.

[11] Dadok J, Sha J. *On embedded flat surfaces in* $\mathbb{S}^3$. J. Geom. Anal. 1997; 7:47-55.

[12] Ferreira WP, Tenenblat K. Hypersurfaces with flat r-mean curvature and Ribaucour transformations. International Journal of Applied Mathematics and Statistics. 2007; 11:38-51.

[13] Kitagawa Y. Periodicity of the asymptotic curves on flat tori in $\mathbb{S}^3$. J. Math. Soc. Jpn. 1988; 40:457-476.

[14] Kitagawa Y. Ridigity of the Clifford tori in $\mathbb{S}^3$. Math. Z. 1988; 198:591-599.

[15] Kitagawa Y. Embedded flat tori in the unit 3-sphere. J. Math. Soc. Jpn. 1995; 47:275-296.

[16] Kitagawa Y. Isometric deformations of flat tori in the 3-sphere withnonconstant mean curvature. Tohoku Math. J. 2000; 52:283-298.

[17] Kitagawa Y. Deformable flat tori in $\mathbb{S}^3$ with constant mean curvature. Osaka J. Math. 2003; 40:103-119. Helicoidal flat surfaces in hyperbolic 3-space. Pacific Journal of Mathematics. 2013; 264(1):195-211.

[18] Manfio F, Santos J, Helicoidal flat surfaces in 3-sphere. Mathematische Nachrichten. 2019; 292:127-136.

[19] Pinkall U. Hopf tori in $\mathbb{S}^3$. Invent. Math. 1985; 81:379-386.

[20] Spivak M. A Comprehensive Introduction to Differential Geometry, vol. 4. Publish or Perish, Inc. Berkeley; 1979.

[21] Tenenblat K. Transformations of manifolds and applications to differential equations. Longman Sci. and Tech; 1998.

[22] Tenenblat K. On Ribaucour transformations and applications to linear Weingarten surfaces. Anais da Academia Brasileira de Ciências (Impresso), ABC. 2002; 74:559-575.

[23] Tenenblat K. Lemes MV. On Ribaucour transformations and minimal surfaces. Matemática Contemporânea. 2005; 29:13-40.

[24] Tenenblat K. Wang Q. Ribaucour Transformations for Hypersurfaces in Space Forms. Annals of Global Analysis and Geometry. 2006; 29:135-162.

[25] Tenenblat K. Wang Q. New surfaces of constant mean curvature in the hyperbolic space. Illinois Journal of Mathematics. 2009; 53:157-185.

[26] Weiner JL. Flat tori in $\mathbb{S}^3$ and their Gauss maps. Proc. Lond. Math. Soc. 1991; 62:54-76.