CONTRIBUCIÓN AL ESTUDIO DEL RAZONAMIENTO ORDINARIO Y LA COMPUTACIÓN CON PALABRAS.

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Al buscar lo imposible el hombre siempre ha realizado y reconocido lo posible. Y aquellos que sabiamente se han limitado a lo que creían posible, jamás han dado un solo paso adelante.
Mijaíl Bakunin (1814-1876)

Resumen

El trabajo es una contribución al desarrollo de los modelos de Conjeturas, Hipótesis y Consecuencias (Modelos CHC), como encargados de formalizar el razonamiento ordinario o de sentido común.

La mayor aportación de este trabajo es la introducción de la posibilidad de manejar la imprecisión típica del lenguaje de los Modelos CHC. Aunque el trabajo es de tipo matemático, todo se plantea bajo un número mínimo de hipótesis para no introducir condiciones que puedan restringir su aplicación.

El primer artículo recogido en este trabajo trata con el problema del significado de las palabras, por lo que puede ser enmarcado en el emergente campo de la Computación con Palabras. Fundamentalmente, trata de analizar qué propiedad intrínseca a un predicado P o al colectivo originado por él, se requiere para obtener una representación matemática a través de una función definida sobre el universo de discurso, donde se aplica el predicado, con imagen en una escala conveniente. Esto permite definir el grado en que un objeto del universo de discurso, x, es P en el lenguaje. El artículo se centra en el estudio de distintas escalas, explicando la aparición de los conjuntos *fuzzy*, conjuntos evaluados sobre intervalos, conjuntos intuicionistas, y los conjuntos *fuzzy* de tipo 2.

Continuando con el problema del significado, se analiza una nueva interpretación de los principios aristotélicos de No-Contradicción y Tercero-Excluído basándose en el concepto de auto-contradicción. El propósito fundamental del segundo artículo recogido en este trabajo, es la caracterización

de la verificación de estos principios en el intervalo unidad. Esto permite extender el estudio al caso de los conjuntos *fuzzy* dotados de álgebras funcionalmente expresables muy generales.

En el tercer artículo, se definen los Modelos CHC sobre un conjunto preordenado. Por lo tanto, el modelo puede aplicarse al caso de los conjuntos *fuzzy* dotados del orden puntual, permitiéndo el estudio del razonamiento conjetural sobre información tanto precisa, como imprecisa. En este caso, el modelo parte de una estructura de consecuencias dada por un operador de consecuencias en sentido de Tarski y una familia de subconjuntos que permiten controlar de distintas formas la consistencia de las premisas y las consecuencias, no admitiendo ninguna premisa falsa, o ninguna auto-contradictoria, o ningún par de premisas contradictorias,... A partir de dicha estructura de consecuencias se definen las conjeturas, hipótesis, especulaciones y refutaciones.

Finalmente, en el útimo artículo englobado en este trabajo, se buscan Modelos CHC no definidos a partir de una estructura de consecuencias. Se contruye el conjunto de conjeturas dependiendo de las distintas interpretaciones de no ser inconsistente con la información aportada por el conjunto de premisas. Dentro del conjunto de conjeturas, se distinguen también las consecuncias, hipótesis y especulaciones.

Debe notarse que mientras las hipótesis y conjeturas son anti-monótonas, las especulaciones son propiamente no-monótonas, al no ser ni monótonas, ni anti-monótonas. Por ello, estos modelos abren una nueva posibilidad para el estudio del razonamiento no-monótono.

Abstract

This work is a contribution to enlarge the Conjectures, Hypotheses and Consequences (CHC) models, which try to formalize commonsense reasoning. Its main contribution is to introduce in these models the possibility to use the imprecision typical of language.Although the paper is of a mathematical character, everything is done under a minimum of hypotheses for not introducing conditions that could restrict its applicability.

The first paper collected in this work deals with the problem of the meaning of words, that can be framed in the basic problems in the new field of Computing with Words. It mainly tries to analyze, which intrinsic properties of a predicate P and the collectives originated by it, are required for obtaining a mathematical representation of it through a function defined in the universe of discourse, where the predicate is stated, to a convenient scale at each case. This allows to compute the extent up to which x is P in the language, for all x in the universe of discourse. The paper focusses on the design of the scale, and considers the case of the Zadeh's fuzzy sets, the interval-valued, the intuitionistic, and the type-2 fuzzy sets.

Continuing with the problem of meaning, it is analyzed a new interpretation of the Aristotelian principles of non-contradiction and excluded-middle based on the concept of self-contradiction, by translating the Aristotelian term 'imposible' by 'self-contradictory'. This is the aim of the second paper collected in the current work. It deals with these 'principles' verification in the case of the unit interval of the real line. Such verification is done in the unit interval for three different preorders, being the first one the restriction

of the usual order of the real line to the unit interval. This allows to extend such study to characterize the 'principles' in the case of fuzzy sets endowed with very general functionally expressible algebras.

In the third paper of this work, the CHC models are defined in a pre-ordered set. So, the results obtained can be applied to the case of fuzzy sets endowed with the usual pointwise ordering, and a way to study conjectural reasoning with both precise and imprecise information is open. The model departs from a structure of consequence given by an operator in the sense of Tarski defined in a family of subsets allowing to control the consistency of the premises and the consequences depending on different interpretations of non-inconsistency (not admitting any false premise, or self-contradictory, or any pair of contradictory premises,...). From them, the corresponding sets of conjectures, hypotheses, speculations and refutations are considered.

Finally, the last contribution of this work searches for CHC models not coming from a consequence operator. The set of conjectures is built depending on different interpretations of being not-inconsistent with the information conveyed by the set of premises, and then consequences, hypotheses and speculations are also obtained. It should be noticed that if hypotheses, and conjectures at large, are anti-monotonic, speculations are non-monotonic since they are neither monotonic, nor ant-monotonic. With all that, the structural study of non-monotonic reasoning is open to be undertaken.

Esquema general del trabajo

La memoria 'Contribución al estudio del razonamiento ordinario y la computación con palabras', presentada para aspirar al grado de doctor por la Universidad de León, está organizada en tres partes además del 'Resumen General' que las antecede.

- En la Parte Primera, se incluyen la 'Introducción' a la tesis, el 'Resumen de los cuatro artículos', y las 'Conclusiones y trabajo futuro'.
- En la Parte Segunda, se incluyen las copias de los cuatro artículos en los que se basa la tesis.
- En la Parte Tercera, además de la lista de publicaciones de la candidata y como información complementaria, se añaden como anexo otras cuatro publicaciones que aún no formando parte del cuerpo de la tesis, están en relación directa con los artículos de la segunda parte.

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Parte I

Memoria

Capítulo 1

Introducción

Háblame para que te vea.
Séneca (4 a.C.-65 d.C.)

El razonamiento ordinario o de sentido común, es aquél que realiza el ser humano de distintas formas: procesando la información que maneja, sea formalmente, sea siguiendo esquemas preestablecidos y de forma meditada, o casi de forma automática sin deparar en los procesos mentales que realiza. Por otro lado, con la lógica se objetiva el razonamiento deductivo a través de sus esquemas formales. ¿Es posible objetivar el razonamiento ordinario? Ésta es la cuestión principal que se aborda en este trabajo, en el que se intenta estudiar modelos formales de razonamiento ordinario.

El razonamiento ordinario rompe la frontera del razonamiento matemático el cual se articula sobre la deducción y puede modelarse por operadores de consecuencias monótonos. Es decir, si a partir de un conjunto de premisas se obtienen ciertas consecuencias, y se incrementa el número de premisas no disminuirá el número de consecuencias. Esto no ocurre con razonamientos sencillos cotidianos, en los que es típico que nueva información destruya viejas conclusiones. Por ejemplo, si un animal es un pájaro, entonces puede deducirse que volará, pero si añadimos la información de que en concreto es un avestruz, el hecho de volar no se mantendrá como consecuencia. No obstante, si añadimos a la premisa de ser pájaro que es además un jilguero, se puede seguir deduciendo que el animal puede volar. Éste es un razonamiento no monótono, esto es que no es ni monótono, ni anti-monótono, no sigue ninguna ley de monotonía. Es decir, si se añaden más premisas no se sabe si se pueden obtener más o menos conclusiones. Por tanto, no podrá modelarse a través de la deducción que sigue la ley de monotonía. De hecho, según John Sowa, sólo el 25 % de los razonamientos ordinarios son deductivos [Sow04].

Dentro del campo de la Inteligencia Artificial es conocido el interés en la ampliación de los sistemas deductivos para la obtención de conclusiones a partir de un cuerpo de conocimiento que deje de preservar la monotonía.

En el razonamiento ordinario distinguimos cuatro tipos básicos de razonamiento: deductivo, abductivo, especulativo y por semejanza. En este trabajo no se estudiará el razonamiento por analogía o semejanza, aunque es una forma bastante común e importante del razonamiento ordinario, ya que el ser humano basa su razonamiento en experiencias vividas y por similaridad a lo ocurrido en el pasado, intenta obtener conclusiones a partir de la información actual que maneja.

Los razonamientos deductivo, abductivo y especulativo se englobarán dentro del llamado razonamiento conjetural. ¿Qué incita a millones de españoles a comprar lotería de Navidad? Evidentemente, no la compran porque se pueda deducir que ganarán el primer premio de la lotería, sino porque el hecho de comprarla no es inconsistente con ganar el premio; ganarlo es contingente, pero no es seguro. Esto es un ejemplo de razonamiento conjetural, y dependiendo de las distintas formas de entender la no inconsistencia, surgen distintos operadores de Conjeturas [TGH10].

En aras de modelizar el razonamiento ordinario el trabajo tendrá dos aspectos, el semántico y el sintáctico, ya que el razonamiento de sentido común se articula lingüísticamente. Así pues, se abordan tanto el problema del significado, como el de obtener modelos formales que traspasen la barrera de la deducción.

Entenderemos por significado la definición de Wittgenstein, "el significado de una palabra es su uso en el lenguaje" [Wit81]. En el uso de la palabra intervienen el contexto y el propósito. Por lo tanto, el significado de cada palabra está lleno de matices e imprecisiones y dada la capacidad de los conjuntos *fuzzy* [Zad65] de recoger la imprecisión serán candidatos a considerar para la representación de los términos lingüísticos. Aún teniendo en cuenta la imprecisión de cada palabra, se presupone la existencia de una semejanza o *parecido de familia* entre todas las interpretaciones de los significados de una palabra. La idea del *parecido de familia* aparece recogida en los 'juegos del lenguaje' en [Wit81], y se formalizará para conjuntos *fuzzy*, relacionándola con el dinamismo del lenguaje. El carácter impreciso del lenguaje natural impulsa a su estudio bajo los conjuntos *fuzzy*. Como lo hace el lingüista George Lakoff [Lak73], quien, mediante la lógica *fuzzy*, aborda problemas de semántica del lenguaje natural. Así mismo, el emergente campo de la Computación con Palabras [Zad96], cuyo creador fue L. A. Zadeh, no deja de ser una evolución de la lógica *fuzzy*, que trata con el problema del significado.

En este trabajo se recoge un modelo para el significado de los predicados y los colectivos asociados a ellos, bajo el estudio del orden introducido en el *universo de discurso* en el que se aplican y la traslación de este órden a una escala $(L, \leq)$, que sea un conjunto parcialmente ordenado. En el caso particular de que dicha escala sea el intervalo unidad con la restricción del orden usual de la recta real, aparecen los conjuntos *fuzzy* de Zadeh.

Respecto al aspecto sintáctico de los modelos de razonamiento, se muestra en [TGHP10] la construcción del modelo a partir de una estructura de conjunto preordenado, debilitando la estructura de ortorretículo donde se definían estos modelos anteriormente [ET00] [TCC01] [Qiu07] [TP06] [TPÁ09], lo que permite considerar el modelo cuando se manejen informaciones imprecisas representadas por conjuntos *fuzzy*. También se formaliza la construcción del modelo partiendo de un operador de consecuencias en el sentido de Tarski [CT89], y se introducen los conjuntos de conjeturas, hipótesis y especulaciones.

Estos modelos podrían llegar a ser una herramienta que permitiese al ser humano verificar la consistencia de sus razonamientos a partir de la información de la que disponen, sin tener en cuenta motivaciones emocionales que pueden llevarle a tener una visión sesgada y realizar razonamientos erróneos. De hecho, como aplicación de estos modelos puede construirse un programa que permita, dados los síntomas del paciente, a través de un razonamiento abductivo, diagnosticar la enfermedad capaz de causarlos, de esta forma se podría llegar antes a enfermedades poco comunes que en muchos casos no se tienen presentes. Realmente, la obtención de algoritmos que permiten obtener hipótesis y especulaciones a partir de un conjunto de premisas es un trabajo que está en curso de realización. Algo que está en el sueño del '¡Calculen!' de Leibniz.

Los aspectos semántico y sintáctico del modelo se unen cuando se construye el modelo a partir del significado que se requiera del término *no inconsistente* para cada problema concreto. Dependiendo de las interpretaciones se obtienen distintos modelos del razonamiento conjetural.

Del mismo modo, bajo el estudio de distintas interpretaciones del término *imposible* surge una nueva interpretación de los principios de No Contradicción y Tercero Excluido. Aristóteles enuncia el principio de no contradicción como "no es posible que un objeto sea a la vez blanco y no blanco". Esta información se traduce en lógica clásica como que la intersección de todo elemento con su negado es falso (cero), y se entienden los principios como axiomas. No obstante, en [Tri09], se recoge una nueva representación de aquel término; se entiende 'imposible' como que la intersección de un elemento y su negado es auto-contradictoria, consiguiendo así que estructuras como las álgebras estándar de conjuntos *fuzzy* o las álgebras de De Morgan verifiquen ambos principios.

Tanto los modelos lingüísticos (la representación de los predicados y colectivos), como los de razonamiento (la elección de la interpretación de la no consistencia) requieren un proceso de diseño [TG10]. La mayor flexibilidad que proporciona el uso de la lógica *fuzzy*, obliga también a prestar atención al uso que queramos hacer de ella y no dar por sentadas todas las propiedades de la lógica clásica.

A lo largo de todo el trabajo se hace notar la relación de la lógica *fuzzy* con el diseño de todos los elementos involucrados en los razonamientos: la representación de las informaciones que se manejen, su consistencia y los distintos tipos de deducción o distintas formas de conjeturar que pueden requerirse dependiendo de los operadores de consecuencias o conjeturas que utilicemos.

Lo expuesto anteriormente se mostrará a través de los siguientes cuatro artículos cuyo principal contenido se resumirá en el siguiente apartado.

1. I. García-Honrado, E. Trillas, *An Essay on the Linguistic Roots of Fuzzy Sets*, Information Sciences 181 4061-4074 (2011).

2. I. García-Honrado, E. Trillas, *Characterizing the Principles of Non Contradiction and Excluded Middle in* $[0,1]$, Internat. J. Uncertainty Fuzz. Knowledge-Based Syst. 2 113-122 (2010).

3. E. Trillas, I. García-Honrado, A. Pradera, *Consequences and Conjectures in Preordered Sets*, Information Sciences 180 (19) 3573-3588 (2010).

4. I. García-Honrado, E. Trillas, *On an Attempt to Formalize Guessing*, Tech. Rep. FSC-2010-11, European Centre for Soft Computing, aceptado en el libro ‘Soft Computing in Humanities and Social Sciences’ (Eds. R. Seising and V. Sanz) Springer-Verlag Berlín (2011).

Capítulo 2

Resumen global de los artículos

Lo maravilloso de aprender algo,
es que nadie puede arrebatárnoslo.
B. B. King (1925-)

En los cuatro artículos que siguen, principalmente se presta atención a los tres problemas teóricos siguientes:

- El del 'significado' y , en consecuencia, qué significa cada función de pertenencia de un conjunto *fuzzy* desde un punto de vista estructural y en relación con el significado contextual del término lingüístico que representa.

- La validez de los llamados 'principios' de Tercero Excluido y No Contradicción en las álgebras de conjuntos *fuzzy*, desde un punto de vista distinto, aunque más general y también cercano al de Aristóteles que el usual. Unos principios históricamente considerados básicos para un correcto razonamiento, y cuyo fallo había permitido la introducción de dudas epistemológicas en la lógica *fuzzy*.

- El estudio estructural del razonamiento ordinario no-analógico, a partir de representaciones en estructuras lo más débiles posibles.

2.1. Un ensayo sobre las raices lingüísticas de los conjuntos *fuzzy*

El artículo [1] aborda el problema de estudiar la 'representación' del significado de los predicados y los colectivos a ellos asociados. El concepto de 'colectivo', que no parece actualmente definible, es sensitivo al contexto (*context-sensitive*) y se intenta representar a través de los $L-$*fuzzy* sets.

En el estudio del significado es relevante la importancia del contexto y propósito. Así pues la notación $\mathcal{C}_X(P; c, up)$ para representar un *colectivo* asociado a un predicado P, en el universo de discurso X, el contexto c y el propósito de su uso up, hace constar la importancia del propósito y el contexto a la hora del diseño del grado y por tanto de la elección de la correspondiente 'escala' de valores.

Al aplicar un predicado P, a un universo de discurso X, se introduce una relación empírica de comparación $\leq_P$, y se trabaja bajo la hipótesis de que esa relación sea un preorden; es decir, una relación reflexiva y transitiva, a la que se llama el 'significado primario' del predicado. Se deja de lado el caso en que $\leq_P$ no sea un preorden a causa de ciertas dificultades técnicas que se explicitan en el artículo. Como el significado no es invariante, sino que tiene una componente 'social', puede ser que para un grupo de m personas se encuentren distintas percepciones del anteriormente mencionado preorden, en cuyo caso se tomará como 'significado primario' para el grupo la intersección de los m preórdenes. El significado primario traduce la intuición de que con su uso los predicados introducen algún orden en el universo de discurso.

El grado del predicado P se define sobre un conjunto parcialmente orde-

[1] "An essay on the linguistic roots of fuzzy sets", por I. García-Honrado y E. Trillas, se publicará próximamente en 'Information Sciences'. Se encuentra *on line* en http://www.sciencedirect.com/science/article/pii/S0020025511002738 .

nado $(L, \leq)$ o escala. Por lo tanto, aunque en la escala se añade la propiedad anti-simétrica, en el caso de existir algunos elementos no comparables bajo el preorden introducido por el predicado, pueden continuar siendo no comparables una vez calculado el grado de verificación de ese predicado que traduce otra componente del uso elemental de P en X.

El grado es un modelo matemático que traslada a una escala el significado primario que el predicado define en el universo de discurso, ya que es una función $\mu_P : X \to L$, que debe verificar, siempre que sean x, y en X tales que si $x \leq_P y$, entonces $\mu_P(x) \leq \mu_P(y)$. Dicho de otra forma, $\leq_P \subset \leq_{\mu_P}$, siendo $x \leq_{\mu_P} y$ si y sólo si $\mu_P(x) \leq \mu_P(y)$, con x, y en X. El grado no es sino una $(L, \leq)-$medida de la verificación de los enunciados elementales 'x es P', y sólo cuando es $\leq_P = \leq_{\mu_P}$ se dice que refleja perfectamente a $\leq_P$, o que refleja perfectamente el significado del predicado P.

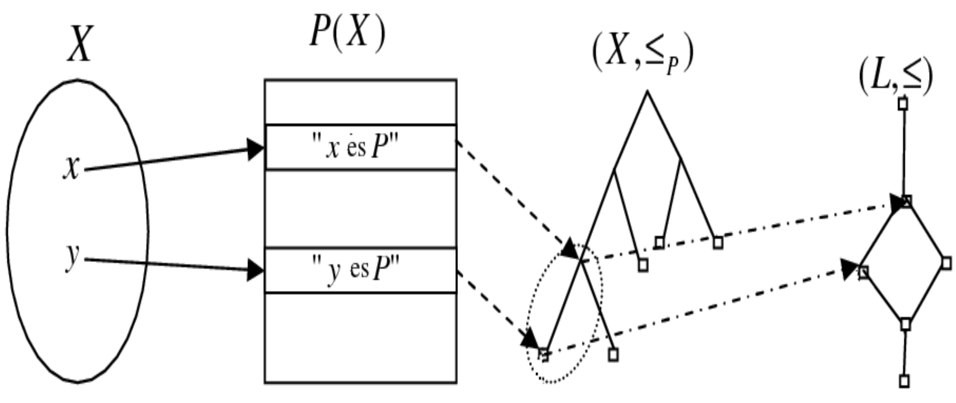

Figura 2.1:

Con la terna $(X, \leq_P, \mu_P)$ se identifica el 'significado' de P en X y debe notarse que esta no es una definición absoluta, sino dependiente de L. No obstante, y si $\leq_P$ es un preorden, se prueba que siempre existe una escala que recoge el significado primario (Teorema 9.1), con un preorden que refleja perfectamente el significado del predicado.

De todos modos, la elección de la escala $(L, \leq)$ más conveniente para el problema, no es trivial y en los problemas tecnológicos, suele ser una escala numérica. En este artículo se muestran y analizan posibles elecciones de los conjuntos parcialmente ordenados $(L, \leq)$.

- Si es $L = [0, 1]$, con el orden usual de la recta real, tenemos un orden total que nos permite comparar todos los elementos de X. El grado sería un *fuzzy* set [Zad65].

- Si es $L = \{[a, b] \subset [0, 1]; a \leq b\}$, el grado estará representado por un intervalo [DK03] [Zad75]. Surgen los *interval-valued* fuzzy *sets* que recogen con precisión la posible imprecisión numérica a la hora de calcular el grado. Y en este caso distinguimos dos órdenes parciales

  - $[a_1, b_1] \leq [a_2, b_2] \Leftrightarrow a_1 \leq a_2$ y $b_1 \leq b_2$,
  - $[a_1, b_1] \leq^* [a_2, b_2] \Leftrightarrow a_2 \leq a_1$ y $b_1 \leq b_2$.

- Si es $L = [0, 1]^X$, el grado estará representado por un conjunto *fuzzy* de tipo 2 [Men07]. Esta definición recoge la imprecisión a la hora de calcular el grado con imprecisión y un caso particular importante acontece si L es el conjunto de los *Fuzzy Numbers* [MJ02], que permite recoger imprecisión numérica evaluada imprecisamente. La definición de orden parcial, a cualesquiera números fuzzy, está recogida en el artículo.

- Si es $L = \{(x, y) \in [0, 1] \times [0, 1]; x + y \leq 1\}$, surgen los conjuntos *fuzzy* intuicionistas o de Atanasov, con el orden $(x_1, y_1) \leq_A (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ y $y_2 \leq y_1$, isomorfos al subconjunto de los intervalos cuyos extremos, a, b verifiquen $a + b \leq 1$, con la relación de contenido. Este conjunto parcialmente ordenado permite representar el grado del predicado P, μ_P, y el de su negación $\mu_{no\ P}$, ya que bajo una operación reversible e involutiva, N, L se puede escribir como $\{(x, y) \in [0, 1] \times [0, 1]; y \leq$

$N(x)\}$. No obstante el término no$P$, no es un término lingüístico, y sí lo es un antónimo de P, aP cuyo grado verifica $\mu_{aP} \leq N(\mu_P) = \mu_{no\ P}$.

Como ejemplo de modelización del significado, se analiza el significado del predicado *probable* utilizando como orden parcial en el que se define el grado, el intervalo $[0, 1]$ con el orden usual de la recta real, y aplicando el predicado en tres universos de discurso con estructuras diferentes: un álgebra de Boole, un retículo ortomodular y el conjunto de los conjuntos *fuzzy* ([TNGH10]).

Tras mostrar posibles formas de representar el significado, se presta atención al importante aspecto del dinamismo del lenguaje. Esto se lleva a cabo con el estudio del "*parecido de familia entre los predicados*". El parecido de familia, *family resemblance* en inglés, proviene de Wittgenstein [Wit81], iniciador de la conocida filosofía del lenguaje. Se puede estudiar ese dinamismo por medio del nuevo concepto de las migraciones de predicados [GHTG10] a otros universos de discurso. Aunque un predicado migre, mantiene unas constantes que permiten reconocerlo aún siendo utilizado en otro universo de discurso. En el intento de caracterizar estas constantes surge la formalización matemática del parecido de familia entre dos predicados [TMS09] y se comprueba que un predicado y su migrado mantienen tal parecido. Hay que observar, sin embargo y naturalmente, que un predicado y su antónimo o predicados contradictorios [TAJ99] no verifican la definición de parecido de familia.

En lógica borrosa, el diseño de todos los elementos involucrados en sus problemas [TG10] es un tema de crucial importancia, ya que los errores de diseño llevan a soluciones no aceptables en muchos casos. Englobando el artículo en la idea del proceso de diseño y mostradas las distintas formas en las que se puede representar un predicado, permite encuadrar el artículo en el campo de la Computación con Palabras, razón por la cual artículos referentes a tales cuestiones son publicados en la revista 'Information Sciences'.

También sirve para el propósito de definir diferentes estructuras sobre las que se pueden obtener modelos de razonamiento ordinario, dependiendo de cómo se represente la información disponible de cada razonamiento concreto, que mayoritariamente, será dada a través del lenguaje con toda la imprecisión que éste conlleva.

2.2. Caracterización de los principios de No Contradicción y Tercero Excluido en $[0, 1]$

El artículo [2] se basa en una nueva interpretación de los conceptos de No Contradicción y Tercero Excluido introducida en [Tri09], a partir del enunciado de los mismos hecho por Aristóteles, quien enunció el principio de No Contradicción como que la coexistencia de 'A *y* no A es imposible' para cualquier enunciado afirmativo A. Clásicamente se tradujo ese 'imposible' por la falsedad de 'A *y* no A', es decir, representando *y* por $\cdot$ y la negación por $'$, el enunciado se compacta en la fórmula $a{\cdot}a' = 0$, con cada representación a de A. La verificación de esta fórmula forma parte de la axiomática de estructuras como las álgebras de Boole o de forma más general, de los ortorretículos, aunque no de las álgebras de De Morgan. Pero en el caso del conjunto de los conjuntos *fuzzy* denotado por $[0, 1]^X = \{\mu; \mu : X \to [0, 1]\}$, la estructura más fuerte en la que se pueden enmarcar es $([0, 1]^X, \text{mín}, \text{máx}, 1 - id)$, un álgebra de De Morgan, donde no se verifican los principios de No Contradicción, ni de Tercero Excluido según su definición clásica de $a{\cdot}a' = 0$ y $(a{\cdot}a')' = 1$. El caso más general, con esta interpretación, fue resuelto previamente probándose la existencia de álgebras de conjuntos fuzzy que no verifican ninguno de los dos principios, que verifican uno u otro y álgebras que verifican los dos [TAP02].

[2] "Characterizing the principles of non contradiction and excluded middle in [0,1]", por I. García-Honrado, E. Trillas en Internat. J. Uncertainty Fuzz. Knowledge-Based Syst., volumen 2, páginas 113-122. Año 2010.

En la nueva interpretación, se traduce 'imposible' por 'auto-contradictorio'. Siendo $\models$, la representación simbólica de la implicación, un elemento A es auto-contradictorio si 'Si A entonces no A', y así el principio de No Contradicción se entiende como

$$a{\cdot}a' \models (a{\cdot}a')' \tag{2.1}$$

y el de Tercero Excluido como $((a{\cdot}a')')' \models (((a{\cdot}a')')')'$, para cada representación a del enunciado A. O, suponiendo que $+$ es la operación dual de $\cdot$, $a + b = (a'{\cdot}b')'$, como

$$(a + a')' \models ((a + a')')'. \tag{2.2}$$

Bajo este nuevo punto de vista, se prueba la existencia de relaciones $\models$ que verifican 2.1 y 2.2 (teoremas 1 y 2). Por lo tanto, se tiene *una relación bajo la cual los* fuzzy *sets verifican estos principios* en su nueva interpretación, no entrando en contradicción las estructuras matemáticas en las que se engloban los conjuntos fuzzy con el postulado de Aristóteles. No obstante, se estudia detenidamente cómo se puede entender esta nueva interpretación de los principios en el intervalo unidad y por ende en los conjuntos fuzzy, bajo las siguientes concreciones de la relación $\models$.

- En el caso usual de traducir $\models$ por $\leq$, donde $\mu \leq \sigma$ con $\mu, \sigma \in [0,1]^X$ si y sólo si $\mu(x) \leq \sigma(x)$, para todo $x \in X$.
- En el caso de traducir $\models$ por el orden φ-*sharpened* $\preceq_\varphi$, definido por $\mu \preceq_\varphi \sigma \Leftrightarrow \begin{cases} 0 \leq \mu(x) \leq \sigma(x) \leq \varphi^{-1}(1/2) \\ \varphi^{-1}(1/2) \leq \sigma(x) \leq \mu(x) \leq 1 \end{cases}$, para todo x en X.
- En el caso de traducir $\models$ por el preorden $\vdash$, definido como $\mu \vdash \sigma \Leftrightarrow |\mu(x) - 0{,}5| \leq |\sigma(x) - 0{,}5|$, para todo x en X.

Con respecto al primer caso, se obtienen teoremas de caracterización para ambos principios, traduciendo la intersección funcionalmente por una función F, la unión por otra función G (ambas sin tener ninguna propiedad especial) y la negación por una negación fuerte $N_\varphi = \varphi^{-1}(1 - \varphi(x))$ (con $\varphi : [0,1] \to [0,1]$, estrictamente monótona y verificando $\varphi(0) = 0$ y $\varphi(1) = 1$).

Los teoremas son los 6,7,9 y 10 del artículo:

- $([0,1], N_\varphi, F)$ verifica $\leq$-NC si y sólo si $F(a, N_\varphi(a)) \leq \varphi^{-1}(1/2)$ para todo $a \in [0,1]$.

- $([0,1], N_\varphi, G)$ verifica $\leq$-EM si y sólo si $\varphi^{-1}(1/2) \leq G(a, N_\varphi(a))$ para todo $a \in [0,1]$.

- $([0,1], 1 - id, F)$ satisface $\leq$-NC, si y sólo si la restricción F^* de F al conjunto $\{(a, 1-a); a \in [0,1]\}$, verifica $F^* \leq Sum/2$

- $([0,1], 1 - id, G)$ satisface $\leq$-EM, si y sólo si la restricción G^* de G al conjunto $\{(a, 1-a); a \in [0,1]\}$, verifica $Sum/2 \leq G^*$

Por lo tanto, para las álgebras estándar de fuzzy *sets en las que F es una t-norma continua y G una t-conorma continua, los principios se verifican siempre.*

En el caso del orden *sharpened* se obtienen también teoremas de caracterización, los Teoremas 12 y 13:

- $([0,1], N_\varphi, F)$ satisface $\preceq_\varphi$-NC, si y sólo si $F(\mu(x), \mu'(x)) = \varphi^{-1}(1/2)$.

- $([0,1], N_\varphi, G)$ satisface $\preceq_\varphi$-EM, si y sólo si $G(\mu(x), \mu'(x)) = \varphi^{-1}(1/2)$.

Análogamente, en el caso del tercer preorden introducido, se obtienen los Teoremas 14 y 15:

- La terna $([0,1]^X, \vdash, 1 - id, F)$ verifica NC para toda función F.

- La terna $([0,1]^X, \vdash, 1 - id, G)$ verifica EM para toda función G.

Así pues, con la nueva interpretación los principios aristotélicos se mantienen en el caso concreto de las álgebras estándar de *fuzzy* sets, llegando en este artículo a caracterizaciones para la verificación de estos principios en el intervalo unidad utilizando diferentes relaciones $\models$. El artículo cierra la polémica sobre la verificación de los principios en las álgebras estándar de conjuntos *fuzzy*.

2.3. Modelos de razonamiento ordinario

A continuación se resumirán las principales contribuciones a los modelos de Conjeturas, Hipótesis y Consecuencias (CHC Models) llevados a cabo en los artículos

1. E. Trillas, I. García-Honrado, A. Pradera, Consequences and conjectures in preordered sets, *Information Sciences* 180 (19) (2010) 3573-3588.

2. I. García-Honrado, E. Trillas, On an attempt to formalize guessing, Tech. Rep. FSC-2010-11, European Centre for Soft Computing, aceptado en *Soft Computing in Humanities and Social Sciences* (Eds. R. Seising and V. Sanz) Springer-Verlag (2011).

Estos modelos aparecen en el año 2000 en el artículo [ET00], donde se definen sobre reticulos orto-complementados, y por lo tanto no se pueden trasladar a ninguna estructura de conjuntos *fuzzy*. En el artículo [TGHP10] se estudiarán los modelos CHC sobre preordenes con negación, estructuras más débiles que permiten incluir álgebras de conjuntos *fuzzy*.

En el artículo [TGH10], se amplían los Modelos CHC construyéndolos de acuerdo a diversas interpretaciones del concepto de consistencia y no únicamente a partir de un operador de consecuencias [CT89], como en [ET09].

2.3.1. Consecuencias y conjeturas en conjuntos preordenados

Los modelos de conjeturas tratan de formalizar los procesos que realiza el ser humano en sus razonamientos. Se equipara conjeturar con razonar a partir de la información de la que se dispone. Por ello, en este artículo se estudia cómo a partir de la deducción, traducida por operadores de consecuencias en el sentido de Tarski, se construye un modelo general para el razonamiento ordinario, que incluye tanto la dedución como la abdución, modelada bajo el conjunto de hipótesis, así como el razonamiento especulativo, modelado con el conjunto de especulaciones.

El primer punto en el que hay que detenerse a la hora de la construcción de estos modelos es la información disponible y en dos vertientes:

- El tipo de proposiciones que se tratan y la estructura en la que pueden modelarse; por ejemplo, si se manejan proposiciones precisas en la que se tiene la incompatibilidad de un elemento y su opuesto (se puede construir el modelo sobre álgebras de Boole o estructuras más débiles como los retículos orto-complementados), o si se manejan proposiciones imprecisas en las que puedan coexistir un elemento y su negado (en este caso se debería construir un modelo sobre álgebras de De Morgan, o de conjuntos *fuzzy*).

- Dentro de la estructura anterior, debe considerarse que información se tiene; es decir, por similitud a la lógica clásica, el conjunto de premisas, P, sobre el que se desarrolle el modelo de conjeturas, supuesta su consistencia.

La contribución de este artículo a estos dos aspectos, es el desarrollo de estos modelos en estructuras que sean simplemente un conjunto con un preorden $(L, \leq)$ y, en los casos que se requiera, añadirle una operación que traduzca la intersección, ínfimo (Inf), y una negación $(')$. La definición de operador de consecuencias en el sentido de Tarski, se efectúa en diferentes espacios $\mathfrak{F}$, que controlarán la consistencia del conjunto de premisas, y concretando así la definición de lo que usualmente se considera una estructura de consecuencias, o sistema deductivo.

Definición 2.3.1. *Sea L un conjunto cualquiera, y $\mathfrak{F} \subset \mathbb{P}(L)$, se dice que $(L, \mathfrak{F}, C)$ es una* estructura de consecuencias, *o* sistema deductivo *siempre que $C : \mathfrak{F} \to \mathfrak{F}$ verifique,*

1. *$P \subset C(P)$, para todo $P \in \mathfrak{F}$ (C es* extensivo*)*
2. *Si $P \subset Q$, entonces $C(P) \subset C(Q)$, para todo $P, Q \in \mathfrak{F}$ (C es* monótono*)*
3. *$C(C(P)) = C(P)$, ó $C^2 = C$, para todo $P \in \mathfrak{F}$ (C es* cerrado*)*

y se dice que C es un operador de consecuencias *(en el sentido de Tarski) para $\mathfrak{F}$ en L.*

En el artículo se consideran los siguientes espacios $\mathfrak{F}$:

1. $\mathfrak{F} = \mathbb{P}(L)$
2. $\mathfrak{F} = \mathbb{P}_0(L)$, siempre que L sea inf-completo, esto es, para todo $P \in \mathbb{P}_0(L)$, existe $Inf P = p_\wedge \in L$, y además se pedirá que $p_\wedge \neq 0$.
3. $\mathfrak{F} = \mathbb{P}_{SC}(L) = \{P \in \mathbb{P}(L);\ \text{para ningún } p \in P:\ p \leq p'\}$
4. $\mathfrak{F} = \mathbb{P}_{NC}(L) = \{P \in \mathbb{P}(L);\ \text{para ningún par } p_1, p_2 \in P:\ p_1 \leq p_2'\}$
5. $\mathfrak{F} = \mathbb{P}_{iC}(L) = \{P \in \mathbb{P}(L);\ \text{para ningún par de subconjuntos finitos } \{p_1, ..., p_r\}, \{p_1^*, ..., p_n^*\} \subset P:\ p_1^* \cdot ... \cdot p_n^* \leq (p_1 \cdot ... \cdot p_r)'\}$, siempre que $\cdot$ sea una operación ínfimo en $(L, \leq)$.

Cuando estas familias existen, se tiene la cadena de inclusiones:

$$\mathbb{P}_{iC}(L) \subset \mathbb{P}_{NC}(L) \subset \mathbb{P}_{SC}(L) \subset \mathbb{P}(L).$$

Estos espacios surgen de las distintas formas de representar la consistencia del conjunto de premisas con el que se trabaje. De la misma forma que se conseguía una nueva interpretación de los principios aristotélicos de No Contradicción y Tercero Excluido, y la misma que nos permitirá obtener modelos de conjeturas sin partir de un operador de consecuencias.

Se estudia la propiedad de consistencia de los operadores de consecuencias, es decir que si $q \in C(P)$, entonces $q' \notin C(P)$, para los siguientes operadores de consecuencias,

- $C_{\leq}(P) = \{q \in L; \exists p \in P : p \leq q\}$, para todo $P \in \mathfrak{F}$.
- $C_{\bullet}(P) = \{q \in L; \exists \{p_1, ..., p_n\} \in P : p_1 \cdot ... \cdot p_n \leq q\}$, para cualquier $P \in \mathfrak{F}$.
- $C_{\wedge}(P) = \{q \in L;\ \inf P \leq q\}$, para cualquier $P \in \mathfrak{F}$.

En cada uno los distintos espacios $\mathfrak{F}$ anteriormente mencionados. Este estudio da lugar a los Teoremas 3.13, 3.20 y 3.25 del artículo, quedando esquemáticamente recogido en la tabla 1 que aparece en el mismo.

Se introduce el operador $C_{\leq}$ que permite calcular las consecuencias sin tener definida la operación de ínfimo, cumpliendo la propiedad especial de que $C_{\leq}(P) = \bigcup\limits_{p \in P} C_{\leq}(p)$.

A partir de las estructuras de consecuencias $(L, \mathfrak{F}, C)$, se calculan los conjuntos de conjeturas, bajo la fórmula

$$Conj_C(P) = \{q \in L, q \in C(P) \ \text{ó}\ q' \notin C(P)\},$$

simplificada bajo la consistencia de $(L, \mathfrak{F}, C)$ a $Conj_C(P) = \{q \in L,\ q' \notin C(P)\}$.

Dentro del conjunto de las conjeturas se define el conjunto de hipótesis,

$$Hyp_C(P) = \{q \in L; \{q\} \in \mathfrak{F} \ y\ P \subset C(\{q\})\}$$

y el de las especulaciones,

$$Sp_C(P) = Conj_C(P) - (Hyp_C(P) \cap C(P)),$$

obteniéndose así una partición del conjuntos de las conjeturas. Para cerrar el modelo, se llamarán refutaciones a aquellos elementos de L que no sean conjeturas.

Se estudia también las estructuras de consecuencias isomorfas y cómo pueden calcularse por medio del isomorfismo los conjuntos de conjeturas, hipótesis y especulaciones (Remarks 3.3 y 4.10).

Gracias al estudio de los modelos sobre conjuntos preordenados, es posible su generalización a estructuras de conjuntos *fuzzy* dotadas del órden puntual ($\mu \leq \sigma$ si y sólo si $\mu(x) \leq \sigma(x)$ para todo x en X), la negación fuerte ($a' = 1 - a$, para todo $a \in [0,1]$) y la operación ínfimo, que aquí es (mín). En el contexto $([0,1]^X, \leq, \text{mín}, ')$, y a lo largo de la sección 5 del artículo, se estudia el comportamiento del operador $C_\leq$ en los siguientes espacios $\mathfrak{F}$:

1. $\mathbb{P}_{SC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu \in P, \exists x \in X : \ \mu(x) > 0{,}5\}$

2. $\mathbb{P}_{NC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu, \sigma \in P, \exists x \in X : \mu(x) + \sigma(x) > 1\}$

3. $\mathbb{P}_{iC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \{\mu_1, \ldots, \mu_r\}, \{\sigma_1, \ldots, \sigma_n\} \in P, \exists x \in X :$ $\text{mín}(\mu_1(x), \ldots, \mu_r(x)) + \text{mín}(\sigma_1(x), \ldots, \sigma_n(x)) > 1\}$

4. $\mathbb{P}_0([0,1]^X) = \{P \subseteq [0,1]^X; \exists x \in X : (InfP)(x) \neq 0\}$

5. $\mathbb{P}_n([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu \in P, \ \mu$ es normalizado$\}$ $= \{P \subseteq [0,1]^X; \forall \mu \in P, \exists x \in X, \mu(x) = 1\}$

Se calculan los conjuntos de conjeturas, hipótesis y especulaciones a partir del operador de consecuencias $C_\leq$. Se iniciará con ello la construcción de un modelo de razonamiento ordinario que permite tratar con la imprecisión que trasladan los conjuntos *fuzzy*.

2.3.2. Un intento de formalizar el proceso de conjeturar

En este artículo se estudia de forma general el proceso de razonamiento ordinario a través de la búsqueda de conjeturas, para lo que se contesta a cuatro cuestiones fundamentales

- En qué conjunto, L estará la información disponible, y se buscarán las conjeturas.
- Con qué estructura algebraica está dotado L.
- Cómo representar la información disponible P, en el conjunto L.
- Cómo definir el conjunto de las conjeturas, bajo las distintas interpretaciones de no ser inconsistente con las premisas.

Respecto al primer punto es destacable la elección del conjunto L, pudiendo ser un conjunto de conjuntos *fuzzy* o de elementos precisos, dependiendo del contexto y características de cada problema concreto para el que se quiera construir el modelo de conjeturas.

Contribuyendo al segundo punto se presentan las *Álgebras Básicas Flexibles*, estructuras algebraicas que no requieren verificar un gran número de propiedades. Éstas se construyen bajo un conjunto parcialmente ordenado en el que se define una operación representando la intersección y otra la unión, que son monótonas y tienen elementos neutro y absorbente. También se define una operación que intercambia el ínfimo y el supremo del retículo, es anti-monótona y representa la negación. Además, se pide que esta estructura contenga una subestructura que sea un álgebra de Boole, para que en el caso que se utilice esta estructura sobre los conjuntos *fuzzy* puedan quedar recogidos los conjuntos clásicos como degeneración de los *fuzzy*. Como anexo, en este artículo se comentan los principios de No Contradicción y Tercero Excluido en el contexto de las Álgebras Básicas Flexibles.

La información disponible o conjunto de premisas P se representa dentro de los espacios $\mathfrak{F}$ que aparecen en el artículo anterior. Pero, además se dan ciertas pautas para conseguir compactar la información de P, bajo un resumen (*résumé*) de P, $(r(P))$. Se muestran tres ejemplos para $r(P)$,

- $r(P) = p_\wedge = p_1 \cdot ... \cdot p_n \in L$,
- $r(P) = p_\vee = p_1 + ... + p_n \in L$, y
- $r(P) = [p_\wedge, p_\vee] = \{x \in L; p_\wedge \leq p \leq p_\vee\}$, con $r(P) \in \mathbb{P}(L)$.

Se prueba que si $r(P) \leq p_\wedge$, el operador $C_r(P) = \{q \in L; r(P) \leq q\}$, es un operador de consecuencias.

Las conjeturas son aquellos elementos que no son inconsistentes con el conjunto de premisas. Por tanto, cabe definirlas a partir de tres interpretaciones del concepto de *no inconsistente*: $r(P) \cdot q \neq 0$, $r(P) \cdot q \nleq (r(P) \cdot q)'$, y $r(P) \nleq q'$, dando lugar a las siguientes definiciones del conjunto de conjeturas:

- $Conj_C(P) = \{q \in L; q' \notin C(P)\}$, siempre que $C(P) \neq L$.
- $Conj_1(P) = \{q \in L; r(P) \cdot q \neq 0\}$
- $Conj_2(P) = \{q \in L; r(P) \cdot q \nleq (r(P) \cdot q)'\}$
- $Conj_3(P) = \{q \in L; r(P) \nleq q'\}$

Con esta visión se da identidad por sí mismo al conjunto de las conjeturas, analizando las propiedades que cumple sin necesidad de su estudio a partir de una estructura de consecuencias. De hecho, se prueba que los operadores de conjeturas $Conj_{C_1}$ y $Conj_{C_2}$ no se pueden obtener a través de un operador de consecuencias. Las propiedades generales que verifican los operadores de conjeturas considerados son:

- $Conj(P) \neq \emptyset$

- $0 \notin Conj(P)$
- $P \subset Conj(P)$
- $C(P) \subset Conj(P)$
- Anti-monotonía:'si $P \subset Q$, entonces $Conj(Q) \subset Conj(P)$'.

En la sección 6 se prueba que para todo $i = 1, 2, 3$, se tiene que $C_\wedge \subset Conj_i$. Por lo tanto dentro del conjunto de conjeturas $Conj_i(P)$ distinguimos,

- Consecuencias, $C_\wedge(P)$
- Hipótesis $Hyp_i(P) = \{q \in Conj_i(P);\ q < p_\wedge\}$
- Especulaciones $Sp_i(P) = \{q \in Conj_i(P);\ q\ NC\ p_\wedge\}$

Obteniéndose la partición del conjunto de las conjeturas

$$Conj_i(P) = C_\wedge(P) \cup Hyp_i(P) \cup Sp_i(P),$$

y la partición del conjunto L gracias al conjunto de Refutaciones, $Ref_i(P) = L - Conj_i(P)$.

Se muestra otra partición del conjunto L, distinguiendo los elementos decidibles de los no decidibles (en inglés $C-$*undecidables*) que se definen como $U_C(P) = \{q \in L;\ q \notin C(P) \& q' \notin C(P)\}$, que a su vez pueden ser divididos en especulaciones e hipótesis $U_C(P) = Sp_i(P) \cup Hyp_i(P)$.

En el artículo se recoge un modelo para la *falsación de hipótesis*, concepto introducido por Popper [Pop63], basándose en que si h es hipótesis para el conjunto de premisas P, se tiene la siguiente cadena: $C(P) \subset C(\{h\}) \subset Conj_C(P)$ y las hipótesis que se falsarán serán aquellas para las que se encuentre un elemento que se siga de P, pero no de $\{h\}$, o un elemento que se siga de $\{h\}$ pero no sea conjeturable a partir de P.

Se hace constar la importancia de las especulaciones en el modelo de razonamiento ordinario , ya que es el único operador de todos los que aparecen que es no-monótono, es decir que no sigue ninguna ley de monotonía, no es ni monótono, ni anti-monótono, así que su estudio se hace más complejo. Además, en los retículos ortomodulares se verifica que $Hyp_3(P) = p_\wedge \cdot Sp_3(P)$ y $C_\wedge(P) = p_\wedge + Sp_3(P)$.

El artículo muestra dos ejemplos modelados por conjeturas:

- La conjetura de Goldbach. Basándose en los cinco axiomas de Peano como conjunto de premisas, decir que todo número par mayor que dos es suma de dos primos, no es incompatible con los axiomas de Peano, ya que no se ha encontrado ningún número par mayor que dos que no pueda escribirse como la suma de números primos, por lo tanto es provisionalmente una conjetura matemática. De hallarse una demostración para este hecho, pasará a ser una consecuencia de los axiomas de Peano, ya que se habrá deducido de los axiomas de los números naturales.

- Otro ejemplo es el construido a partir de los sucesos que pueden acontecer al lanzar un dado. En este caso todos los posibles resultados son conjeturables, obtener cualquier resultado no entra en contradicción con la naturaleza del dado, y para más precisión son hipótesis, no hay ni especulaciones, ni consecuencias distintas a la premisa ('que salga un de las seis caras'), ya que ningún suceso distinto al total, es seguro sino contingente.

Capítulo 3

Conclusiones y trabajo futuro

Toda la imaginería
que no ha brotado del río,
barata bisutería.
Antonio Machado (1875-1939)

3.1. Relacionado con los orígenes lingüísticos de los conjuntos *fuzzy*

La investigación llevada a cabo en este trabajo pretende ser una contribución al estudio del significado, el llamado 'nudo gordiano' de la Inteligencia Artificial.

En primer lugar, se muestran formas de modelizar la actuación de un predicado en un *universo de discurso*. Es decir, el *uso primario* de un predicado en un universo, o *à la* Wittgenstein, cual es su significado. Se modela el *significado elemental* o *uso primario*, a través de una relación que traslada cuantitativamente, y a veces de forma perceptiva, la forma de actuar del predicado en el universo de discurso.

En segundo lugar, se muestra el concepto de grado, es decir, hasta dónde un objeto verifica la propiedad nombrada por el predicado; de alguna manera, el modo cuantitativo de medir esa magnitud. Se formaliza el grado, tras conocer la relación modelada por el *uso primario* del predicado. De esta forma se clarifica cual es la propiedad intrínseca de una función en L^X para considerarla una representación del predicado y, así representar el colectivo por él definido.

En tercer y último lugar, se atiende al problema práctico de la elección de la escala en la cual varía el grado. Mostrándose, para el caso concreto de un predicado gradual las posibles representaciones a través de conjuntos *fuzzy*, conjuntos *fuzzy* de tipo 2, o conjuntos *fuzzy* evaluados en intervalos.

El artículo se enmarca dentro de la revisión de la actual artillería disponible de la Lógica *fuzzy*, por lo que se puede considerar útil en el campo más amplio de la Computación con Palabras.

Como futuro trabajo dentro de este campo enumeramos:

- El análisis empírico de la relación $\leq_P$, para comprobar cuando es un preorden, así como qué hacer cuando no lo es.
- Propiedades generales que una operación $(\cdot)$ debería verificar para obtener $\mu_{P\&Q} = \mu_P \cdot \mu_Q$, bien sobre el mismo universo de discurso, o bien sobre universos distintos.
- Propiedades generales que una operación $(+)$ debería verificar para obtener $\mu_{P \text{ or } Q} = \mu_P + \mu_Q$, bien sobre el mismo universo de discurso, o bien sobre universos distintos.
- Propiedades generales que la relación fuzzy R debería verificar para obtener $\mu_{Si\ P\ entonces\ Q} = R \circ (\mu_P \times \mu_Q)$, bien sobre el mismo universo de discurso, o bien sobre universos distintos.

- Generalizar el concepto de migración lingüística de predicados, a la migración de oraciones complejas que involucren conectivos o condicionales.

- Analizar el concepto de 'significado grupal', relacionándolo con el parecido de familia o migración, para intentar capturar la noción de 'significado social'.

3.2. Relacionado con los principios aristotélicos

Para Aristóteles la ley de No-Contradicción era un principio del pensamiento. Lo enunciaba como que la oración 'A y no A es imposible' es válida universalmente y no necesita ser probada, es decir, la enunciaba como un axioma. Pero la flexibilidad de los conjuntos *fuzzy* permite que un elemento pueda ser A con cierto grado y no A con otro grado.

Si la ley se traduce por 'A y no A es falso', su validez depende de las interpretaciones del término falso, y cómo se representen en un marco formal. Si la ley se entiende como 'A y no A es auto-contradictorio', su validez también depende de las interpretaciones del término auto-contradictorio, y como se representen en un marco formal. Obviamente, en ambos casos también influyen las características del marco formal elegido.

¿Qué interpretación de imposible es preferible? ¿En qué marco formal es más adecuada cada una de ellas? Estas cuestiones no tienen una respuesta inmediata. Por ejemplo, en el marco de los ortorretículos los términos falso y auto-contradictorio, son equivalentes siempre que falso se represente por el primer elemento del retículo, 0, y siempre que un elemento x es auto-contradictorio se represente por $x \leq x'$. De todos modos, en otros marcos como las álgebras de De Morgan o las álgebras estándar de conjuntos *fuzzy*, existen muchos elementos no nulos que son autocontradictorios. Por lo tanto, en el artículo se estudia la desigualdad, $a{\cdot}a' \leq (a{\cdot}a')'$, sabiendo que en los ortorretículos es equivalente a $a \cdot a' = 0$.

Respecto al principio de Tercero-Excluido, partiendo de la forma de enunciarlo de Aristóteles como 'A o no A es verdadero', se traduce algebraicamente por $(a + a')' \leq ((a{\cdot}a')')' = 0$, y en términos de auto-contradicción, por $((a{\cdot}a')')' \leq (((a{\cdot}a')')')'$, que coincide con la ley de No-Contradicción si la operación $'$ es involutiva ($a'' = a$).

La mayor contribución del artículo es que se garantiza la verificación de los principios de No-Contradicción y Tercero-Excluido, bajo pocas condiciones sobre los conectivos *y*, *o*, siempre que sean funcionalmente expresables y que el complemento se traduzca por una negación involutiva. En el campo de la filosofía de la ciencia, esto permite que los conjuntos *fuzzy* se asienten sobre una base sólida.

Como futuro trabajo señalaremos

- El estudio de los principios en distintas BFA.
- Bajo qué sistemas deductivos podría deducirse $a + a'$ a partir de a y a'?
- Bajo qué sistemas deductivos podría considerarse que $a \cdot a'$ es decidible a partir de a y a'?

3.3. Relacionado con el tema de los modelos de conjeturas

La capacidad cerebral de conjeturar es crucial en la evolución de la especie Homo. Sin ella, unida a la capacidad del lenguaje, posiblemente el Homo no habría destacado respecto a otros animales, ni constituido las organizaciones sociales, económicas e incluso religiosas propias del ser humano. Uno de los hechos distintivos de la especie Homo Sapiens es el arte de guiar sus conjeturas hacia una meta. Incluso la investigación científica y tecnológica no deja de ser una actividad humana que se basa en una gestión altamente articulada de conjeturas. En palabras de Jenófanes de Colofón "Todo no es sino un tejido de conjeturas" [Pop63].

Aunque las consecuencias y las hipótesis, así como varios tipos de razonamiento no-monótono, se consideran como un tema de elevada importancia entre lógicos, filósofos, informáticos y probabilistas, ningún acercamiento a la formalización del concepto de conjetura había aparecido anteriormente. En el marco de los ortorretículos, las conjeturas se definen como elementos no inconsistentes con un conjunto de premisas (no inconsistente), que refleje la información disponible. Es decir, las conjeturas son los elementos del ortorretículo que son posibles una vez conocido un *résumé* del conjunto de premisas. Como casos particulares de conjeturas, distinguimos: consecuencias (conjeturas seguras o necesarias), hipótesis (conjeturas contingentes explicativas) y especulaciones(conjeturas contingentes elucubrativas o especulativas).

Hay que destacar que ni las especulaciones, ni las hipótesis, pueden tratarse como cuerpos de información. El proceso de obtener consecuencias se enmarca dentro de la deducción. El de obtener hipótesis en la abducción y el de obtener especulaciones en el razonamiento especulativo, y todos ellos bajo el término 'razonamiento'. Obviamente, en las ciencias formales y en el contexto de las demostraciones el rey de los razonamientos es el deductivo.

3.3.1. Consecuencias y Conjeturas en conjuntos preordenados

Álgebras como las de De Morgan no verifican las hipótesis de trabajo hechas en ([ET00]) y ([ET09]), ya que en ellas no se cumplen las leyes de No-Contradicción y Tercero-Excluido como en los ortorretículos. Esta falta se solventa en este artículo ya que como hipótesis de trabajo se manejan conjuntos preordenados dotados con una negación y en los que en caso de ser necesaria se les dota de una operación ínfimo. Así pues, en este artículo se estudian propiedades de los modelos CHC construidos sobre conjuntos preordenados, que son estructuras más débiles que aquellas en las que se definían los modelos anteriormente. Para mantener algunas propiedades de los modelos, el artículo considera operadores consistentes de consecuencias.

Se estudian en detalle tres operadores de consecuencias definidos sobre distintas familias de subconjuntos de premisas útiles para garantizar la consistencia de las premisas:

- $C_{\leq}$, que permite obtener como consecuencia todo elemento que se sigue de alguna premisa.
- $C_{\bullet}$, que permite obtener como consecuencia todo elemento que se sigue de algún número finito de premisas.
- $C_{\wedge}$, que permite obtener como consecuencia todo elemento que se sigue de todas las premisas.

Las conclusiones más relevantes que se obtienen son:

- La formalización del modelo sobre estructuras débiles en cuanto al número de propiedades que han de verificar.
- El conseguir poder tratar informaciones imprecisas, abriendo una puerta a la lógica *fuzzy*.
- Abrir una nueva puerta para el estudio del razonamiento no-monótono, ya que se prueba la anti-monotonía de las conjeturas e hipótesis y se muestra que las especulaciones son no-monótonas, es decir que no siguen ninguna ley de monotonía.
- Se formaliza el concepto de *falsación de hipótesis* propuesto por Popper [Pop63].

3.3.2. Un ensayo de la formalización del razonamiento conjetural

Previamente se definen los operadores de conjeturas a partir de operadores de consecuencias consistentes, por lo tanto se sitúa previamente al de conjeturar, el proceso de deducción, y conjeturar se puede entender como una extensión de deducir. Después de la publicación de varios artículos ([ET00], [ET02], [ET09], [TPÁ09], [TCC01], [TGHP10], [AFP01]) bajo esta idea, permanecía la duda de la existencia de operadores de conjeturas independientes de operadores de consecuencias. Este artículo despeja tal duda; mostrando las propiedades típicas de los operadores de conjeturas, se consideran la antimonotonía y la propiedad de contener tanto al conjunto de premisas como a uno de consecuencias, y se definen tres diferentes operadores de conjeturas a partir de diferentes interpretaciones de la no-inconsistencia. Solamente en uno de estos tres casos se puede considerar que las conjeturas se obtienen a través de un operador de consecuencias en el sentido de Tarski. De todos modos, en el contexto de las álgebras de Boole estas interpretaciones son equivalentes, ya que los tres operadores de conjeturas se reducen exclusivamente a uno.

Adicionalmente, se prueba que los tres operadores de conjeturas extienden el conjunto de las consecuencias para el operador $C_\wedge$.

3.3.3. Trabajo futuro

Entre las cuestiones que requieren un estudio futuro en este campo, destacamos:

- El cómo representar la información que aportan las premisas, para clarificar el concepto de *résumé* de la información.

- Establecer definitivamente la definición de operador de conjeturas de forma axiomática como se hace en el caso de los operadores de consecuencias de Tarski.

- Conectar los Modelos CHC con el Razonamiento Analógico, o al menos, con el Razonamiento basado en casos y dado por un índice de semejanza.

- Clarificar cómo proceder cuando aparece nueva información de distinto tipo. Por ejemplo, lo que se hace en [GHRdST11], cuando se añade una premisa imprecisa a un conjunto de premisas precisas.

- Estudiar bajo qué mínimas condiciones, se puede enunciar, como ocurre en los retículos ortomodulares, que todas las hipótesis son reducibles [TPÁ09].

- Introducir algún tipo de medida en el modelo que permita comparar las conjeturas y conocer así cual es la conjetura mejor, o al menos cuales son las menos 'malas'.

- Avanzar más allá en el estudio del razonamiento conjetural con conocimiento impreciso, es decir, cuando el mismo sea representable en álgebras de conjuntos *fuzzy*. Más en general, plantearlo en un marco de BFA.

Capítulo 4

Conclusions and Future Work

Uncertainty is an uncomfortable position, but certainty is an absurd position.
Voltaire (1694-1778)

4.1. Related to the topic "Linguistic Roots of Fuzzy Sets"

The research, that is a small contribution to the so-called problem of meaning, the 'Gordian' knot of Artificial Intelligence, had a triple goal.

First, to show that there are actually mathematical ways of modeling the action of a predicate on universes of discourse that are sets. That is, how the predicate is primarily used in the universe, or, and *à la* Wittgenstein, which is its 'meaning'. *Elementary meaning* is here modeled through a relation translating a 'qualitative', and sometimes perceptive-based form of how the predicate works in the universe of discourse.

Second, to show that the concept of 'degree', the 'extent' up to which an object satisfies the property named by the predicate, in some form, a way towards 'quantitatively' measuring such extent, is formalizable once known the relation modeling the primary use of the predicate. In this way, it is clarified *which is the intrinsic property a function in* L^X *should verify to represent a* $L-$set on X. If *collectives are generated* by predicates but cannot be well defined, *L-sets are well defined* by particular representations of predicates.

Third, to reflect on the practical problem of choosing at each case the scale in which the degrees can vary. In the particular case the predicate is numerically measurable, some special types of fuzzy sets, like Zadeh's, type-2, or interval-valued fuzzy sets, do appear.

The Thesis tries to contribute to the revision of the current armamentarium of Fuzzy Logic that seems necessary to go through the new *Computing with Words* proposed by Zadeh.

Among the *questions that deserve future study*, let us cite the following:

- To analyze the case in which the empirical relation $\leq_P$ is not a preorder.

- Which are the general properties an operation $(\cdot)$ should have to allow $\mu_{P\&Q} = \mu_P \cdot \mu_Q$, either in the same or different universes of discourse?

- Which are the general properties an operation $(+)$ should have to allow $\mu_{P \text{ or } Q} = \mu_P + \mu_Q$, either in the same or different universes of discourse?

- Which are the general properties a relation R should have to allow $\mu_{If\ P\ then\ Q} = R \circ (\mu_P \times \mu_Q)$, either in the same or different universes of discourse?

- To go further with the new concept of the *linguistic predicate migration* and, even more, with the migration of complex statements involving either connectives, or conditionals.

- Analyze the concept of 'group meaning' at the light of either family resemblance, or the particular case of migration, to be able to capture a notion of 'social' meaning.

4.2. Related to the topic of the Aristotelian Principles

For Aristotle the law of non-contradiction was, actually, a 'principle' of thought. Although he stated that the statement 'A and not A is impossible', is universally valid and non susceptible to proof, within Fuzzy Logic either it is a theorem, or it is proven false.

For instance, if the law of non contradiction is read in the form 'A and not A is false', its validity will depend on the interpretation of the term 'false', and on how it is represented in a given formal framework. If such law is posed by 'A and not A is self-contradictory', its validity will depend on the interpretation of 'self-contradictory', and on how it is represented in a formal framework. Of course, in both cases the validity of principles also will depend on the characteristics of the chosen formal framework.

Which one of these two interpretations of the Aristotelian term 'impossible' is preferable? In which formal framework each one is preferable? These questions do not have an immediate answer. For example, within the framework of ortholattices there is equivalence between 'false' and 'self-contradictory', provided the first term is represented by the first element 0 of the lattice, and the second by the definition $x \leq x'$. Notwithstanding, within the framework of DeMorgan algebras, and also in that of the standard algebras of fuzzy sets, there are many non-null self-contradictory elements. The Thesis studies the validity of $a \cdot a' \leq (a \cdot a')'$ that, in ortholattices, is equivalent to $a \cdot a' = 0$.

Concerning the principle of Excluded Middle, after Aristotle usually taken in the form 'A or not A is true', the paper adopts the algebraic interpretation $(a+a')' = ((a \cdot a')')' = 0$, from which, in terms of self-contradiction, follows $((a \cdot a')')' \leq (((a \cdot a')')')'$, coincidental with that of Non-Contradiction if the operation *not*, denoted by $'$, is involutive ($a'' = a$).

The most important conclusion of the paper, is that the two principles of Non-Contradiction and Excluded-Middle are guaranteed under very few conditions on the connectives *and*, *or*, once functionally expressed, and provided the complement is given by a strong negation function. For most philosophers of Science, this result places Fuzzy Sets on non-trembling grounds.

As *future aspects to be studied*, the following three are paramount

- In which BFA the principles hold?
- Under which deductive systems *follows* $a + a'$ from a and a'?
- Under which deductive systems is $a \cdot a'$ *decidable* form a and a'?

4.3. Related to the topic of Conjectures

In the course of millennia the brain's capability of conjecturing resulted extremely important for the evolution of the species Homo. Without articulate language and partially articulate guessing, possibly Homo would have neither prevailed over the rest of animals, nor constituted the social, religious and economic organizations typical of humankind. And one of the most distinguishing features of Homo Sapiens is the act, and especially the art, of reasoning, or goal-oriented managing conjectures. Even more, scientific and technological research is a human activity that manages guessing in a highly articulated way. In words of Xenophanes of Colophon 'All is but a woven of guesses' [Pop63].

Although consequences and hypotheses, as well as several types of non- monotonic reasoning, deserved a good deal of attention by logicians, philosophers, computer scientists, and probabilists, no attempt at formalizing the concept of conjecture appeared before [ET00] was published. In the framework of an ortholattice, conjectures were defined there as those elements non-inconsistent with a given set of (non-inconsistent) premises reflecting the available information. That is, conjectures are those elements in the ortholattice that are "possible", once a résumé of the information given by the premises is known. This is the basic definition of which consequences (or safe, necessary conjectures), hypotheses (or explicative contingent conjectures), and speculations (or lucubrative, speculative contingent conjectures) are particular cases, in agreement with W. Whewell words 'Deduction is a necessary part of Induction'.

It should also be pointed out that neither the set of hypotheses, nor that of speculations, can be taken as bodies of information. Processes to obtain consequences perform deductive reasoning, or deduction. Those to obtain hypotheses perform abductive reasoning, or abduction, and those to obtain speculations perform speculative reasoning, a term that is also more generally applied to obtaining either hypotheses or speculations, and then results close to the term "reasoning". Of course, in Formal Sciences and in the context of proof, the king of reasoning processes is deduction.

4.3.1. Consequences and Conjectures in Preordered Sets

Algebras as the De Morgan ones do not fit in the working hypotheses made in ([ET00]) and ([ET09]), since they do not verify the Non-Contradiction and Excluded-Middle laws. This lack is overcome in this Thesis: now the only necessary underlaying structure is a preordered set endowed with a negation, that can be enriched with an inf-operation or upgraded to an inf-$*$-complete poset. So, this paper studies some properties of CHC models built on preordered sets that are weaker structures than the others where CHC models had been studied before. Furthermore, in order to keep some properties that hold in stronger structures, this Thesis considers *consistent* operators of consequences.

In addition, three different consequence operators have been analyzed in detail, defining them on different families of subsets useful to control the consistency of the premises:

- $C_{\leq}$, which only provides as consequences those elements 'following' *from some premise*;
- $C_{\bullet}$, which provides as consequences those elements 'following' *from the conjunction of any finite number of premises*;
- $C_{\wedge}$, which considers the elements 'following' *from the conjunction of all the premises.*

The operators $C_{\leq}$ and $C_{\bullet}$ actually define *partial consequences* of the set of premises.

Among the most relevant conclusions of this Thesis, the following can be cited:

- Conjectures are formalized in very week algebraic structures.
- Such formalization opens the door to work with Fuzzy Logic.
- Conjectures at large are proven to be anti-monotonic, hypotheses to be also anti-monotonic, but speculations are without any law of monotonicity. This opens the door to a new way of considering Non-monotonic Reasoning.

- The Popper's well know process of 'falsification of hypotheses' is formalized in the new framework.

4.3.2. On an Attempt to Formalize Guessing

Defining the operators of conjectures only by means of consistent consequence ones has the drawback of placing deduction before guessing, when it can be supposed that guessing is more common and general than deduction. After the publication of some papers ([ET00], [ET02], [ET09], [TPÁ09], [TCC01], [TGHP10], [AFP01]) on the subject, it yet remained the doubt on the existence of operators of conjectures obtained without operators of consequences, and *this Thesis liberates from such doubt by showing that to keep some properties* that seem to be typical of the concept of conjecture, it suffices to only consider operators that are extensive and monotonic, but without enjoying the closure property. It is reached three operators of conjectures by considering (like it was done in [ET02]), three different ways of defining non-inconsistency by means of non-self-contradiction. Of these three ways, only one of them conducts to reach conjectures directly through logical consequences. Of course, in the framework of Boolean algebra the three operator collapse into a single one.

What results important is that, notwithstanding, also those operators for conjectures that do not come from consequence ones, do contain a subset of consequences given by the known operator $C_\wedge$.

4.3.3. Future work

Among the *questions deserving future study*, let us cite the following:

- A deep study of how to represent the information conveyed by the premises, that is , to reach a clear concept of what can be called a *résumé* of information.

- To definitively establish the kind of 'abstract' operators that, like it happens with those of consequences and those shown in 4.3.2, allow to define the concept of conjecture. That is, to obtain a structural definition of conjecture, like that obtained by Tarski with respect to a deductive system.
- To link CHC Models with Analogical Reasoning or, at least, with Case-Based Reasoning given by a similarity index.
- To clarify how to proceed when new information of a different type appears. For instance, like it is done in [GHRdST11], when an imprecise premise should be add to the set of precise premises.
- Which are the minimum conditions on which it can be stated, as it happens in orthomodular lattices, that all the hypotheses are reducible [TPÁ09]?
- Introduce some kind of measure that allows to compare conjectures, and allows to take a decision of which is the best or which are the best conjectures.
- Go further in the analysis of conjectural reasoning with imprecise knowledge, that is, with knowledge representable in algebras of fuzzy sets.

Bibliografía

[AFP01] C. Vaucheret, A. Fernandez Pineda, E. Trillas. Additional comments on conjectures, hypotheses, and consequences in orthocomplemented lattices. *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, pp. 107–114, 2001.

[CT89] J. L. Castro and E. Trillas. Sobre preordenes y operadores de consecuencias de Tarski. *Theoria*, 4(11):419–425, 1989.

[DK03] G. Deschrijver and E. Kerre. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, 133:227–235, 2003.

[ET00] E. Castiñeira, E. Trillas, S. Cubillo. On conjectures in orthocomplemented lattices. *Artificial Intelligence*, 117:255–275, 2000.

[ET02] Ana Pradera and Enric Trillas. A reflection on rationality, guessing and measuring. En *Proceedings IPMU*, pages 777–784, Annecy, 2002.

[ET09] M. Monserrat, E. Trillas and M. Mas. Conjecturing from consequences. *International Journal of General Systems*, 38:567–578, 2009.

[GHRdST11] I. García-Honrado, A. R. de Soto, and E. Trillas. Some (unended) queries on conjecturing. En *World conference on Soft Computing 2011*, San Francisco, United States, 2011.

[GHTG10] I. García-Honrado, E. Trillas and S. Guadarrama. Grado de parentesco entre predicados. En *Actas XV Congreso Español sobre Tecnologías y Lógica Fuzzy*, pp. 163–168, 2010.

[Lak73] G. Lakoff. Hedges: A study in meaning criteria and the logic of fuzzy concepts. *Journal of Philosophical Logic*, 2(4):458–508, 1973.

[Men07] J. M. Mendel. Type-2 fuzzy sets and systems: an overview. *IEEE Computational Intelligence Magazine*, 2:20–29, 2007. Se publicará en Fuzzy Sets and Systems.

[MJ02] J.M. Mendel and R.I. John. Type-2 fuzzy sets made simple. *IEE-Transactions on Fuzzy Systems*, 10(2):117–127, 2002.

[Pop63] Karl Popper. *Conjectures and Refutations*. Rutledge & Kegan Paul, London, 1963.

[Qiu07] D. Qiu. A note on trillas' chc models. *Artificial Intelligence*, 171:239–254, 2007.

[Sow04] John F. Sowa. The challenge of knowledge soup. *Publicado por Homi Bhabha Centre for Science Education Tata Institute of Fundamental Research*, Mumbai, India, pp. 55–91, 2004.

[TAJ99] E. Trillas, C. Alsina and J. Jacas. On contradiction in fuzzy logic. *Soft Computing*, 3:197–199, 1999.

[TAP02] E. Trillas, C. Alsina, and A. Pradera. Searching for the roots of non-contradiction and excluded-middle. *International Journal of General Systems*, 31(5):499–513, 2002.

[TCC01] E. Trillas, E. Castiñeira, and S. Cubillo. Averaging premises. *Mathware and Soft Computing*, 8:83–91, 2001.

[TG10] E. Trillas and S. Guadarrama. Fuzzy representations need a careful design. *International Journal of General Systems*, 39(3):329–346, 2010.

[TGH10] E. Trillas and I. García-Honrado. On an attempt to formalize guessing. Technical Report FSC-2010-11, European Centre for Soft Computing, 2010. Aceptado en Soft Computing in Humanities and Social Sciences (Eds. R. Seising and V. Sanz) Springer-Verlang (available upon request).

[TGHP10] E. Trillas, I. García-Honrado and A. Pradera. Consequences and conjectures in preordered sets. *Information Sciences*, 180(19):3573–3588, 2010.

[TMS09] E. Trillas, C. Moraga and A. Sobrino. On 'family resemblances' with fuzzy sets. En *Proceedings IFSA-EUSFLAT*, pp. 306–311, 2009.

[TNGH10] E. Trillas, T. Nakahama and I. García-Honrado. Fuzzy probabilities: Tentative discussions on the mathematical concepts. En *IPMU'10*, pp. 139–148, Dortmund, Germany, 2010.

[TP06] E Trillas and A. Pradera. An orthogonality-based classification of conjectures in ortholattices. *Mathware & soft computing*, 13(1):71–87, 2006.

[TPÁ09] E. Trillas, A. Pradera and A. Álvarez. On the reducibility of Hypotheses and Consequences. *Information Sciences*, 179(23):3957–3963, 2009.

[Tri09] E. Trillas. Non contradiction, excluded middle, and fuzzy sets. Editado por DiGesù, Pal, and Petrosino, *International Work-*

shop of Fuzzy Logic and Applications, pp. 1–11, Palermo, Italy, 2009. Springer (LNAI 5571).

[Wit81] L. Wittgenstein. *Philosophical Investigations*. Basil Blackwell, Oxford, 3rd edition, 1981.

[Zad65] L. A. Zadeh. Fuzzy Sets. *Information and Control*, 8:338–353, 1965.

[Zad75] L. A. Zadeh. The concept of linguistic variable and its application to aproximate reasoning, parts i,ii,iii. *Information Sciences*, 8,8,9:199–249,301–357,43–80, 1975.

[Zad96] L. A. Zadeh. Fuzzy logic = computing with words. *IEEE Trans. Fuzzy Systems*, 4:103–111, 1996.

Parte II

Publicaciones presentadas

Capítulo 5

An Essay on the Linguistic Roots of Fuzzy Sets

A menudo he dicho que todo poema resuelve algo para mí en la vida. Voy tan lejos como para decir que cada poema es una fugaz aportación en contra de la confusión del mundo ... Capaces de transformar a orden el desorden. Y los poemas que hago son pequeños fragmentos de orden.
John F. Sowa (1940-)

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An essay on the linguistic roots of fuzzy sets ☆

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A R T I C L E I N F O

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A B S T R A C T

This paper mainly tries to show that the membership function of a fuzzy set labeled *P* does show some intrinsic property related with how *P* is actually managed in the universe of discourse. Its final goal is to analyze an answer to the question, which intrinsic but simple property allows a function to represent a fuzzy set labeled *P*? The presented property exhibits that the membership function just 'measures' in some scale the extent up to which *x* is *P* in language, for all *x* in the universe of discourse.

Such study is done in a form allowing to consider how to represent the 'collective' originated by a predicate reflecting a collective noun. As particular cases of what is presented, and when the degrees can be some kinds of numerical subsets, the Zadeh's fuzzy sets, the interval-valued, the intuitionistic, and the type-2 fuzzy sets, appear as particular cases and to some extent are discussed. A 'unification' of all different kinds of fuzzy sets based on a linguistic origin is achieved.

1. Introduction

The evolution of fuzzy logic towards Natural Language computing, or *Computing with Words* in Zadeh's terminology [37,39], requires an intensive theoretic focusing in the admissible representations of the meaning conveyed by linguistic expressions. This is, in itself, a very complex problem that neither in computer science, nor in linguistics, nor in philosophy, is yet even sufficiently well posed and, less again, solved. It is a situational problem that strongly depends on the context in which the linguistic expressions are uttered or written, as well as on the current purpose for which its composing words, connectives, modifiers, etc., are jointly used. From all that, it easily follows the necessity of a careful design [27,24] of the representation in fuzzy terms of those systems or reasonings described in linguistic terms and not only in mathematical ones.

In this line, the clarification of the links between the meaning of unary and binary imprecise predicates and *L*-fuzzy sets, fuzzy relations, connectives, and linguistic modifiers, seems to be of a paramount importance for the representation of systems that, once described its behavior in Natural Language expressions, are not simpler than those currently considered in fuzzy control. Of no less importance is the representation of complex ordinary reasonings purposed and presented in Natural Language.

This paper, that continues those in references [21–23], is a kind of essay that tries to shed some light on those links by means of mathematical representations in algebraic frameworks as simple as possible and, hence, sufficiently general to allow the study of a wide spectrum of dynamical systems and reasonings expressed in Natural Language. With respect to this paper, it can be said that 'In the beginning was the word' (in the first verse in the Gospel of John, 1:1–5).

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2. Basic concepts

2.1

Correctly managing a language at least implies to know what its expressions mean, that is, how to use them properly at each situation. In particular, 'I learned how and where to correctly use predicate *P*' implies 'I know the meaning of *P*'. Following Ludwig Wittgenstein [31], 'the meaning of a word is its use in the language' and, of course, the first problem for capturing the meaning of a predicate lies in how to describe and represent its use. Without 'representation' there is neither room for a scientific type of study, nor the possibility of establishing a useful 'Computing with Words' like that advocated by Zadeh ([37]).

A predicate *P* means nothing by itself, it only can mean something when acting on some universe of objects, *X*, the universe of discourse, of which it will be here supposed is a set (in the mathematically naïve sense of Halmos [11]). Such action is made though the elemental statements '*x* is *P*', and to know this action implies to know, at least, some of the basic rules by which the statements '*x* is *P*', '*y* is *P*',... are related.

It will be considered that a predicate *P* is a name given to a property *p* eventually verified, up to some extent, by the objects in *X*. In general, the notions of property and object are inextricable woven together since an object instantiates properties, and properties are what objects have or instantiate [15].

This paper only deals with those predicates reflecting 'collective nouns', that is, generating *collectives* in the universe they act in, or work, or are used. Actually, collectives are in the language, in the same way in which, for instance, in a big horses farm, *P* = '*short*' allows to talk of *the short horses* in the farm.

Once some elemental treats of the use of *P* on *X* are described, a way for representing the meaning of *P* in *X* is introduced and the concept of the degree up to which *x* is *P*, is tentatively defined. This is done in a form making clear that several values places, or scales, for the degrees are possible. Depending on the scales for the degree, fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc., do appear respectively (see [21–23]). A kind of 'unification' of Zadeh's, interval-valued, Attanasov's and type-2 fuzzy sets, when based on linguistic genesis is achieved. A someway antecedent of what is here presented can be found in the nice and interesting paper [5].

2.2

It should be noticed that predicates appearing in the language were usually introduced by naming a property exhibited by some elements in a 'universe of discourse'. After this, it is frequently the case that the considered predicate *migrates* to another universe of discourse, and that its use results in some form distorted, but showing 'family resemblance' with its former use.

Hence, the use we analyze of a predicate is with reference to a given universe of discourse. The resemblance of uses is also taken into account, yet an initial study of them can be found in [8,29].

This paper does not deal with the processes going from a collective towards a predicate naming it, but from a predicate on a universe towards the 'representation' in mathematical terms of the collective it can originate. Most of the predicates originating collectives are such that these collectives are not sets by lacking sharp boundaries. This reflects what is commonly stated as 'the greys' their uses show, that is, the graduation under which the elements in the universe of discourse do verify the property named by the predicate. In this sense, the intuitive idea the word 'collective' tries to express is not representable with the same kind of uniqueness than that expressed by the word 'set', although evidently 'sets' are particular cases of 'collectives'. The *L*-sets introduced in next Section 6 are, at its turn, a mathematical way of representing the collectives originated by predicates that are 'gradable' in some way. The word 'collective' reflects an abstract concept, and the corresponding *L*-sets are representational concretions or precisiations of it.

2.3

Words are for describing some reality, be it perceived in the real, or in an intellectual world, be it invented in a fictitious one, in a static or dynamic way, partially or totaly, and usually such reality is presented in the form of some 'information'. Those realities are never isolated ones, but placed in a context that can produce either restrictions, or softenings, in the word's use. Hence, the meaning or use of the words depends on the corresponding purpose for which they are used in a given context.

The use of words depends on the reality to be described with them, on the context in which it is inscribed, and on the purpose for managing it. This is a relevant part of how words work. For instance, in the context of an experiment in which a parameter or variable takes its values between 0 and 10 units of something, it can be good enough to either use the word 'small' in a loose sense, or to use it in a very specific one like 'less or equal than 4', that can be viewed as a restriction of 'small'.

3. Primary or elemental meaning of *P* in *X*

3.1

If someone states 'I do manage *P* on *X*', she/he should recognize when (for $x, y \in X$) it is '*x* is *equally P than y*', or it is not. It seems also obvious that he/she should at least recognize when '*y* is *less P than x*' or when '*x* is *more P than y*', similarly to how in the Montessori's learning method [18], children learn the concept of length by ordering several sticks of different sizes.

By taking the two (sometimes, perception-based) relations (a) x is *equally P than y*, (b) x is *less P than y*, the corresponding algebraic relations in X are obtained.

1. $=_P$, for (a), with $=_P \subset X \times X$
2. $\leqslant_P$, for (b), with $\leqslant_P \subset X \times X$,

and $y \leqslant_P^{-1} x \ (\iff x \leqslant_P y)$ can be identified with 'y is *more P than x*'. Of course, if $x = y$, it can be accepted that $x =_P y$, but not reciprocally in general.

3.2

Considering the *working hypotheses*

3. $=_P = \leqslant_P \cap \leqslant_P^{-1}$

4. $\leqslant_P$ is a preorder (a reflexive and transitive relation),it is $=_P$ an equivalence relation giving the quotient-set $X/=_P$, with classes $[x] = \{y \in X; x =_P y\} = \{y \in X; x \leqslant_P y \ \& \ x \leqslant_P^{-1} y\}$. Under hypothesis (3), the only relation to be taken into account is $\leqslant_P$.

It should be remarked that what follows could be done without supposing (4), but then some technical troubles do appear. For this reason such case is not yet considered (see [21,23]). Provided the transitivity of $\leqslant_P$ is not acceptable, it yet remains the possibility of alternatively considering some other relation weaker than transitivity.

Definition 3.1. $\leqslant_P$ is the primary, or elemental, meaning of P in X, and P is meaningless in X if $\leqslant_P = \emptyset$.
Of course, if P is meaningless in X, it is also $\leqslant_P^{-1} = \emptyset$ and $=_P = \emptyset$: There is no way of 'organizing' the universe of discourse X.

Remarks 3.2.

1. If S = *small*, and $X = [0,10]$, S is usually acting on X under the rule '$x \leqslant_S y \Leftrightarrow y \leqslant x$ in the linear order of $\mathbb{R}$'. Hence, $\leqslant_S^{-1} = \leqslant$, and '$x =_P y \Leftrightarrow x = y$'.
2. When X is a non-numerical, or a non-structured set, it is not so easy to obtain $\leqslant_P$. For instance,
   - If X is a collection of paintings, and Q = beautiful, a way for establishing $\leqslant_Q$ is by means of a group of experts that fix a set of attributes $a_1, \ldots, a_n$ of the paintings, and by defining

     $x \leqslant_Q y \iff (x,y) \in a_1, \ldots, (x,y) \in a_n,$

     where $(x,y) \in a_i$ means that painting y shows (in the view of the experts) attribute a_i more than painting x shows it. Of course,

     $x \leqslant_Q^{-1} y \iff (y,x) \in a_1, \ldots, (y,x) \in a_n,$

     and $x =_Q y \Leftrightarrow (x,y) \in a_1, (y,x) \in a_1, \ldots, (x,y) \in a_n, (y,x) \in a_n$.
     Notice that $\leqslant_Q$ is a preorder if, for all $x \in X$, it is $(x,x) \in a_i$, $1 \leqslant i \leqslant n$, and if $(x,y) \in a_i$ and $(y,z) \in a_i$, then $(x,z) \in a_i$, $1 \leqslant i \leqslant n$.
   - If X is the set of inhabitants in a very big city, and P = *short*, it is usually stated

     $x \leqslant_P y \iff$ Height of $y \leqslant$ Height of x,

     and $x =_P y \Leftrightarrow$ Height of x = Height of y, once the heights are measured with a given accuracy. Obviously, $\leqslant_P$ is a preorder.
3. In $X = \mathbb{N}$, the predicate P = 'transparent' is meaningless ($\leqslant_P = \emptyset$), unless if it is possible to define 'n is transparent' by either a necessary and sufficient condition, or giving a relation like,

   n is less transparent than m.

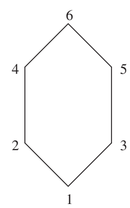
4. Only when X is endowed with some specific structure, it is possible to precisely define how P acts in X by means, for example, of some precise rules for its use. This is the case, for instance, of P = even in the set $X = \mathbb{N}$ of the natural numbers, where 'n is even' $\iff$ $n = \dot{2}$. Defining

   $n \leqslant_P m \iff n = 2p \ \& \ m = 2q \ \& \ p \leqslant q,$

   from which it follows $n =_P m \Leftrightarrow p = q \Leftrightarrow n = m$.
5. It can be said that some predicates are compatible with some structure in X, when it exists. For instance, if P = probable acts on a σ-algebra $\mathfrak{a}$ of events, before introducing any probability in $\mathfrak{a}$, mathematicians would say 'A is less probable than B' if and only if $A \subset B$. Hence, 'probable' is compatible with the poset $(\mathfrak{a}, \subset)$ and, of course, the empty set $\emptyset$ is 'less probable' than any $A \in \mathfrak{a}$ that, at its turn, is less probable' than the maximum element in $\mathfrak{a}$. Anyway, even if it could be easily accepted that $\subset$ is included in $\leqslant_{probable}$, the identification of both relations is not clear enough. In cases like this, the

identification of $\leqslant_P$ with the order of the structure in X can be an artificial reduction of the 'meaning' of P. Anyway, predicates act on any kind of universe X, be it previously structured or not. In addition and intuitively, the action of P introduces some organization, structure, or order, in the universe of discourse X: that given by the binary relation $\leqslant_P$, be it a preorder or not. Rationality, expressed by a discourse, introduces some structure in the world in which the discourse is based.

6. By freely defining $\leqslant_P$ it is opened the possibility of 'creating' new predicates P. For example, in $X = \{1,2,3,4,5,6\}$ with the preorder given by the graph, or equivalently the matrix, with which '$n \leqslant_P m \Leftrightarrow n \leqslant m$ and something else', it results that P is a restriction of S = *small* in X, that is, it exists a binary relation $B \subset X \times X$ such that $\leqslant_P = \leqslant_S \cap\ B$. Of course, $=_P = \leqslant_P \cap \leqslant_P^{-1}$ is nothing else than the equality of the numbers in X. Predicate P can be named either by a new word like P = *kakanboo*, or by an old one in reason of some similarity of the corresponding meanings, for instance, P = *cutted-small*.



$$[\leqslant_P] = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. The concept of degree

4.1

Once the structure $(X, \leqslant_P)$ is established, an important question lies in how to measure, and where, up to which extent x is P (for all x in X). Provided a partially ordered set $\mathfrak{L} = (L, \leqslant)$ can be, in some form, associated to $(X, \leqslant_P)$, a function $\mu_P : X \to L$ is an $\mathfrak{L}$–degree for P in X, if

$$x \leqslant_P y \Rightarrow \mu_P(x) \leqslant \mu_P(y).$$

Obviously, if $x =_P y$ it follows $\mu_P(x) \leqslant \mu_P(y)$ and $\mu_P(y) \leqslant \mu_P(x)$, that is, $\mu_P(x) = \mu_P(y)$ (see [21,23,22,25]). The set L^X can be viewed as a repository of potential degrees for predicates, and the poset $\mathfrak{L}$ as a scale for the degrees.

Theorem 4.1. *Provided $\leqslant_P$ is a preorder, there exist a poset $\mathfrak{L} = (L, \leqslant)$ and an $\mathfrak{L}$–degree μ_P, naturally linked with $(X, \leqslant_P)$.*

Proof. Take the quotient set $X/=_P$, and translate $\leqslant_P$ to its classes by

$$[x] \leqslant_P^* [y] \Longleftrightarrow x \leqslant_P y.$$

This definition does not depend on the chosen representatives of the classes and is a partial order. Hence, $\mathfrak{L}_P = (X/{=_P}, \leqslant_P^*)$ is a poset naturally linked to $(X, \leqslant_P)$.

Take any poset $(L, \leqslant)$ isomorphic to $\mathfrak{L}_P$, and define $\mu_P : X \to L$ by $\mu_P(x) = r$, with r the element in L that corresponds to the class $[x]$ by the isomorphism with $(X/{=_P}, \leqslant_P^*)$. Obviously,

$$x \leqslant_P y \Longleftrightarrow [x] \leqslant_P^* [y] \Rightarrow \mu_P(x) \leqslant \mu_P(y).$$

Hence, μ_P is an L–degree for P in X. □

Thus, the concept of degree is not an empty one.

For example, if $X = [0,6]$, and P = *close to four*, it can be stated

$$x \leqslant_P y \Longleftrightarrow x \leqslant y, \text{ if } x, y \in [0,4] \text{ and } y \leqslant x \text{ if } x, y \in [4,6],$$

and any L–degree μ_P for P = *close to four*, will be a non-decreasing function between 0 and 4, and decreasing between 4 and 6.

4.2

Once an L–degree μ_P is defined in $(X, \leqslant_P)$, it can be considered the new relation $\leqslant_{\mu_P} \subset X \times X$, defined by

$$x \leqslant_{\mu_P} y \Longleftrightarrow \mu_P(x) \leqslant \mu_P(y).$$

Obviously: $x \leqslant_P y \Rightarrow x \leqslant_{\mu_P} y$, or $\leqslant_P \subset \leqslant_{\mu_P}$, that is, *the relation $\leqslant_{\mu_P}$ is larger than the relation $\leqslant_P$*. When $\leqslant_P = \leqslant_{\mu_P}$ it can be said that μ_P *perfectly reflects* the primary meaning, or use, of P in X. When $\leqslant_P \subsetneq \leqslant_{\mu_P}$, μ_P only reflects partially the primary use of P.

The pair $(\leqslant_P, \leqslant_{\mu_P})$ could be called the *meaning*, or *use*, of P in X. It should be noticed that the meaning, as defined in this way, is not of an absolute character but relative to the scale $\mathfrak{L}$ in which the degree μ_P takes its values.

Remarks 4.2.

1. It is $x \leqslant_{\mu_P} x$ since $\mu_P(x) = \mu_P(x)$, and if $x \leqslant_{\mu_P} y$ and $y \leqslant_{\mu_P} z$, from $\mu_P(x) \leqslant \mu_P(y)$, and $\mu_P(y) \leqslant \mu_P(z)$, follows $\mu_P(x) \leqslant \mu_P(z)$ or $x \leqslant_{\mu_P} z$. That is, $\leqslant_{\mu_P}$ is also a preorder. Obviously, $\leqslant_{\mu_P}$ is not always antisymmetric.
2. Zadeh's degrees do appear when $L = [0,1]$ and $\leqslant$ is the usual order of $\mathbb{R}$ ([32,38]). In this case what often is first established is a $[0,1]$-degree μ_P, and hence, what is known is $\leqslant_{\mu_P}$, but not $\leqslant_P$.
3. With $=_P = \leqslant_P \cap \leqslant_P^{-1}$, it results $\mu_P(x) = \mu_P(y) \iff x =_{\mu_P} y$. Obviously, $x = y \Rightarrow x =_P y \Rightarrow \mu_P(x) = \mu_P(y) \iff x =_{\mu_P} y$. What is obtained is the chain of inclusions: $= \subset =_P \subset =_{\mu_P}$.
4. Any constant function $\mu_r : X \to L$, $\mu_r(x) = r \in L$ for all $x \in X$, 'can be taken' as a degree for any structure $(X, \leqslant_P)$, since $x \leqslant_P y \Rightarrow r \leqslant r$. Anyway, these functions are only admisible as $\mathfrak{L}$–degrees for the predicates P_r = constantly r, in which case $\leqslant_{P_r} = \leqslant_{P_r}^{-1} = =_{P_r}$, since $x \leqslant_{P_r} y \iff \mu_P(x) = \mu_P(y) = r$, for all x, y in X. Constant predicates are not meaningless, since $\leqslant_{P_r} = =_{P_r} = X \times X \neq \emptyset$.
   If the poset $\mathfrak{L}$ has a minimum element $\alpha \in L$, and a maximum element $\omega \in L$, then there are the two degrees $\mu_\alpha(x) = \alpha$, $\mu_\omega(x) = \omega$, for all $x \in X$, corresponding to the constant predicates P_α = constantly α, and P_ω = constantly ω, respectively.
5. For any poset $(L, \leqslant)$, it exists a totally ordered set (toset) $(L^*, \leqslant^*)$, and an injection $\varphi : L \to L^*$ (see [34]) such that

   $$a \leqslant b \Rightarrow \varphi(a) \leqslant^* \varphi(b).$$

   That is, any poset $(L, \leqslant)$ can be embedded in a toset $(L^*, \leqslant^*)$. In this abstract sense, any L–degree could be defined as a degree taking its values in a totally ordered set.
6. If $(L, \leqslant)$ is a toset, given x, y in X, it is either $\mu_P(x) \leqslant \mu_P(y)$, or $\mu_P(y) \leqslant \mu_P(x)$, that is $x \leqslant_{\mu_P} y$, or $y \leqslant_{\mu_P} x$. In this case, the preorder $\leqslant_{\mu_P}$ is a total preorder.

5. A comment on group meaning

The meaning of words is not fixed for all people and all context. For example, in a dinner with three commensals the deliciousness of a dessert plate could easily result in three different orderings of such plate. Since language is a social phenomenon, also meaning is such, and it is possible to tentatively say something on the meaning of predicates for a group of people in, of course, a given context.

For a group of people $G = \{p_1, \ldots, p_m\}$, a predicate P on X can show m primary meanings $\leqslant_{P,i}$, $1 \leqslant i \leqslant m$. Since

$$\left(\bigcap_{i=1}^{m} \leqslant_{P,i}\right) = \leqslant_{P,G}$$

is not empty (all $\leqslant_{P,i}$ are reflexive), it can be taken

Primary meaning of P *on* X *for the group* $G = \leqslant_{P,G}$.

Notice that $=_{P,G} = \left(\bigcap_{i=1}^{m} =_{P,i}\right)$ is an equivalence, and provided all $\leqslant_{P,i}$ are preorders, $\leqslant_{P,G}$ is also a preorder.

Since $\left(\bigcap_{i=1}^{m} \leqslant_{P,i}\right)^{-1} = \bigcap_{i=1}^{m} \leqslant_{P,i}^{-1}$, provided $(=_{P,i}) = \leqslant_{P,i} \cap \leqslant_{P,i}^{-1}$ for all $1 \leqslant i \leqslant m$, then

$$(=_{P,G}) = \leqslant_{P,G} \cap \leqslant_{P,G}^{-1} = \bigcap_{i=1}^{m} \left(\leqslant_{P,i} \cap \leqslant_{P,i}^{-1}\right) = \bigcap_{i=1}^{m} =_{P,i}.$$

If m $\mathcal{L}$ – degrees $\mu_P^{(i)}$ are known for each primary meaning $\leqslant_{P,i}$, since

- $x =_{P,G} y \Leftrightarrow x =_{P,1} y \,\&\ldots\&\, x =_{P,m} y$,
- $x \leqslant_{P,G} y \Leftrightarrow x \leqslant_{P,1} y \,\&\ldots\&\, x \leqslant_{P,m} y$,

for each function $\Phi : L^m \to L$, non-decreasing in each place i between 1 and m (for example, if $a \leqslant b$ then $\Phi(a, x_2, \ldots, x_m) \leqslant \Phi(b, x_2, \ldots, x_m)$), or *Aggregation Function*, it results

- $x \leqslant_{P,G} y \Rightarrow \Phi\left(\mu_P^{(1)}(x), \ldots, \mu_P^{(m)}(x)\right) \leqslant \Phi\left(\mu_P^{(1)}(y), \ldots, \mu_P^{(m)}(y)\right)$,

that allows to take

$$\mu_P^G(X) = \Phi\left(\mu_P^{(1)}(x), \ldots, \mu_P^{(m)}(x)\right), \text{ for all } x \in X,$$

as an $\mathcal{L}$ – degree of P on X for the group G. The meaning for the group G results from aggregating its people's meanings. For something closely related with this idea, see [10,2].

6. *L*-sets

6.1

Given a triplet $(X, \leqslant_P, \leqslant_{\mu_p})$, and in a more general way than in [9], it is possible to *represent* the 'collective' that *P generates* in *X*.

Such representation, allows to translate into mathematical terms the 'collective of the *P*s in *X*', or *L*-set (by following [9]), noted by $\mathbb{P}$, and defined by the change of notation:

1. $x \in_r \mathbb{P}$ (read: x belongs to $\mathbb{P}$ with degree $r \in L$) $\Leftrightarrow \mu_P(x) = r$.
2. $\mathbb{P} = \mathbb{Q} \iff \mu_P(x) = \mu_Q(x)$, for all $x \in X$.

It is obvious that $\mathbb{P}$ is equivalent to μ_P. Hence, the *L*-set concept is relative to the poset $(L, \leqslant)$ and the chosen *L*–degree, $\mu_P \in L^X$. There is not, in general, a unique *L*-set in *X* defined by *P*.

If *P* is meaningless in *X*, from $\leqslant_P = \emptyset$ it follows that for no $\mathfrak{L}$ a degree does exist: *meaningless predicates P do not define any L-set* $\mathbb{P}$ *in the universe of discourse.*

It should be distinguished the before mentioned predicate P_α, with degree μ_α, that gives the *L*-set $\mathbb{P}_\alpha$ characterized by $x \in_\alpha \mathbb{P}_\alpha$ for all $x \in X$, and the predicate P_ω whose corresponding *L*-set $\mathbb{P}_\omega$ is characterized by $x \in_w X$, for all $x \in X$. For all the constant predicates, the corresponding *L*-sets $\mathbb{P}_r$ are unique, and $\mathbb{P}_\alpha, \mathbb{P}_\omega$, are their limiting cases.

The definition:

3. $\mathbb{P} \subset \mathbb{Q}$ (*L*-set $\mathbb{P}$ is included in *L*-set $\mathbb{Q}$) $\Leftrightarrow \mu_P(x) \leqslant \mu_Q(x)$, for all $x \in X$, obviously gives a reflexive, antisymmetric and transitive relation. That is, $\subset$ is a partial order, under which

$$\mathbb{P}_\alpha \subset \mathbb{P} \subset \mathbb{P}_\omega,$$

for all *L*-set $\mathbb{P}$. Obviously, it is $\mathbb{P}_r \subset \mathbb{P}_s$ if and only if $r \leqslant s$. Consequently, the *L*-set $\mathbb{P}_\alpha$ can be identified with the empty *L*-set, and the set $\mathbb{P}_\omega$ with the total set, that is, with the classical sets $\emptyset$ and *X*, respectively. The set L^X can be identified as that of all 'potential' *L*-sets in *X*.

6.2

In the case the poset $\mathfrak{L}$ has the elements α, ω, the set $L_0 = \{\alpha, \omega\} \subset L$ gives the poset $\mathfrak{L}_0 = (L_0, \leqslant)$, and the functions $\mu \in L_0^X$ are those for which either $\mu(x) = \alpha$, or $\mu(x) = \omega$, for any $x \in X$.

The mapping $\varphi : L_0^X \to \mathcal{P}(X),\ \varphi(\mu) = \mu^{-1}(\omega) \subset X$ is bijective, since:

1. $\varphi(\mu) = \varphi(\sigma) \Leftrightarrow \mu^{-1}(\omega) = \sigma^{-1}(\omega) \Leftrightarrow \mu = \sigma$
2. If $A \in \mathcal{P}(X)$, with $\mu_A(x) = \begin{cases} \omega & \text{if } x \in A \\ \alpha & \text{if } x \notin A \end{cases}$, is $\mu_A \in L_0^X$ and $\varphi(\mu_A) = A$.

Of course, it is $A = B \Leftrightarrow \mu_A = \mu_B$, and $A \subset B \Leftrightarrow \mu_A(x) \leqslant \mu_B(x)$, for all $x \in X$. It is also $\varphi(\mu_\alpha) = \emptyset$, $\varphi(\mu_\omega) = X$. Hence, it can be said that *in the set L^X of the L-sets in X, it is included the classical power set $\mathcal{P}(X)$ of X.*

If *P* is a predicate on *X* such that $\mu_P \in L_0^X$, it is said that *P* is *precise*, *crisp*, or *rigid*. The corresponding (classical) *L*-sets are obviously unique.

6.3

In the classical case of crisp sets ($\mathfrak{L}$-sets in $\{0,1\}^X$, or $\{\alpha,\omega\}^X$), that the set $\{0,1\}^X$ contains all potential classical subsets of *X*, is expressed by the so-called *specification axiom* ([11]): For each binary predicate *P*, it exists a single subset $\mathbb{P} \subset X$ whose elements are all the $x \in X$ verifying the property denoted by *P*. For non-binary predicates *P*, it should be previously determined which is the more suitable poset $\mathfrak{L}$ in which the degree can vary and, if possible, verifying $\leqslant_P = \leqslant_{\mu_P}$.

After what has been said in Sections 4 and 6, it seems to be clarified that a function $\mu \in L^X$ only can represent a predicate *P* on *X* (namely, a particular use of *P* on *X*) *provided μ verifies the intrinsic property of being an $\mathfrak{L}$–degree for P*. This is the sense of considering L^X as the set of all 'potential' *L*-sets in *X*, like $\{0,1\}^X$ is the set of all potential crisp sets in *X*. Any function $\mu \in L^X$ only 'defines' an $\mathfrak{L}$-set whenever it exists a predicate *P* such that $\mu = \mu_P$.

Remark 6.1. From a pure mathematics point of view, the form in which the *L*-sets are here introduced, is just a 'naïve' form that cannot be considered an 'axiomatic' one, whose existence is actually an open problem.

7. A remark on the concept of collective

A predicate P acting on a universe of discourse X under a given context (c) and a purpose for its use (up), generates a collective $\mathcal{C}_X(P;c,up)$ that does not have a unique representation by a L-set but, in principle, many that depend on the poset taking for the degrees. L-sets are *precisiations* of the linguistic concept $\mathcal{C}_X(P;c,up)$ in the sense of Zadeh [35].

For instance, if X= Inhabitants of the European Union, P = tall, c_1 = High School, up_1 = Selection of players for an EU contest of Basketball, c_2 = EU's Basketball teams up_2 = Selection of players for an EU's Basketball team to play in a world's champions league, it is

$$\mathcal{C}_X(P;c_1,up_1)\neq\mathcal{C}_X(P;c_2,up_2).$$

Concerning a direct scientific study of collectives, the following comments are in order:

1. L-sets are nothing else than 'crisp' concepts (functions) reflecting with the highest possible precisition made at each case, the use of imprecise predicates. With the difference that the Earth's atmosphere is not a classical set, collectives in X are like clouds in the atmosphere, but not always classical subsets in X (see [19]). Under this metaphor, collectives are 'vague entities' submitted to both internal and external dynamisms making very difficult their characterization in formal terms.
2. Without a definition for the identity of collectives, $\mathcal{C}_X(P;c_1,up_1)=\mathcal{C}_Y(Q;c_2,up_2)$, it is difficult (if not impossible) to directly manage the collective's concept in a formal way.
3. L-sets, for which identity there is a clear definition, facilitate an indirect way to begin with the study of collectives.

8. Particular types of L-sets

8.1

When L = [0,1], and $\leqslant$ is the total order of $\mathbb{R}$, with the poset $([0,1],\leqslant)$ the obtained L-sets are the well known Zadeh's fuzzy sets [32]. Since [0,1] is totally ordered, fuzzy sets show the special feature that the preorder $\leqslant_{\mu_P}$ is 'total', since for any x, y in X it is either $x\leqslant_{\mu_P}y$, or $y\leqslant_{\mu_P}x$. Hence, the degrees up to which x is P and y is P, are always comparable. This property, obviously not always fulfilled by any predicate P in any universe of discourse X, makes difficult that μ_P does perfectly reflect the primary use of P on X, that is, to have $\leqslant_P=\leqslant_{\mu_P}$.

8.2

There are, of course, other ways of partially ordering the elements in [0,1]. For instance, by means of the so-called 'sharpened' order, introduced in [14],

$$a\leqslant_S b \iff \begin{cases} a\leqslant b\leqslant 0.5,\\ 0.5\leqslant b\leqslant a,\end{cases}$$

a partial order with maximum 0.5 and the two minimal elements 0 and 1. The poset $([0,1],\leqslant_S)$ could serve to obtain L-sets allowing to represent the predicate F = *fuzzy* in X = [0,1], provided it can be defined

$$a\leqslant_F b \iff a\leqslant_S b,$$

with degree given by any function $\mu_F:[0,1]\to[0,1]$, such that: If $a\leqslant_S b$, then $\mu_F(a)\leqslant\mu_F(b)$, and $\mu_F(0)=\mu_F(1)=0$, $\mu_F(0.5)=1$, since 0.5 is the maximum, and 0 and 1 are the minimals in $([0,1],\leqslant_S)$. From this can follow the so-called 'fuzzy entropies' defined in $[0,1]^X$, a set to which the order $\leqslant_S$ can be easily extended (see [11]). Notice that substituting 0.5 by any $n\in(0,1)$, the relation

$$a\leqslant^* b \iff \begin{cases} a\leqslant b\leqslant n,\\ n\leqslant b\leqslant a\end{cases},$$

is also a partial order with maximum n, and the minimals 0 and 1.

By taking $L_\alpha=[0,1]\cup\{\alpha\}$, adding a new element α such that $\alpha\leqslant_S x$ for all $x\in[0,1]$, it is also $(L_\alpha,\leqslant_S)$ a partially ordered set, but with the minimum α. Hence, the corresponding L-sets in $\{\alpha,0.5\}^X$ can be taken as the crisp sets.

8.3

The set $L=\{(x,y)\in[0,1]\times[0,1]\ ;\ x+y\leqslant 1\}$, endowed with the Atanassov's relation ([1]):

$$(x_1,y_1)\leqslant_A(x_2,y_2)\iff x_1\leqslant x_2 \text{ and } y_2\leqslant y_1$$

(implying $(x_1,y_1) =_A (x_2,y_2) \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$), is a poset with minimum (0,1), and maximum (1,0). Hence, L-sets in $\{(0,1),(1,0)\}^X$ can be identified with the crisp sets, and the L-sets $\mu : X \to L$, $\mu(x) = (\mu_1(x),\mu_2(x))$ with $\mu_1(x) \leqslant 1 - \mu_2(x)$, do coincide with the so (wrongly) called 'intuitionistic' fuzzy sets [1]. In this case, is

$$x \leqslant_\mu y \Longleftrightarrow \mu(x) \leqslant \mu(y) \Longleftrightarrow \mu_1(x) \leqslant \mu_1(y) \text{ and } \mu_2(y) \leqslant \mu_2(x),$$

with $x =_\mu y \Leftrightarrow \mu_1(x) = \mu_1(y)$ and $\mu_2(x) = \mu_2(y)$.

Remark 8.1. To well manage a predicate P in X, it is necessary to also manage at least one of its antonyms. It should be recalled that there are predicates for which no antonym is known (non-regular terms), in this case it is usually taken the negation of the predicate as its antonym (see [28]).

Intuitionistic fuzzy sets (see 8.3) correspond to the interpretation $\mu_P(x) = (a_P(x), b_P(x))$, with $a_P(x) + b_P(x) \leqslant 1$, or $b_P(x) \leqslant 1 - a_P(x) = a_{notP}(x)$, with not P defined by the particular strong negation $1 - id$. Thus, b_P can be interpreted as the degree of an antonym, or opposite, of P, and μ_P does correspond with a degree for the pair or predicates (P, *antonymP*) necessary to linguistically manage P.

Actually, it could perhaps be more suitable to think in the simultaneous linguistic management of the triplet (P, *not P*, *antonym P*) to correctly use P. In this sense, it is possible to think on the set L of triplets $(x_1,x_2,x_3) \in [0,1]^3$ such that $x_2 \leqslant x_3$, endowed with the binary relation

$$(x_1,x_2,x_3) \leqslant (y_1,y_2,y_3) \Longleftrightarrow x_1 \leqslant y_1,\ y_2 \leqslant x_2,\ y_3 \leqslant x_3,$$

obviously reflexive, antisymmetric, and transitive. That is, $(L,\leqslant)$ is a poset with minimum (0,1,1) and maximum (1,0,0), that can serve to define L-sets by functions

$$\mu_P : X \to L,\ \mu_P(x) = (a_P(x), b_P(x), c_P(x)),\ \textit{with } a_P(x) \leqslant b_P(x),\ \textit{for all } x \in X.$$

Such L-sets could be called 'linguistic-fuzzy sets', and are yet more general than intuitionistic fuzzy sets.

8.4

The set of intervals $L = \{[a,b] \subset [0,1]\ ;\ a \leqslant b\}$, endowed with the relation $\leqslant$ defined by

$$[a_1,b_1] \leqslant [a_2,b_2] \Longleftrightarrow a_1 \leqslant a_2 \text{ and } b_1 \leqslant b_2,$$

gives the poset $(L,\leqslant)$, whose maximum element is $[1,1] = \{1\}$, and whose minimum is $[0,0] = \{0\}$.

The corresponding L-sets, defined by $\mu : X \to L$, $\mu(x) = [a(x),b(x)]$, with $a(x)$ and $b(x)$ in [0,1], and such that $a(x) \leqslant b(x)$, for all $x \in X$, can be identified with the so-called *interval-valued* fuzzy sets ([35,6]). At its turn, Zadeh's fuzzy sets can be identified with those *interval-valued* $\mu \in L^X$, such that $\mu(x) = [a(x),a(x)] = \{a(x)\}$, and crisp subsets do appear when $a(x) \in \{0,1\}$.

Remark 8.2. There are, of course, other ways of partially ordering the set L of intervals. For instance,

$$[a_1,b_1] \leqslant^* [a_2,b_2] \Longleftrightarrow a_2 \leqslant a_1 \text{ and } b_1 \leqslant b_2,$$

also giving a poset $(L,\leqslant^*)$, with the maximum element [0,1]. Nevertheless, it is obvious that $(L,\leqslant^*)$ has neither minimum, nor minimals. Anyway, adding the empty set $\emptyset$ to L, $L^* = L \cup \{\emptyset\}$, and with the assumption $\emptyset \leqslant^* [a,b]$, for all $[a,b] \in L$, the new poset $(L^*,\leqslant^*)$ does have maximum ([0,1]), and minimum ($\emptyset$).

8.5

Once known that μ_P can vary in [0,1], it is not always clear enough if the degree will actually be measurable by numbers or intervals in [0,1]. The uncertainty in the knowledge of the values $\mu_P(x)$, that is, the uncertainty associated with their determination, could result in the reasonable kind of statements that, for instance, '$\mu_P(x)$ is around $a(x)$', for some $a(x) \in [0,1]$, conducting to take as L the set of 'fuzzy numbers' in [0,1] ([16]).

Fuzzy numbers ([13,39]) are Zadeh's fuzzy sets defined by function $\mu_{(s)} : [0,1] \to [0,1]$ such that

$$\mu_{(s)}(x) = \begin{cases} 1, & \text{if } x = s \\ L(x), & \text{if } x \in [s-\varepsilon, s) \\ R(x), & \text{if } x \in (s, s+\delta) \\ 0, & \text{otherwise,} \end{cases}$$

with ε, $\delta > 0$, and where L (*left side* of $\mu_{(s)}$) is a non-decreasing function $L : [s-\varepsilon, s] \to [0,1]$, and R (*right side* of $\mu_{(s)}$) is a decreasing function R: $(s, s+\delta] \to [0,1]$.

If FN is the set of these fuzzy numbers, there is the problem of how to obtain a poset $(FN, \leqslant)$ extending the order of the real numbers. A possible definition for such $\leqslant$ is:

$$\mu_{(r)} \leqslant \mu_{(s)} \iff r \leqslant s, \text{ and } a_r(\alpha) \subset a_s(\alpha), \ r, s \text{ in } \mathbb{R},$$

with $a_t(\alpha) = \{x \in [0,1] \,;\, \alpha \leqslant \mu_t(x)\}$, the α-cuts of μ_t $(\alpha \in [0,1])$.

With the poset $(FN, \leqslant)$, the L-sets defined by degrees $\mu : X \to FN$ can be identified with a particular case of the so-called type-2 fuzzy sets ([16,6,7]).

The fuzzy numbers μ_r with $L(X) = R(X) = 0$ for all x, that is

$$\mu_r(x) = \begin{cases} 1, & \text{if } x = r \\ 0, & \text{if } x \neq r \end{cases},$$

can be identified with the numbers $r \in [0,1]$, and Zadeh's fuzzy sets result to be a particular case of type-2 fuzzy sets. In the same way, the set $\{0,1\}$ can be viewed as a part of FN, and also crisp subsets of X can be viewed as type-2 fuzzy sets.

Remark 8.3. It is obvious that any partial ordering between intervals in $[0,1]$ can be employed to define a partial ordering in FN. This is the case, for example, of the order $\leqslant^*$ defined in 8.4, allowing to define

$$\mu_r \leqslant^* \mu_s \iff r \leqslant s, \text{ and } a_r(\alpha) \leqslant^* a_s(\alpha).$$

8.6

The degree is a Kolmogorov–Kappos probability (see [12]) in the very special case in which $(L, \leqslant)$ is the unit interval with the linear order of the real line, $(X, \leqslant_P)$ with $P = Probable$, is not only a preordered set but one strongly structured as a boolean algebra with intersection $\cdot_P$, union $+_P$, complement $'_P$, minimum 0_P, maximum 1_P, and there is a function $\mu_P : X \to [0,1]$ such that.

- $\mu_P(1_P) = 1$.
- If $x \cdot_P y = 0_P$, then $\mu_P(x +_P y) = \mu_P(x) + \mu_P(y)$,

from which, as it is well known ([12]), follows

$$x \leqslant_P y \Rightarrow \mu_P(x) \leqslant \mu_P(y).$$

Hence, the 'probability' μ_P is a degree. In this case, it can be said that 'x is P' has the probability $\mu_P(x)$. Actually, the same can be said if $(X, \leqslant_P)$ is weakly structured in the form of an orthomodular lattice ([3]), by substituting $x \cdot_P y = 0_P$ by $x \leqslant_P y'^P$.

In both science and technology, the linguistic predicate 'probable' [20] is applied to the elements in a boolean algebra $\mathfrak{a} \subset \mathcal{P}(X)$ in relation with the poset $([0,1], \leqslant)$ and, as it was said in Remark 3.2(5), provided

$$A \leqslant_{probable} B \iff A \subset B, \text{ with } A, B \text{in } \mathfrak{a}.$$

In such case, the degree to measure up-to-which value 'A is probable', is taken as a probability $\mu_{probable} : \mathfrak{a} \to [0, 1]$, and it can be said that 'A is probable' with degree, or probability, $\mu_{probable}(x)$.

Although no subset of $[0,1]^X$ different from $\{0,1\}^X$ is a boolean algebra (not even an ortholattice [26]), if the predicate 'probable' is applied to some fuzzy sets in X, its extent can be sometimes measured by a degree with the properties of a probability. For instance, if $X \subset \mathbb{R}$, and $[0,1]^X_*$ is the set of all $\mu \in [0,1]^X$ that are Lebesgue integrable, the function $\mu_{probable} : [0,1]^X_* \to [0,1]$, defined by:

$$\mu_{probable}(\mu) = \int_X \mu d\lambda,$$

(see [36]) verifies all the Kolmogorov's properties of a probability, included: $\mu \leqslant \sigma \Leftrightarrow \mu(x) \leqslant \sigma(x)$, for all $x \in X \Rightarrow \mu_{probable}(\mu) \leqslant \mu_{probable}(\sigma)$, and it can be said that 'μ is probable' up to the probability-degree $\mu_{probable}(\mu)$. Hence, it can be said that there are also probabilities for fuzzy events. What is not yet clarified enough is the concept of 'fuzzy probabilities', that is, functions assigning to some fuzzy sets a fuzzy number in $[0,1]^{[0,1]}$ satisfying the properties of a probability (see [33]). The mathematical study of fuzzy probabilities of fuzzy events is actually an important open theoretical problem [30].

9. On the practical election of the poset and the degree

After what has been said, it remains the important open practical problem of determining in each case which is the most suitable scale $\mathfrak{L} = (L, \leqslant)$, as well as the associate degree $\mu_P \in L^X$, to represent the use of P on X, that is, to obtain a meaning $(\leqslant_P, \mu_P)$ of P on X. Of course, the best case is reached when μ_P perfectly reflects $\leqslant_P$, and at this respect next theorem is relevant:

Theorem 9.1. *If* $(X, \leqslant_P)$ *is a preordered set, its natural degree perfectly reflects* $\leqslant_P$.

Proof. The natural degree comes from the natural poset $(X/=_P, \leqslant^*)$ presented in Section 4.1, defining $\mu_P: X \to X$, by $\mu_P(x) = [x]$. Hence,

$$x \leqslant_{\mu_P} y \iff [x] \leqslant^* [y] \iff x \leqslant_P y,$$

that is, $\leqslant_{\mu_P} = \leqslant_P$. □

Thus, *a way of obtaining an L-degree perfectly reflecting* $\leqslant_P$ *is by means of a poset* $(L, \leqslant)$ *isomorphic to* $(X/=_P, \leqslant^*)$, like it is done in theorem 4.1.

Notwithstanding, since to completely know the perceptive/empirical relation $\leqslant_P$ is not always easy, it is not usual to completely know the quotient $(X/=_P, \leqslant^*)$, and this makes difficult to know a good poset. Hence, to have a good enough poset $\mathfrak{L}$ where defining a degree, it is relevant to know as much as possible about:

1. The *use* of P on X, and *how* P is managed on X.
2. The *purpose* for which P is used on X, *why* P is handled on X.
3. The context surrounding (1) and (2).

From the actual extent of this knowledge, and from the way of establishing the degrees, it could be induced if it is possible to numerically measure the extent up to which x is P, and, for instance, if L can be either the unit interval, the Atanassov's pairs of numbers in $[0,1]$, or a poset of fuzzy numbers. The last two cases could be selected when the uncertainty involved in determining each value $\mu_P(x)$ is such that, for instance, it only can be safely said that $\mu_P(x)$ varies between 0.3 and 0.5, or that $\mu_P(x)$ is 'around 0.4', that is $\mu_P(x) \in [0.3, 0.5]$, or $\mu_P(x)$ = a fuzzy number 'around 0.4'. Of course, in the case in which $\mu_P(x)$ is a fuzzy number, its shape should be adequately designed.

With respect to determining $\mathfrak{L}$ and μ_P, the complexity of the context plays a central role, as well as how are acquired the data for approaching the degrees that could come, for instance, from how the degree's estimation is done (by asking expert people, by subjective estimation, by comparison with some prototypes, etc.). In short, L-sets should be obtained by a careful process of design.

10. Additional remarks

10.1

There is a point deserving some comment. That concerning the *truth of statements*, of which it should be recalled is nothing else than a degree-up-to-which statements can be considered to be *true*, and that is relative to the poset in which they can take their values. In this sense, *to describe the degrees of true* for an statement can be done accordingly with the purpose of reflecting how true is it, indeed up to which degree it agrees with a reality perceived in some way. That is, by the meaning of the predicate 'true' applied to the involved statements.

Although *Truth* is a concept not properly belonging to Science, in the scientific language the term 'truth' is often used. Usually it refers to some compatibility with the available information on a given reality, and this is the sense in which 'true' is here interpreted.

In particular, statements 'x is P' are only true of false, that is, either totally according with reality, or not according at all with it, if $\leqslant_P$ is such that the quotient set $X/=_P$ only has at most two classes. That is, if the poset $(X/=_P, \leqslant_P^*)$ is isomorphic with a part of $(L_0, \leqslant)$, where it is $L_0 = \{\alpha, \omega\}$ and $\leqslant$ is given by $\alpha \leqslant \alpha$, $\omega \leqslant \omega$, $\alpha < \omega$. This corresponds to the classical-bivaluate case in which only *true* statements ($\omega = 1$), and *false* statements ($\alpha = 0$), are accepted. Of course, the bivaluate case can be only accepted if there is a total confidence in the reality and a perfect, or clear-cut, perception or description of it. In general this is not always the situation, and more than the two degrees α, ω are necessary for the degree of *true*, but it is also necessary to agree to which poset $(L, \leqslant)$ belongs the degrees of *true*. For instance, if truth is multiple-valued, interval-valued, of fuzzy valued, something that, in principle, could be forced by the context in which the statements are used.

10.2

Sometimes a imprecise predicate P on a universe X is approached by another P_1 with a bivaluate use. For instance, the predicate P = *small* with the multiple-valued use given by $\mu_P(x) = 1 - x$ on $X = [0,1]$, can be approached by P_1 = less than 3, with the bivaluate use given by μ_{P_1} equal to 1 in $[0,3]$, and 0 in $(3,10]$, that could be considered as a 'restricted' representation of P. Sometimes a predicate P in X 'moves' to a different universe Y where it is designated by Q, but keeping some similarity with P. For instance, Q= short in $Y = [0,10]$ can be viewed as a movement or linguistic migration of P = *small* in $X = [0,1]$, with $\mu_P(x) = 1 - x$, in which case it could be accepted that $\mu_Q(y) = \mu_P(y/10) = 1 - y/10$.

These ideas can be related with the Wittgenstein's concept of *family resemblance* [31].

11. Family resemblance with *L*-sets

In the path towards searching for models for Natural Language, it is needed to capture the linguistic relationships between the linguistic terms in the best possible way than possible. The Wittgenstein's concept of family resemblance could allow to know if there exists some similarity between *L*-sets representing a particular predicate, but in different contexts, and with different purposes.

The concept of *family resemblance* [31] reflects the family's air between the uses of some words, but in fuzzy logic and up to now, it was only considered for some gradable predicates represented by Zadeh's fuzzy sets [8,29]. In this paper, the definitions given there will be enlarged for pairs of *L*-sets, defined on different universes of discourse. The new definition, although it is only valid when the universes are totally ordered, allows to approach the concept of **migration** of a predicate [8] as a particular case of *family resemblance* that is important for the evolution of the use of words. Notice that by the language's own dynamism it is frequent to apply a word to a new universe of discourse once it is first introduced in another universe.

Let $\mu_P : X \to L_1$ be a representation of the use of a predicate *P*, or *L*-set, $\mathbb{P}$, and let $\mu_Q : Y \to L_2$ be a representation of a use of the predicate *Q*, or *L*-set, $\mathbb{Q}$. Both *L*-sets can be easily compared, if the universes *X* and *Y* are totally 'ordered' and there exists an isomorphism $f : Y \to X$, allowing to check, in the universe of discourse *X*, if both predicates show some kind of 'resemblance'.

First of all, let us introduce some instrumental definitions.

Definition 11.1

- Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two posets. A non-decreasing function in $A \subset X$, $f:X \to Y$, is a mapping verifying that if $x \leqslant_X y$, then $f(x) \leqslant_Y f(y)$, for all $x, y \in A$. In addition, if *f* is onto and one-to-one, it is an isomorphism.
- Let *X* be a set and $(L, \leqslant)$ be a poset with a maximum 1. For any mapping $\mu : X \to L$, it is $S(\mu) = \{x \in X\ ;\ \mu(x) = 1\}$. If the poset $(L, \leqslant)$ is with a minimum 0, it is $Z(\mu) = \{x \in X\ ;\ \mu(x) = 0\}$.

Now, with these concepts it can be introduced the following definitions,

Definition 11.2. Let *X*, *Y* be two universes of discourse endowed, respectively, with total orders $\leqslant_X$ and $\leqslant_Y$, and $f:Y \to X$ an isomorphism. Let $(L_1, \leqslant_{L_1})$ and $(L_2, \leqslant_{L_2})$ be two posets with minimum and maximum. $\mu \in L_1^X$ and $\sigma \in L_2^Y$ are said to be in the **relation of family resemblance**, denoted by $(\mu, \sigma) \in \mathbf{fr}$, whenever:

1. $Z(\mu) \cap f(Z(\sigma)) \neq \emptyset$, $S(\mu) \cap f(S(\sigma)) \neq \emptyset$
2. σ is non-decreasing in $A \subset X$ iff $\mu \circ f$ is non-decreasing in *A*.
3. σ is decreasing in $A \subset X$ iff $\mu \circ f$ is decreasing in *A*.

This definition generalizes the following one given in [29,8] which only deals with fuzzy sets in the same universe of discourse, and is actually a particular case of Wittgenstein's idea [31].

Definition 11.3. With $X \subset \mathbb{R}$, the relation of family resemblance, $\mathbf{fr} \subset [0,1]^X \times [0,1]^X$, is defined by $(\mu, \sigma) \in \mathbf{fr}$ if and only if,

1. $Z(\mu) \cap Z(\sigma) = \{x \in X; \mu(x) = 0\} \cap \{x \in X; \sigma(x) = 0\} \neq \emptyset$, $S(\mu) \cap S(\sigma) = \{x \in X; \mu(x) = 1\} \cap \{x \in X; \sigma(x) = 1\} \neq \emptyset$.
2. μ is non-decreasing in $A \subset X$ iff σ is non-decreasing in *A*.
3. μ is decreasing in $A \subset X$ iff σ is decreasing in *A*.

From this definition follows:

- For no negation $\mu' = N \circ \mu$ (see [23]), is $(\mu, \mu') \in \mathbf{fr}$. Because the pair (μ, μ') does not verify the points 2 and 3 in definition 11.2. So, two contradictory fuzzy sets do not verify the relation of family resemblance.
- For no opposite ${}'\mu = \mu \circ \alpha$ (see [23]), is $(\mu, {}'\mu) \in \mathbf{fr}$. A predicate *P* represented by μ and its antonym built as $\mu \circ \alpha$, with α a symmetry (i.e. $\alpha : X \to X$, such that $\alpha \circ \alpha = id$ and $\alpha(1_X) = 0_X$), does not verify the relation of family resemblance since if the pair $(\mu, {}'\mu)$ verifies properties 2 and 3 of definition 11.2, then ${}'\mu$ introduces the same order in the universe of discourse, and this is contradictory with the concept of antonym.

The relation **fr** of family resemblance is reflexive, $(\mu, \mu) \in \mathbf{fr}$, for all $\mu \in L_1^X$. It is also symmetric, since $(\mu, \sigma) \in \mathbf{fr}$ implies,

1. $Z(\mu) \cap f(Z(\sigma)) \neq \emptyset$, $S(\mu) \cap f(S(\sigma)) \neq \emptyset$
2. σ is non-decreasing in $A \subset Y$ iff $\mu \circ f$ is non-decreasing in *A*.
3. σ is decreasing in $A \subset Y$ iff $\mu \circ f$ is decreasing in *A*.

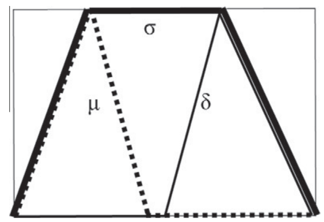


Fig. 1. Non transitivity.

which is equivalent to

1. $f^{-1}(Z(\mu)) \cap Z(\sigma) \neq \emptyset$, $f^{-1}(S(\mu)) \cap S(\sigma) \neq \emptyset$
2. $\sigma \circ f^{-1}$ is non-decreasing in $A \subset X$ iff μ is non-decreasing in A.
3. $\sigma \circ f^{-1}$ is decreasing in $A \subset X$ iff μ is decreasing in A.

So, $(\sigma, \mu) \in$ **fr**.

Notwithstanding, **fr** is not transitive, since there are μ, σ, δ such that $(\mu, \sigma) \in$ **fr**, $(\sigma, \delta) \in$ **fr**, but $(\mu, \delta) \notin$ **fr**. For example, in the Fig. 1, it is $S(\mu) \cap S(\delta) \neq \emptyset$. The lack of transitivity seems in agreement with the people family's air translated by **fr**.

A particular case of L-sets in family resemblance is given by the next concept.

Definition 11.4. Let X, Y be two universes of discourse endowed respectively with total orders $\leqslant_X$ and $\leqslant_Y$. Let $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ be two posets with minimum and maximum. $\sigma \in L_2^Y$ is said to be a **migration** of $\mu \in L_1^X$, whenever:

- There exists an isomorphism $f: Y \to X$, such that $f(Z(\sigma)) \cap Z(\mu) \neq \emptyset$, and $f(S(\sigma)) \cap S(\mu) \neq \emptyset$.
- There exists a non-decreasing function $F: L_1 \to L_2$, that verifies the boundary conditions $F(0_1) = 0_2$, and $F(1_1) = 1_2$.
- It is $\sigma = F \circ \mu \circ f$.

Hence, if $\sigma = F \circ \mu \circ f$ is a migration of μ, it is immediate that it is $(\mu, \sigma) \in$ **fr**.

A second step in the study of family resemblance is to capture the degree of family resemblance between L-sets. Up to now only some of these degrees are defined in the case of Zadeh's fuzzy sets (see [8,29]), but, the general problem of graduating the relation of family resemblance is an open one.

Remarks 11.5

- Provided is $X \subset \mathbb{R}$, if P on (X, c_1, up_1), and Q on (X, c_2, up_2) (see Section 7), can be taken as synonyms with $\mu_Q = F \circ \mu_P \circ \varphi$, it can be supposed that $C_X(P; c_1, up_1) = C_X(Q\,; c_2, up_2)$, but the reciprocal is, at least dubious.
- If P and Q show some family resemblance, it seems reasonable that the corresponding collectives do show some kind of similarity relationship. Anyway, and right now, only something can be said for L-sets.

12. Conclusion

12.1

This paper, presenting a kind of linguistic and semantic genesis of fuzzy sets, has a triple goal. First, to show that there are actually ways of modeling the action of a predicate on universes of discourse that is a set. That is, how the predicate is primarily used in the universe, or which is its 'meaning'. To such an end, it is introduced a binary relation representing when an object verifies the property named by the predicate less than another object verifies it. Such relation just corresponds to a 'qualitative', and sometimes perceptive's based form of how the predicate works in the universe of discourse.

Second, to show that the concept of 'degree' (the extent up to which an object satisfies the property), in some form a way towards 'quantitatively' measuring such extent, is formalizable once known the relation modeling the primary use of a predicate. If this relation is a preorder, it always exists a poset and an associate degree perfectly reflecting the primary use, and naturally linked to it. It is also shown how a simple change in the way of speaking can represent as an L-set the collective, the predicate generates in the universe of discourse. Such L-set is not an absolute concept, but one relative to the scale where the degree can take its values. In addition, some light is shed on which is the intrinsic property a function in L^Xshould verify to represent an L-set on X. If *collectives are generated* by predicates, *L-sets are defined* by a particular representation of collectives.

Third, to reflect on the practical problem of choosing at each case the scale in which the degrees can vary. In the particular case the predicate is numerically measurable, it can appear some special types of fuzzy sets, like type-2 or interval-valued fuzzy sets. In particular, and from a linguistic point of view, the so-called 'intuitionistic' fuzzy sets could come from simultaneously considering the given predicate and one of its antonyms, as well as from the supposition that the negation of the predicate is modeled by the strong negation $1 - id$. At this respect, and since a good linguistic management of a predicate entails simultaneously that of its negation and one its antonyms, a new, model for numerically measurable predicates is introduced by triplets of numbers in the unit interval, representing the corresponding degrees of *P*, not *P* and an antonym of *P*.

12.2

It is to be remarked that since all systems described by means of linguistic expressions involving imprecise terms (like rules in fuzzy control), should be carefully designed ([27]), before starting with the design's process, the designers should decide by which kind of *L*-sets the predicates in the linguistic expressions can be represented. This is a crucial decision upon which connectives, opposites, conditionals, etc., will be represented in one or in another form, a decision that conditionates the final design of the system in, let us say, 'formal' terms. At this point, it should be noticed that in all those cases in which $\leqslant_P$ is not known but a degree μ_P such that, $\leqslant_P \subset \leqslant_{\mu_P}$ is accepted, the designers are actually considering an 'excessive' meaning for *P*.

Anyway, simplicity is of an upmost value in both science and technology, and Occam's Razor should be always taken into account, (*Don't introduce more entities than those strictly necessary*), although with Menger's addition, '*but not reduced to the point of inadequacy*' [17]. In that sense, and in the authors view, Zadeh's fuzzy sets are, in general, the most simple election with which, for instance, also the degree of not satisfying the predicate, and the degree of an opposite to it, can be perfectly taken into account. Usually, it makes unnecessary the system's modelling by intuitionistic fuzzy sets. Even more, since as it is proven in [7], intuitionistic fuzzy sets are isomorphic to interval-valued fuzzy sets.

Nevertheless, like with almost all numerical functions, there is a problem that can conduct to the modelling by means of either type-2 fuzzy sets, or interval valued fuzzy sets. This problem arises from the contextual uncertainty coming form imprecision that, when there is no reasonable way of reducing it for what concerns to fix a concrete value for the degree up to which '*x* is *P*', could allow to take such degree as an interval, or a fuzzy number. Anyway, the context in which the problem is inscribed could make this uncertainty not totally avoidable. For instance, with interval-valued fuzzy sets there remains the uncertainty related to determine the limiting points of the intervals, and with type-2 fuzzy sets that of fixing the parameters of the fuzzy numbers.

The uncertainty in the determination of the values of a function $\mu_P \in [0,1]^X$ representing the use of *P*, is not only an important practical problem, but also a theoretical one. It could happen, for instance, that a first approximate design μ_P^* of the use of *P* results to be contradictory with a second one μ_P^{**} $(\mu_P^{**} \leqslant \mu_P^{*\prime})$, conducting to think that μ_P^*, or μ_P^{**}, or both, are not a good approximation for the degree of *P*. Hence, at least when the degree of *P* is in [0,1], the uncertainty coming from the imprecise use of *P* deserves to be studied. The uncertainty coming from imprecision is a relevant and open subject of research.

At least from a practical point of view, it can be more suitable to devote more time to approach the degree by a single number in [0,1] (for instance, by means of some kind of aggregation function), and using Zadeh's fuzzy sets in the benefit of simplicity, even if it is not to be forgotten the respective meanings of the terms 'simplicity' and 'simplification'.

In general there are not algorithmic processes allowing to obtain the values $\mu_P(x)$. It suffices to remember that in the case of the rigid predicate *P* = 'transcendental' on $X = \mathbb{R}$, to know if $\mu_P(x)$ is either 0, or 1, is a very difficult mathematical problem [4].

Nevertheless, there are cases in which the representation of *P* by some kind of either type-2, or interval-valued fuzzy sets, is the more suitable. For instance, if *X* is a continuous domain but it is not possible to define with precision enough the values in [0,1] of a $\leqslant_P$-degree, provided it is clear that *P* is numerically measurable, a degree μ_P with values in either the set of fuzzy numbers or in that of intervals, could be suitable. Specially if it is $\leqslant_P = \leqslant_{\mu_P}$, that is, if μ_P perfectly reflects the primary use of *P* on *X*, in which case the corresponding poset where μ_P takes its values is isomorphic to that in $X/=_P$ naturally linked to the use of *P* on *X*.

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References

[1] K.T. Atanassov, Intuitionistic Fuzzy Sets, Physica-Verlag, 1999.
[2] J. Barranquero, S. Guadarrama, Collecting fuzzy perceptions from non-expert users. In IEEE World Congress on Computational Intelligence, Barcelona, Spain, pp. 932–939, 2010.
[3] G. Bodiou, Thorie dialectique des probabilits (englobant leurs calculs classique et quantique), Gauthier-Villars, 1964.
[4] Edward B. Burger, Robert Tubbs, Making Transcendence Transparent: An Intuitive Approach to Classical Transcendental Number Theory, Springer, USA, 2004.

[5] G. de Cooman, Evaluation sets and mappings, the order-theoretic aspect of the meaning of properties, in: E. Kerre (Ed.), Introduction to the Basic Concepts of Fuzzy Set theory and Some of its Applications, Comunication and Cognition (1991) 159–213.
[6] G. Deschrijver, E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 133 (2003) 227–235.
[7] G. Deschrijver, E.E. Kerre, On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision, Inform. Sci. 77 (2007) 1860–1866.
[8] I. García-Honrado, E. Trillas, and S. Guadarrama. Grado de parentesco entre predicados. In Actas XV Congreso Español sobre Tecnologías y Lógica Fuzzy, Punta Umbra, Spain, pp. 163–168, 2010. (In Spanish).
[9] J. Goguen, *L*-fuzzy sets, J. Math. Anal. Appl. 10 (1967) 145–174.
[10] S. Guadarrama, D.P. Pancho, Using Soft Constraints to interpret descriptions of shapes. In Semantic Computing (ICSC), 2010 IEEE Fourth International Conference on, Pittsburgh, USA, pp. 341–348, 2010.
[11] P.R. Halmos, Naïve Set Theory, Van Nostrand, 1960.
[12] D.A. Kappos, Probability Algebras and Stochastic Spaces, Academic Press, 1969.
[13] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic, Prentice Hall PTR, 1995.
[14] A. De Luca, S. Termini, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, Inform. Control 20 (4) (1972) 301–312.
[15] C. McGinn, Logical Properties, Clarendon Press, 2000.
[16] J.M. Mendel, R.I. John, Type-2 fuzzy sets made simple, IEEE Trans. Fuzzy Syst. 10 (2) (2002) 117–127.
[17] K. Menger, A counterpart of Occam's Razor in pure and applied mathematics; semantic uses. Selected Papers in Logic Foundations, Didactics, Economics, pp. 331–349. Reidel, 1979.
[18] See any Google's entrance on 'Montessori Methods', for instance, <http://en.wikipedia.org/wiki/Montessori_method>.
[19] Karl R. Popper, Objective Knowledge: An Evolutionary Approach, Clarendon Press, Oxford, 1972.
[20] E. Trillas, On the words not-probable and improbable. In Proceedings Eighth International Conference IPMU, Madrid, Spain, (2):780–783, 2000.
[21] E. Trillas, On the use of words and fuzzy sets, Inform. Sci. 176 (11) (2006) 1463–1487.
[22] E. Trillas, On the genesis of fuzzy sets, Agora 27 (1) (2008) 7–33.
[23] E. Trillas, On a model for the meaning of predicates, in: R. Seising (Ed.), Fuzzy sets theory Philosohpy and criticism, Springer, 2009, pp. 175–205. chapter 9.
[24] E. Trillas. Why do fuzzy representations need a careful design? In Computational intelligence in decision and control: Proceedings of the 8th International Conference FLINS, pp. 189–197, 2008.
[25] E. Trillas, C. Alsina, A reflection on what is a membership function, Mathware Soft Comput. 6 (1999) 201–215.
[26] E. Trillas, C. Alsina, A. Pradera, On a class of Fuzzy Set Theories, in: Proceedings FUZZ-IEEE, London, pp. 1–5, 2007.
[27] E. Trillas, S. Guadarrama, Fuzzy representations need a careful design, Int. J. General Syst. 39 (3) (2010) 329–346.
[28] E. Trillas, C. Moraga, S. Guadarrama, S. Cubillo, E. Castieira, Computing with Antonyms, in: M. Nikravesh et al. (Eds.), Forging New Frontiers: Fuzzy Pioneers I, Springer, 2007, pp. 133–153.
[29] E. Trillas, C. Moraga, A. Sobrino, On 'family resemblances' with fuzzy sets. In Proceedings IFSA-EUSFLAT, Lisbon, pp. 306–311, 2009.
[30] E. Trillas, T. Nakahama, I. García-Honrado, Fuzzy probabilities: Tentative discussions on the mathematical concepts. In IPMU'10, Dortmund, Germany, pp. 139–148, 2010.
[31] L. Wittgenstein, Philosophical Investigations, 3rd ed., Basil Blackwell, Oxford, 1981.
[32] L.A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965) 338–353.
[33] L.A. Zadeh, Probability measure of fuzzy events, J. Math. Anal. Appl. 23 (1968) 421–427.
[34] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[35] L.A. Zadeh, The concept of linguistic variable and its application to aproximate reasoning, Part I, Inform. Sci. 8 (1975) 199–249.
[36] L.A. Zadeh, Fuzzy probabilities, Inform. Process. Manage. 20 (3) (1984) 363–372.
[37] L.A. Zadeh, From computing with numbers to computing with words, in: P.P. Wang (Ed.), Computing with Words, John Wiley & Sons, 2001, pp. 35–68.
[38] L.A. Zadeh, Is there a need for fuzzy logic?, Inform Sci. 8 (2008) 199–249.
[39] L.A. Zadeh, A note on Z-numbers, Inform. Sci. 8 (2011) 2923–2932.

Capítulo 6

Characterizing the Principles of Non Contradiction and Excluded Middle in [0,1]

A pesar de que la creencia en la ley de contradicción es un pensamiento, la ley de contradicción en sí misma no es un pensamiento, sino un hecho.
Bertrand Russell (1872-1970)

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CHARACTERIZING THE 'PRINCIPLES' OF NON CONTRADICTION AND EXCLUDED MIDDLE IN $[0,1]^*$

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Under an interpretation of the principles of non-contradiction and excluded-middle based on the concept of self-contradiction, this paper mainly deals with the principles' verification in the case of the unit interval of the real line. Such verification is done in the three following cases: (1) The unit interval is totally ordered by the restriction to it of the usual order of the real line, (2) the unit interval is partially ordered by the sharpened order, and (3) the unit interval is under a new particular preorder. The first case is immediately extended to characterize the case of fuzzy sets.

Keywords: Non-contradiction; excluded middle; fuzzy sets; self-contradiction; unit interval.

1. Introduction

In the field of Philosophy there is a big amount of papers dealing with the so-called 'principles' of Non-contradiction (NC) and Excluded-Middle (EM) that were introduced by Aristotle. Nevertheless, in the fields of Logic, Mathematics, and Computer Science, these principles did not deserve too much attention. In such fields, the principles are either laws of the corresponding structure, or are checked to fail.

For example, in ortholattices $(L, \cdot, +, '; 0, 1)$ (and hence in orthomodular lattices and boolean algebras), the principles are embodied as axioms in the forms $a{\cdot}a' = 0$ (NC), and $a + a' = 1$ (EM), and because of the laws $0' = 1$, $a'' = a$, and $a{\cdot}b = (a' + b')'$, one of them is equivalen to the other. In the case of De Morgan algebras they simply fail, like it happens in the case of the unit interval $[0,1]$ with $\cdot = \min$, $+ = \max$, and $' = 1 - id$.

*Characterizing the 'principles' NC and EM in $[0,1]$.

In the standard algebras of fuzzy sets $([0,1]^X, T, S, N)$, intersections are represented by continuous t-norms, T, unions by continuous t-conorms, S, and negations by strong negations, N.[3] The formulas $\mu\cdot\mu' = \mu_0$ (NC) (with $\mu_0(x) = 0,\ \forall x \in X$), $\mu + \mu' = \mu_1$ (EM) (with $\mu_1(x) = 1,\ \forall x \in X$), can be translated into the functional equations $T(\mu(x), N(\mu)(x)) = 0$, and $S(\mu(x), N(\mu)(x)) = 1$, for all x, which respective solutions are[4]:

NC: $T = W_\varphi$, where $W_\varphi(a,b) = \varphi^{-1}\max(0, \varphi(a) + \varphi(b) - 1)$ is a t-norm in the family of Łukasiewicz (So, $\varphi : [0,1] \to [0,1]$ is an order automorphism, that is, it is continuous and strictly increasing function verifying $\varphi(0) = 0$ and $\varphi(1) = 1$), and $N_\varphi \leq N$.

EM: $S = W^*_\psi$, where $W^*_\psi(a,b) = \psi^{-1}\min(1, \psi(a) + \psi(b))$ is a t-conorm in the family of Łukasiewicz (with ψ an order automorphism), and $N \leq N_\psi$.

Hence, the two principles only hold in the cases in which $T = W_\varphi$, $S = W^*_\psi$, $N_\varphi \leq N \leq N_\psi$.

The principle NC was stated by Aristotle as ‘A and not A is imposible’,[1] and in the above cases the term ‘imposible’ is translated into the term ‘false’. The principle EM was finally translated by ‘It is always A or not A’, and the term ‘always’ furtherly translated into ‘true’.

This paper is based on translating ‘imposible’ by self-contradictory,[5] the term ‘always’ by the self-contradiction of the corresponding ‘not’, and all that without presupposing any particular property for the connectives ‘and’, ‘or’, and ‘not’.[8] That is,

NC: ‘A and not A is self-contradictory’
EM: ‘not (A and not A) is self-contradictory’

2. Posing the General Problem

Let it L be a non-empty set, $' : L \to L$ a mapping, $* : L \times L \to L$ an operation, and consider the triplet $(L,\ ', *)$. Eventually, elements a in L do represent statements A, mapping $'$ does represent ‘not’, and the operation $*$ does represent either the conjunction ‘and’ (in which case $*$ is written $\cdot$), or the disjunction ‘or ’ (in which case $*$ is written $+$). For example,

- $a'\cdot b$, does represent an statement ‘not A and B’
- $a' + b'$, does represent an statement ‘not A or not B’
- $(a')' = a''$ does represent an statement ‘not (not A)’, etc.

Consider a relation $\models \subset L \times L$, eventually representing If/then. Elements a, b in L are $\models$-contradictory if $a \models b'$ (If A, then not B). An element $a \in L$ is $\models$-self-contradictory if $a \models a'$ (If A, then not A).

Definition 1. A triplet $(L, {}', \cdot)$ is $\models$-non contradictory ($\models NC$, for short) if all its elements of the form $a{\cdot}a'$ are $\models$-self contradictory. That is, if

$$a{\cdot}a' \models (a{\cdot}a')', \ \forall a \in L.$$

Theorem 1. *For any triplet $(L, {}', \cdot)$ there is at least a relation for which the triplet is $\models NC$.*

Proof. Such relation is, obviously, the one given by the set of pairs in $L \times L$, $\{a{\cdot}a', (a{\cdot}a')'; a \in L\} = \models_{NC}$. □

Theorem 2. *$(L, {}', \cdot)$ is $\models NC$, if and only if $\models_{NC} \subset \models$.*

Proof. Obvious. □

Hence, $(L, {}', \cdot)$ is not $\models NC$ if and only if $\models_{NC} \not\subseteq \models$.

Definition 2. A triplet $(L, {}', +)$ is $\models$-excluded middle ($\models EM$, for short) if all its elements of the form $(a + a')'$ are $\models$-self contradictory. That is, if

$$(a + a')' \models ((a + a')')', \ \forall a \in L.$$

Theorem 3. *For any triplet $(L, {}', +)$ there is at least a relation for which the triplet is $\models EM$.*

Proof. Such relation is, obviously, the one given by the set of pairs in $L \times L$, $\{(a + a')', ((a + a')')'; a \in L\} = \models_{EM}$. □

Theorem 4. *$(L,', +)$ is $\models EM$, if and only if $\models_{EM} \subset \models$.*

Proof. Obvious. □

Hence, $(L,', +)$ is not $\models_{EM}$ if and only if $\models_{EM} \not\subseteq \models$.

Remarks 5.

(1) If $(L,', \cdot)$ is not $\models NC$, it can not be $\models^* NC$, with $\models^* \subset \models$. Since if $\models^* NC$, it is $\models_{NC} \subset \models^*$, and follows the absurd $\models_{NC} \subset \models$.

(2) If $(L,', +)$ is not $\models EM$, it can not be $\models^* EM$, with $\models^* \subset \models$. Since if $\models^* EM$, it is $\models_{EM} \subset \models^*$, and follows the absurd $\models_{EM} \subset \models$.

(3) Hence, if $(L,', \cdot)$ is not $\models EM$, but it is $\models^* EM$, either $\models \subset \models^*$, or both $\models$ and $\models^*$ are not comparable under set's inclusion. Analogously, if $(L,', \cdot)$ is not $\models NC$, but it is $\models^* NC$, either $\models \subset \models^*$, or $\models$ and $\models^*$ are non-comparable under set's inclusion.

(4) If $\models \subset \models^*$, and $\models_{NC} \subset \models^*$, the triplet is $(\models_{NC} \cup \models)NC$.

(5) If $\models \subset \models^*$, and $\models_{EM} \subset \models^*$, the triplet is $(\models_{EM} \cup \models)EM$.

Provided the semantic relation of entailment between statements 'If A, then B', can be represented by means of an operation $\to: L \times L \to L$ (implication), and a relation $\models_\to NC$ is defined by something like

$$a \models_\to b \Leftrightarrow a \to b \text{ is 'such and such'},$$

the interesting semantic problem on NC and EM lies in the verification of $\models_\to NC$, and $\models_\to EM$.

For example, if $(L, \cdot, +, ')$ is a boolean algebra, where $a \to b = a' + b$, it is

$$a \models_\to b \Leftrightarrow a \to b = 1 \Leftrightarrow a \leq b, \text{ with } a \leq b \Leftrightarrow a{\cdot}b = a,$$

and since $a{\cdot}a' = 0$, $(a{\cdot}a')' = 1$, $(a + a')' = 1' = 0$, and $(a + a')'' = 0' = 1$, it is obviously follows $a{\cdot}a' \leq (a{\cdot}a')'$ and $(a + a')' \leq (a + a')''$. That is, any boolean algebra verifies $\leq NC$, and $\leq EM$. Notice that, in this case (and in that of any ortholattice) is

$$\models_{EM} = \models_{NC} \subset \leq .$$

If $L = [0, 1]$, $a' = 1 - a$, $\cdot = \min$, $+ = \max$, and $a \to b = a' + b = \max(1 - a, b)$, since $a \to b = 1 \Leftrightarrow a = 0$ or $b = 1$, it is $\models_\to = \{(a, 1); a \in [0, 1]\} \cup \{(0, b); b \in [0, 1]\}$. In this case, $\models_{NC} = \{(\min(a, 1 - a), \max(a, 1 - a)); a \in [0, 1]\}$, since $(a{\cdot}a')' = 1 - \min(a, 1 - a) = \max(a, 1 - a)$.

From, $\min(a, 1 - a) \leq \max(a, 1 - a)$, follows $\models_{NC} \subset \leq$, that is, $(L, ', \cdot)$ verifies $\leq NC$.

Nevertheless, it is not $\models_{NC} \subset \models_\to$: $(\min(0.3, 0.7), \max(0.3, 0.7)) = (0.3, 0.7) \in \models_{NC}$, but $(0.3, 07) \notin \models_\to$, hence $(L, ', \cdot)$ does not verify $\models_\to NC$. With the semantics given by $a \to b = a' + b$, NC is not verified: $\models_\to NC$ does not hold.

Since $\max(a, b) = 1 - \min(a, 1 - a)$, it is $(a + a')' = 1 - \max(a, b) = \min(a, 1 - a) = a{\cdot}a'$, and from $a'' = 1 - (1 - a) = a$, follows $a'' = 1 - (1 - a) = a$, follows $((a + a')')' = (a{\cdot}a')'$: $\models_{EM} = \models_{NC} = \{(a, 1 - a); a \in [0, 0.5]\} \cup \{(a - 1, a); a \in [0.5, 1]\} \subset \leq$.

Hence, $(L, ', +)$ verifies $\leq EM$, but does not verify $\models_\to EM$ because of $\models_{EM} \not\subseteq \models_\to$. The semantic problem has a negative solution.

With the semantics given by any R-implication $J_T(a, b) = Sup\{z \in [0, 1]; T(a, z) \leq b\}$, with a continuous t-norm T, from $J_T(a, b) = 1 \Leftrightarrow a \leq b$, results $\models_\to = \leq$, and, in this case, $(L, ', \min, \max)$ is $\models_\to NC$ and $\models_\to EM$: the semantic problem has a positive solution.

Remark 1. For the case of the validity of the principles NC and EM in the case of three-valued logics and related with the implication $\to$, see Ref. 6.

3. The Case of the Ordered Unit Interval Endowed with a Strong Negation

Let $L = [0, 1]$, $\models \, = \, \leq$, and $'$ given by an strong negation N ($a' = N(a)$). As it is well known, it always exists[7] an order automorphism φ of $([0, 1], \leq)$, such that

$N(a) = N_\varphi(a) = \varphi^{-1}(1 - \varphi(a))$, for all $a \in [0,1]$. For which operations $\cdot = F$, $+ = G$, $\leq NC$ and $\leq EM$ do hold?

Theorem 6. *$([0,1], N_\varphi, F)$ verifies $\leq NC$ if and only if $F(a, N_\varphi(a)) \leq \varphi^{-1}(1/2)$ for all $a \in [0,1]$.*

Proof. Follows from the chain of equivalences:
$F(a, N_\varphi(a)) \leq N_\varphi(F(a, N_\varphi(a))) \Leftrightarrow \varphi(F(a, N_\varphi(a))) \leq 1 - \varphi((F(a, N_\varphi(a)))) \Leftrightarrow 2\varphi(F(a, N_\varphi(a))) \leq 1 \Leftrightarrow F(a, N_\varphi(a)) \leq \varphi^{-1}(1)$. □

Theorem 7. *$([0,1], N_\varphi, G)$ verifies $\leq EM$ if and only if $\varphi^{-1}(1/2) \leq G(a, N_\varphi(a))$ for all $a \in [0,1]$.*

Proof. Follows from the chain of equivalences:
$N_\varphi(F(a, N_\varphi(a))) \leq F(a, N_\varphi(a)) \Leftrightarrow 1 - \varphi((G(a, N_\varphi(a)))) \leq \varphi(G(a, N_\varphi(a))) \Leftrightarrow 1 \leq 2\varphi(G(a, N_\varphi(a))) \Leftrightarrow \varphi^{-1}(1) \leq G(a, N_\varphi(a))$. □

Remarks 8.

(1) t-norms T are among functions F in Theorem 6: It follows from $T \leq \min \leq \max \leq S$, for any t-conorm S, and by taking $S = N \circ T \circ (N \times N)$, in which case

$$T(a, N(a)) \leq N(T(N(a), N(N(a)))) = N(T(N(a), a)),$$

and $T(a, N(a)) \leq \varphi^{-1}(1/2)$, if $N = N_\varphi$.

(2) t-conorms S are among functions G in Theorem 7. It follows from $T \leq S$, for any t-norm T, and by taking $T = N \circ S \circ (N \times N)$, in which case

$$N(S(N(a), a)) \leq S(a, N(a)),$$

and $\varphi^{-1}(1/2) \leq S(a, N(a))$ if $N = N_\varphi$.

(3) In the cases in which $\models_\rightarrow \, = \, \leq$, and $\cdot = F$ verifies Theorem 6, it should hold $\models_\rightarrow NC$.

(4) In the cases in which $\models_\rightarrow \, = \, \leq$, and $+ = G$ verifies Theorem 7, it should hold $\models_\rightarrow EM$.

Theorem 9. *$([0,1], 1 - id, F)$ satisfies $\leq NC$, if only if the restriction F^* of F to the set $\{(a, 1-a); a \in [0,1]\}$, verifies $F^* \leq Sum/2$.*

Proof. Since $N = 1 - id$ is the strong negation for $\varphi = id$, it is $\frac{Sum}{2}(a, 1-a) = \frac{1}{2}$, and $F(a, 1-a) \leq \frac{1}{2} = \frac{Sum}{2}(a, 1-a)$ □

Theorem 10. *$([0,1], 1 - id, G)$ satisfies $\leq EM$, if and only if the restriction G^* of G to the set $\{(a, 1-a); a \in [0,1]\}$, verifies $Sum/2 \leq G^*$.*

Proof. Since $\frac{Sum}{2}(a, 1-a) = \frac{1}{2} \leq G(a, 1-a)$. □

Remark 2.

(1) If $F \leq \frac{Sum}{2}$, it holds the Theorem 9, and then it follows $\leq NC$. For instance, for all t-norm T is $T \leq \min \leq \frac{Sum}{2}$.

(2) If $\frac{Sum}{2} \leq G$, it holds the Theorem 10, and then it is $\leq EM$. For instance, for all t-conorm S is $\frac{Sum}{2} \leq \max \leq S$.

4. The Case of Fuzzy Sets

Let $[0,1]^X$ be endowed with functionally expressible operations,

$$\mu{\cdot}\sigma(x) = F(\mu(x), \sigma(x))$$

$$\mu + \sigma(x) = G(\mu(x), \sigma(x))$$

$$\mu'(x) = N(\mu(x))$$

for all $x \in X$, and $F : [0,1]\times[0,1] \to [0,1]$, $G : [0,1]\times[0,1] \to [0,1]$, $N : [0,1] \to [0,1]$ (strong negation).

For the case $[0,1]^X$ is with the partial pointwise ordering

$$\mu \leq \sigma \Leftrightarrow \mu(x) \leq \sigma(x),\ \forall x \in X,$$

the results in Sec. 3 are immediately applicable.

Let us suppose that $(\mu \to \sigma)(x,y) = J(\mu(x), \sigma(y))$ with a T-conditional such that,

$$a \leq b \Leftrightarrow J(a,b) = 1,$$

and define $\mu \models_{\to} \sigma \Leftrightarrow \mu \leq \sigma$.

Theorem 11. *The algebras of fuzzy sets* $([0,1]^X, F, G, N)$ *do verify* $\leq NC$ *and* $\leq EM$*, if and only if*

$$F(a, N(a)) \leq \varphi^{-1}(1/2) \leq G(a, N(a)),$$

for all $a \in [0,1]$*, and provided* $N = N_\varphi$.

Proof. Immediate after Theorems 6 and 7. □

Obviously, all standard algebras of fuzzy sets do verify Theorem 11, since in such case F is a continuous t-norm, and G is a continuous t-conorm. What remain an open problem is the case in which either $\cdot$, or $+$, are not functionally expressible.

Apart from the partial pointwise order for fuzzy sets in $[0,1]^X$, it can be defined the φ-sharpened order,

$$\mu \preceq_\varphi \sigma \Leftrightarrow \begin{cases} 0 \leq \mu(x) \leq \sigma(x) \leq \varphi^{-1}(1/2) \\ \varphi^{-1}(1/2) \leq \sigma(x) \leq \mu(x) \leq 1 \end{cases}$$

The greatest fuzzy set is $\mu_{\varphi^{-1}(1/2)}$, the function constantly equal to $\varphi^{-1}(1/2)$ the fix point of the strong negation N_φ. The order $\preceq_\varphi$ is not a total order, since there are elements that are non comparable, for example μ and σ if $\mu \leq \mu_{\varphi^{-1}(1/2)}$ and $\sigma \geq \mu_{\varphi^{-1}(1/2)}$.

Theorem 12. *$([0,1], N_\varphi, F)$ satisfies $\preceq_\varphi$ NC, if and only if $F(\mu(x), \mu'(x)) = \varphi^{-1}(1/2)$.*

Proof. $F(\mu, N_\varphi(\mu)) \preceq_\varphi N_\varphi(F(\mu, N_\varphi(\mu))) \Leftrightarrow$

$$\begin{cases} 0 \leq F(\mu, N_\varphi(\mu)) \leq N_\varphi(F(\mu, N_\varphi(\mu))) \leq \varphi^{-1}(1/2) \\ \varphi^{-1}(1/2) \leq N_\varphi(F(\mu, N_\varphi(\mu))) \leq F(\mu, N_\varphi(\mu)) \leq 1. \end{cases}$$

From $F(\mu, N_\varphi(\mu)) \leq N_\varphi(F(\mu, N_\varphi(\mu))) \leq \varphi^{-1}(1/2)$, we have

$$F(\mu, N_\varphi(\mu)) \leq \varphi^{-1}(1/2), \tag{1}$$

and

$$N_\varphi(F(\mu, N_\varphi(\mu))) \leq \varphi^{-1}(1/2), \tag{2}$$

since, applying the strong negation N_φ to equation (1) we have $N_\varphi(\varphi^{-1}(1/2)) = \varphi^{-1}(1/2) \leq N_\varphi(F(\mu, N_\varphi(\mu)))$, and from equation (2) and the antisymmetric property, it is $N_\varphi(F(\mu, N_\varphi(\mu))) = \varphi^{-1}(1/2)$, or equivalently $F(\mu, N_\varphi(\mu)) = \varphi^{-1}(1/2)$.

The reciprocal is obvious, since $F(\mu, N_\varphi(\mu)) = \varphi^{-1}(1/2)$ is self-contradictory. □

Theorem 13. *$([0,1], N_\varphi, G)$ satisfies $\preceq_\varphi$ EM, if and only if $G(\mu(x), \mu'(x)) = \varphi^{-1}(1/2)$.*

Proof. $N_\varphi \circ G(\mu, N_\varphi(\mu)) \preceq_\varphi N_\varphi \circ N_\varphi \circ G(\mu, N_\varphi(\mu)) = G(\mu, N_\varphi(\mu)) \Leftrightarrow$

$$\begin{cases} 0 \leq N_\varphi \circ G(\mu, N_\varphi(\mu)) \leq G(\mu, N_\varphi(\mu)) \leq \varphi^{-1}(1/2) \\ \varphi^{-1}(1/2) \leq G(\mu, N_\varphi(\mu)) \leq N_\varphi \circ G(\mu, N_\varphi(\mu)) \leq 1. \end{cases}$$

From $N_\varphi \circ G(\mu, N_\varphi(\mu)) \leq G(\mu, N_\varphi(\mu)) \leq \varphi^{-1}(1/2)$, we have

$$G(\mu, N_\varphi(\mu)) \leq \varphi^{-1}(1/2), \tag{3}$$

and

$$N_\varphi \circ G(\mu, N_\varphi(\mu)) \leq \varphi^{-1}(1/2), \tag{4}$$

since, applying the strong negation N_φ to the equation (3) we have $N_\varphi(\varphi^{-1}(1/2)) = \varphi^{-1}(1/2) \leq N_\varphi(G(\mu, N_\varphi(\mu)))$, and from equation (4) and the antisymmetric property, it is $N_\varphi(G(\mu, N_\varphi(\mu))) = \varphi^{-1}(1/2)$, or equivalently $G(\mu, N_\varphi(\mu)) = \varphi^{-1}(1/2)$.

The reciprocal is obvious, since $G(\mu, N_\varphi(\mu)) = \varphi^{-1}(1/2)$ is self-contradictory. □

There are many functions F satisfying the condition $F(a, N(a)) = \varphi^{-1}(1/2)$. *They can be characterized as follows:* the image of the pairs $\{(a, N(a)); a \in [0,1]\}$ is fixed and equals to $\varphi^{-1}(1/2)$, but the function's values can be taken arbitrarily for the other pairs of points in $[0,1] \times [0,1]$.

With the strong negation $N = 1 - id$, and $F(\mu, N(\mu)) = G(\mu, N(\mu)) = Sum(\mu, N(\mu))/2$, the two principles of Non Contradiction and Excluded Middle do hold.

Taking $\varphi = id$, we get $\varphi^{-1}(1/2) = 1/2$ and the order $id-$sharpened is the classical sharpened order. This order is related with the concept of fuzzy entropy, introduced by DeLuca and Termini in Ref. 2.

5. The Case of the Unit Interval Endowed with Other Orderings

5.1. *φ-sharpened order*

The φ-sharpened order can be translated into the unit interval $[0,1]$, for all $a, b \in [0,1]$,

$$a \preceq_\varphi b \Leftrightarrow \begin{cases} 0 \leq a \leq b \leq \varphi^{-1}(1/2) \\ \varphi^{-1}(1/2) \leq b \leq a \leq 1\,. \end{cases}$$

The only functions F that verify NC and EM will be those such that $F(a, N_\varphi(a)) = \varphi^{-1}(1/2)$.

For fuzzy sets with the pointwise order ($\mu \leq \sigma \Leftrightarrow \mu(x) \leq \sigma(x)$, for all $x \in X$), the only functions F that verify NC and EM will be those such that $F(\mu(x), N_\varphi(\mu(x))) = \varphi^{-1}(1/2)$, for all $x \in X$.

5.2. *A preorder,* $\vdash$

The relation,

$$x \vdash y \Leftrightarrow |x - 0.5| \leq |y - 0.5|,\ \forall x, y \in [0,1],$$

is a preorder, since $\vdash$ is reflexive and transitive, but it is not an order since it is not antisymmetric:

$0.2 \vdash 0.8$, because $|0.2 - 0.5| = 0.3 \leq |0.8 - 0.5| = 0.3$, and
$0.8 \vdash 0.2$, because $|0.8 - 0.5| = 0.3 \leq |0.2 - 0.5| = 0.3$
but, $0.2 \neq 0.8$.

Theorem 14. *The triplet* $([0,1]^X, \vdash, 1 - id, F)$ *verifies NC for all function* F.

Proof. $F(a, 1-a) \vdash 1 - F(a, 1-a) \Leftrightarrow |F(a, 1-a) - 0.5| \leq |1 - F(a, 1-a) - 0.5| \Leftrightarrow |F(a, 1-a) - 0.5| \leq |0.5 - F(a, 1-a)|$, what is always verified since $|F(a, 1-a) - 0.5| = |0.5 - F(a, 1-a)|$. □

Theorem 15. *The triplet* $([0,1]^X, \vdash, 1 - id, G)$ *verifies EM for all function* G.

Proof. $1 - G(a, 1-a) \vdash 1 - (1 - G(a, 1-a)) \Leftrightarrow |1 - G(a, 1-a) - 0.5| \leq |1 - (1 - G(a, 1-a)) - 0.5| \Leftrightarrow |0.5 - G(a, 1-a)| \leq |G(a, 1-a) - 0.5|$, what is always verified since $|G(a, 1-a) - 0.5| = |0.5 - G(a, 1-a)|$. □

In the case of fuzzy sets $[0,1]^X$, the preorder is translated as follows, for any $\mu, \sigma \in [0,1]^X$,

$$\mu \vdash \sigma \Leftrightarrow |\mu(x) - 0.5| \leq |\sigma(x) - 0.5|, \ \forall x \in X.$$

With this definition, and as a corollary of Theorems 14 and 15, any quartet $([0,1]^X, \vdash, 1 - id, F)$ verifies NC and EM.

6. Conclusion

Aristotle stated that the statement 'A and not A is impossible', is universally valid and non susceptible to proof. For Aristotle the law of non-contradiction is, actually, a 'principle' of thought.

It is not this the place to comment on the meaning and role of this kind of very general principles. We limit ourselves to note that we can fruitfully interpret them in a more narrow form. For instance, if the law of non contradiction is read in the form 'A and not A is false', its validity will depend on the interpretation of the term 'false', and on how it is represented in a given formal framework. If such law is posed by 'A and not A is self-contradictory', its validity will depend on the interpretation of 'self-contradictory', and on how it is represented in a formal framework. Of course, in both cases the principles' validity also will depend on the characteristics of the chosen formal framework.

Which one of these two interpretations of the aristotelian term 'impossible' is preferable? In which formal framework each one is preferable? These questions do not have an immediate answer. For example, within the framework of ortholattices there is equivalence between 'false' and 'self-contradictory', provided the first term is represented by the first lattice's element 0, and the second by the definition $x \leq x'$. Notwithstanding, within the framework of DeMorgan algebras, and also in that of the standard algebras of fuzzy sets, there are non-null self-contradictory elements.

Hence, those that are at least partially aristotelian, could prefer the interpretation conducting to the law's validity in more and less restrictive frameworks, and for what has been proven in this paper and in,[5],[6] and,[8] this new interpretation could be preferable to the first. For those that are completely non-aristotelian, the first could be preferable to the second.

References

1. Aristotle, *Metaphysics*, Book IV, Translated by W. D. Ross, eBooks@Adelaide, 2007.
2. A. De Luca and S. Termini, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Information and Control* **20**(4) (1972) 301–312.

3. A. Pradera, E. Trillas, S. Guadarrama, and E. Renedo, On fuzzy set theories, in P. P. Wang, D. Ruan, and E. Kerre (eds.), *Fuzzy Logic. A Spectrum of Theoretical and Practical Issues*, *Studies in Fuzziness and Soft Computing*, Vol. 215, Springer, 2007, pp. 15–47.
4. E. Trillas and C. Alsina, Elkan's theoretical argument reconsidered, *Int. J. Approx. Reason.* **26** (2001) 145–152.
5. E. Trillas, C. Alsina, and A. Pradera, Searching for the roots of non-contradiction and excluded-middle, *Int. J. General Systems* **31**(5) (2002) 499–513.
6. E. Trillas, C. Moraga, and E. Renedo, On Aristotle's NC and EM principles in three-valued logic, Proceedings IFSA-EUSFLAT, 2009, pp. 879–884.
7. E. Trillas, Sobre funciones de negación en la teoría de conjuntos difusos, *Stochastica* **3**(1) (1979) 47–59 (in Spanish). Reprinted (English version) in *Advances of Fuzzy Logic*, S. Barro *et al.* (eds.), Press of the Universidad de Santiago de Compostela, 1998, pp. 31–43.
8. E. Trillas, Non contradiction, excluded middle, and fuzzy sets, in *Int. Workshop of Fuzzy Logic and Applications*, eds. Di Gesù, Pal and Petrosino, Springer (LNAI5571), 2009, pp. 1–11.

Capítulo 7

Modelos de Razonamiento Ordinario

El género humano tiene,
para saber conducirse,
el arte y el razonamiento.
Aristóteles (384 a.C.-322 a.C.)

- E. Trillas, I. García-Honrado, A. Pradera, *Consequences and Conjectures in Preordered Sets*, Information Sciences 180 (19) 3573-3588 (2010).

- I. García-Honrado, E. Trillas, *On an Attempt to Formalize Guessing*, Tech. Rep. FSC-2010-11, European Centre for Soft Computing, aceptado en Soft Computing in Humanities and Social Sciences (Eds. R. Seising and V. Sanz) Springer-Verlag (2011).

7.1. Consequences and Conjectures in Preordered Sets

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Consequences and conjectures in preordered sets

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ABSTRACT

In a preordered set, or preset, consequence operators in the sense of Tarski, defined on families of subsets, are introduced. From them, the corresponding sets of conjectures, hypotheses, speculations and refutations are considered, studying the relationships between these sets and those previously defined on ortholattices. All the concepts introduced are illustrated with three particular consequence operators, whose behavior is studied in detail. The results obtained are applied to the case of fuzzy sets endowed with the usual pointwise ordering.

1. Introduction

One of the most distinguishing features of human beings is the act, and especially the art, of reasoning or goal-oriented managing conjectures. Reasoning and conjecturing are joint brain activities, very difficult to separate one from the other. Good-guesswork and rationality might even be synonyms, and, actually, scientific and technological research is an activity that manages guessing in a highly articulated way.

Traditionally, logic dealt with deductive reasoning, that is, with ways of obtaining safe, necessary, conclusions from a set of premises translating some previous information. Even more, sometimes a logic is defined as a pair (L, C), where L is set of statements, and C an operator of consequences, allowing to pass from some subsets of L to the corresponding sets of safe conclusions, or logical consequences. Anyway, Artificial Intelligence did show the interest of obtaining not so safe conclusions from a given body of knowledge. Processes to obtain consequences perform deductive reasoning, or deduction. Those to obtain hypotheses perform abductive reasoning, or abduction, and those to obtain speculations perform inductive reasoning, or induction. These three processes can be embodied in the term "conjecturing", that results close to the term "reasoning".

The so-called CHC models (shortening 'Consequences, Hypotheses and Conjectures', see [9]), were introduced in [12] with the aim of providing a formal framework allowing to study how to conjecture from a given set of premises. The seminal paper [12] was followed by papers [4,11,5,10,9,14], where different aspects of the models were investigated in depth.

CHC models are defined within the framework of ortholattices, and both the set of conjectures (which is partitioned into three different subsets, made of consequences, hypotheses and speculations), and the set of refutations, are described by

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means of the lattice natural order. Recently, the paper [13] has proposed a generalization where conjectures are defined starting from an abstract operator of consequences (in the sense of Tarski) rather than using the lattice order. Despite this improvement, the model still presents some major drawbacks. The most important one is that it is still only valid for ortholattices, thus excluding other important structures such as De Morgan algebras, used for example when dealing with fuzzy sets. On the other hand, the only operator of consequences that, till now, has been studied in depth, denoted as $C_\wedge$, presents the problem that it allows to derive consequences and conjectures from something (the infimum of the premises) that is not necessarily a premise.

In addition, Qiu [9] compared some basic properties of consequence and conjecture operators in orthocomplemented lattices, orthomodular lattices, residuated lattices, and boolean algebras. These comparisons show that some results holding in an algebraic structure may not hold in another one, and as shown in [7,8] some properties of classical finite automata hold if and only if the truth-value lattices underlying the logic satisfy different distributive laws in which one distributivity implies another one but the contrary implication may not hold.

What follows tries to (partially) avoid the above mentioned troubles by proposing a further generalization and simplification of the model that presents three main characteristics. First, the framework is enlarged from ortholattices to the more general structures of preordered sets endowed with a negation. Second, consequence operators are allowed to be defined on different families of subsets rather than exclusively on one of them. Finally, new consequence operators are proposed, studying in detail their behavior as well as the one of the associated sets of conjectures.

The paper is organized as follows. After a review of the basic concepts that are used in the paper (Section 2), Section 3 proposes a new definition of consequence operators within preordered sets, and studies the behavior of three particular operators. Next, Section 4 generalizes the concepts of conjectures and refutations to preordered sets, starting from the abstract consequence operators introduced in the previous section. Finally, Section 5 analyzes the case of fuzzy sets, and Section 6 ends with some conclusions and pointers to future work.

2. Basic concepts

In the following some basic notions regarding preordered sets are briefly recalled (see e.g. [1]):

Definition 2.1 (*Preorder*). Given a set L, a binary relation $\leqslant\ \subseteq L \times L$ is a *preorder* on L provided the two following conditions hold:

1. $a \leqslant a$, for all $a \in L$ (reflexivity).
2. If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$, for all $a,b,c \in L$ (transitivity).

The pair $(L,\leqslant)$ is said to be a *preordered set* or a *preset*.

Definition 2.2. Let $(L,\leqslant)$ be a preset. Then:

- If $\leqslant$ is antisymmetric, i.e., for any $a,b \in L$ it is $a = b$ whenever $a \leqslant b$ and $b \leqslant a$, then the pair $(L,\leqslant)$ is said to be a *partially ordered set* or a *poset*. A poset satisfying the condition $[a \leqslant b$ or $b \leqslant a]$ for all $a,b \in L$ is a *totally ordered set*.
- An element 0 in L is said to be a *first element* of the preset if $0 \leqslant a$ for all $a \in L$.
- An element 1 in L is said to be a *last element* of the preset if $a \leqslant 1$ for all $a \in L$.
- Given a subset $S \subseteq L$, it is said that $a \in L$ is an *infimum* (respectively a *supremum*) of S if
  1. $a \leqslant x$ ($x \leqslant a$) for all $x \in S$.
  2. If $c \in L$ is such that $c \leqslant x$ ($x \leqslant c$) for all $x \in S$, then $c \leqslant a$ ($a \leqslant c$).
- L is said to be inf-*-complete (respectively sup-*-complete) if every non-empty subset S of L has an infimum (supremum). It is said that L is *complete* when it is both inf-*-complete and sup-*-complete.
- A binary operation $*$: $L \times L \to L$ is said to be an *inf-operation* (respectively a *sup-operation*) if for all $a,b \in L$, $a*b$ is an infimum (supremum) of $\{a,b\}$.
- A *lattice* is a poset endowed with both an inf-operation (usually denoted as $\cdot$) and a sup-operation (usually denoted as +). If the lattice has first and last element, then it is a *bounded lattice*.
- A unary operation $'$: $L \to L$ is said to be a *negation* if it verifies the two following conditions:
  1. If $a \leqslant b$, then $b' \leqslant a'$ for all $a,b \in L$.
  2. If the preset has a unique first element *0* and a unique last element 1, then $0' = 1$ and $1' = 0$.
- A bounded lattice $(L,\cdot,+;0,1)$ is an *ortholattice* $(L,\cdot,+,';0,1)$ once it is endowed with a negation $'$ verifying the *non-contradiction* law, $a \cdot a' = 0$ for any $a \in L$, and the *involutive* law, $(a')' = a$ for any $a \in L$. Recall that ortholattices verify other well-known properties, such as the *excluded-middle* law, $a + a' = 1$ for any $a \in L$, or the *De Morgan* laws, $(a \cdot b)' = a' + b'$ and $(a + b)' = a' \cdot b'$ for any $a,b \in L$. An *orthomodular lattice* is an ortholattice verifying the so-called *orthomodular law*: for all $a,b \in L$, if $a \leqslant b$, $b = a + a' \cdot b$, or, equivalently, $a = b \cdot (a + b')$.
- A *Boolean algebra* is a distributive ortholattice, i.e., an ortholattice verifying the distributive laws $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $a + (b \cdot c) = (a + b) \cdot (a + c)$ for all $a,b,c \in L$. A *De Morgan algebra* is a bounded distributive lattice endowed with a negation verifying the involutive law and the De Morgan laws.

Remark 2.3

- Recall that if $(L, \leqslant)$ is a poset, then the infimum and the supremum of any subset $S \subseteq L$, if they exist, are unique. They are denoted, respectively, as *Inf S* and *Sup S*. Note in addition that antisymmetry is not only sufficient but also necessary in order to guarantee that each subset has at most one infimum (supremum). Indeed, if the antisymmetry property is not fulfilled, then there exist two different elements $a, b \in L$ such that $a \leqslant b$ and $b \leqslant a$, and this means that there exists at least a subset $S = \{a, b\}$ of L, with two infimum, a and b (and two supremum, again a and b).
- In Definition 2.2, the concept of inf-$*$-complete and sup-$*$-complete are not the usual ones (see [1]). Hence, along this paper it is not supposed, for instance, the existence in L of Inf $\emptyset$.

3. Consequences in presets

Consequence operators in the sense of Tarski constitute a well-known mechanism for deriving conclusions from a given set of premises (see e.g. [3] or [13] for the basic notions and further references). In this section we propose a generalization of the standard definition, allowing to define these operators on a family $\mathfrak{F} \subseteq \mathbb{P}(L)$, and we illustrate it with some particular examples.

3.1. Definitions and main properties

Definition 3.1 (*Structure of consequences*). Let L be a set and let $\mathfrak{F} \subseteq \mathbb{P}(L)$. It is said that $(L, \mathfrak{F}, C)$ is a *structure of consequences*, or, alternatively, that C is an *operator of consequences* (in the sense of Tarski) for $\mathfrak{F}$ in L, provided $C : \mathfrak{F} \to \mathfrak{F}$ verifies the three following properties:

1. $P \subseteq C(P)$, for all $P \in \mathfrak{F}$ (C is *extensive*).
2. If $P \subseteq Q$, then $C(P) \subseteq C(Q)$, for all $P, Q \in \mathfrak{F}$ (C is *monotonic*).
3. $C(C(P)) = C(P)$ for all $P \in \mathfrak{F}$, or $C^2 = C$ (C is a *clausure*).
   Sometimes, and because consequences are usually only reached from finite sets of premises, the so-called axiom of compacity,
4. For all $P \in \mathfrak{F}$, there exists a finite set $\{p_1, p_2, \ldots, p_n\} \subseteq P$, such that $\{p_1, p_2, \ldots, p_n\} \in \mathfrak{F}$ and $C(P) = C(\{p_1, p_2, \ldots, p_n\})$,

is also added, although it does not always hold.

Obviously, if L is finite, any consequence operator is compact. Note also that $C(P) \subseteq C(C(P))$ follows from 1 and 2, implying that property 3 can be reduced to $C(C(P)) \subseteq C(P)$ for all $P \in \mathfrak{F}$. The definition of operator of consequences proposed in [13] is recovered by just considering that L is a complete ortholattice and choosing $\mathfrak{F} = \mathbb{P}_0(L) = \{P \in \mathbb{P}(L);\ InfP \neq 0\}$.

In the case of singleton sets of premises $P = \{p\}$, for simplicity reasons the notation $C(p)$ will sometimes be used to refer to $C(\{p\})$.

Example 3.2. In [12] L was a complete ortholattice $(L, \cdot, +, '; 0, 1)$, $\mathfrak{F}$ was taken as $\mathbb{P}_0(L) = \{P \in \mathbb{P}(L); InfP \neq 0\}$, and the consequence operator $C_\wedge(P) = \{q \in L; Inf\, P \leqslant q\}$ was defined. Notice that, in general, axiom 4 does not hold in this structure of consequences.

Remark 3.3. Two structures $(L, \mathfrak{F}, C)$, and $(L^*, \mathfrak{F}^*, C^*)$ are isomorphic when there exists a bijective mapping $\mathfrak{f} : L \to L^*$, such that

1. If $P \in \mathfrak{F}$, then $\mathfrak{f}_e(P) \in \mathfrak{F}^*$, with $\mathfrak{f}_e$ the extension of $\mathfrak{f}$ to $\mathbb{P}(L)$, that is, $\mathfrak{f}_e(P) = \{\mathfrak{f}(p) \in L^*; p \in P\} \subseteq \mathbb{P}(L^*)$.
2. It is $\mathfrak{f}_e \circ C = C^* \circ \mathfrak{f}_e$, that is, $\mathfrak{f}_e(C(P)) = C^*(\mathfrak{f}_e(P))$, for all $P \in \mathfrak{F}$. Equivalently, $C^* = \mathfrak{f}_e \circ C \circ \mathfrak{f}_e^{-1}$.

Consequence operators can be compared in the following way: given two operators C and C^*, it is said that $C \subseteq C^*$ if for all $P \in \mathfrak{F}$ it is $C(P) \subseteq C^*(P)$. In this sense, the smallest operator of consequences is $C_0(P) = Id_{\mathfrak{F}}(P) = P$, that is, $C_0 \subseteq C$ for any consequence operator C in $(L, \mathfrak{F})$. If $L \in \mathfrak{F}$, $C_1(P) = L$ for all $P \in \mathfrak{F}$ is the greatest operator of consequences since, obviously, for any consequence operator C it is $C \subseteq C_1$. If $L \notin \mathfrak{F}$, the greatest operator of consequences is $C_1(P) = \cup_{Q \in \mathfrak{F}} Q$ for all $P \in \mathfrak{F}$, provided $\cup_{Q \in \mathfrak{F}} Q \in \mathfrak{F}$. In these cases it is $C_0 \subseteq C \subseteq C_1$ for any consequence operator C.

When L is a preset endowed with a negation $'$, the following important property of consequence operators may be established:

Definition 3.4 (*Consistency*). Let $(L, \leqslant, ')$ be a preset endowed with a negation $'$ and let $\mathfrak{F} \subseteq \mathbb{P}(L)$. A consequence operator $C : \mathfrak{F} \to \mathfrak{F}$ is *consistent for* $P \in \mathfrak{F}$ when for any $q \in C(P)$ it is $q' \notin C(P)$. It is said that C is *consistent* in $\mathfrak{F}$ when it is consistent for all $P \in \mathfrak{F}$.

Consistency for a given P states that if q "follows deductively" from P ($q \in C(P)$), then it will not be the case that also $q' = not\ q$ "follows deductively" from P ($q' \notin C(P)$).

Example 3.5. The consequence operator cited in Example 3.2 is clearly consistent in $\mathbb{P}_0(L)$, since $q,q' \in C_\wedge(P)$ would entail *Inf* $P \leqslant q \cdot q' = 0$ and hence *Inf* $P = 0$, which is contradictory with the fact $P \in \mathbb{P}_0(L)$.

Remark 3.6. Note that if $(L, \mathfrak{F}_1, C)$ and $(L, \mathfrak{F}_2, C)$ are two structures of consequences such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, then if C is consistent in $\mathfrak{F}_2$, it is also consistent in $\mathfrak{F}_1$. Equivalently, C can not be consistent in $\mathfrak{F}_2$ if it is not consistent in $\mathfrak{F}_1$.

The concept of $\cdot$-compatibility, introduced in [3] for consequence operators, can be generalized to the case of consequence structures in the following way:

Definition 3.7 (*Compatibility*). Let $(L, \mathfrak{F}, C)$ be a structure of consequences and let $\cdot : L \times L \to L$ be a binary operation on L. The operator C is $\cdot$-compatible in $\mathfrak{F}$ if the two following conditions hold for any $a,b \in L$:

1. For any $P \in \mathfrak{F}$, $a \in C(P)$ and $b \in C(P)$ imply $a \cdot b \in C(P)$.
2. If $\{a \cdot b\} \in \mathfrak{F}$, then $\{a,b\} \subseteq C(a \cdot b)$.

When the operation $\cdot$ is commutative and associative (as it happens, for example, for any inf-operation), it appears that the consequences obtained with a $\cdot$-compatible operator from a finite set of premises are the same as the ones obtained after applying the operation $\cdot$ to the premises:

Theorem 3.8. *Let $(L, \mathfrak{F}, C)$ be a structure of consequences and let $\cdot : L \times L \to L$ be an associative and commutative binary operation. If C is $\cdot$-compatible in $\mathfrak{F}$, then for any $p_1, \ldots, p_n \in L$ such that $\{p_1, \ldots, p_n\}, \{p_1 \cdot p_2 \cdots p_n\} \in \mathfrak{F}$, $C(\{p_1,p_2,\ldots,p_n\}) = C(p_1 \cdot p_2 \cdots p_n)$.*

Proof. Since C is a consequence operator, $\{p_1,p_2,\ldots, p_n\} \subseteq C(\{p_1,p_2,\ldots,p_n\})$, and this, by property 1 in Definition 3.7, implies $p_1 \cdot p_2 \cdots p_n \in C(\{p_1,p_2,\ldots,p_n\})$, and hence $C(p_1 \cdot p_2 \cdots p_n) \subseteq C^2(\{p_1,p_2,\ldots,p_n\}) = C(\{p_1,p_2,\ldots,p_n\})$. By 2 in Definition 3.7, $\{p_1,,p_2,\ldots,p_n\} \subseteq C(p_1 \cdot p_2 \cdots p_n)$, and hence $C(\{p_1,p_2,\ldots,p_n\}) \subseteq C^2(p_1 \cdot p_2 \cdots p_n) = C(p_1 \cdot p_2 \cdots p_n)$. $\square$

We end this section recalling the concept of *filter* [1]:

Definition 3.9 (*Filter*). Let $(L,\leqslant,\cdot)$ be a preset endowed with an inf-operation. A *filter* is a set $F \subseteq L$verifying the two following properties for any $x,y \in L$:

- $x,y \in F \Rightarrow x \cdot y \in F$
- $x \in F$ & $x \leqslant y \Rightarrow y \in F$

Given an element $p \in L$, the set $\{q \in L; p \leqslant q\}$ is a filter, called the *principal filter generated by p*. Note also that when dealing with sets of consequences, the first requirement in the above definition is equivalent to the first condition for $\cdot$-compatibility (Definition 3.7). In the following section we will see that there are some consequence sets $C(P)$ which are filters, but that this is not necessarily the case.

3.2. Some particular consequence operators

In the sequel we will consider a preset endowed with a negation, $(L,\leqslant,')$, and, in order to define consequence structures, we will deal with the following families $\mathfrak{F} \subseteq \mathbb{P}(L)$:

1. $\mathfrak{F} = \mathbb{P}(L)$
2. $\mathfrak{F} = \mathbb{P}_{SC}(L) = \{P \in \mathbb{P}(L);\ \text{forno}\ p \in P:\ p \leqslant p'\}$
3. $\mathfrak{F} = \mathbb{P}_{NC}(L) = \{P \in \mathbb{P}(L); \text{forno} p_1, p_2 \in P:\ p_1 \leqslant p_2'\}$
4. Provided $\cdot$ is an inf-operation in $(L,\leqslant)$, $\mathfrak{F} = \mathbb{P}_{iC}(L) = \{P \in \mathbb{P}(L); \text{fornofinitesubsets}\ \{p_1, \ldots, p_r\}, \{p_1^*, \ldots, p_n^*\} \subseteq P:\ p_1^* \cdots p_n^* \leqslant (p_1 \cdots p_r)'\}$.
5. Provided L is an inf-*-complete poset and 0 = *Inf* L, $\mathfrak{F} = \mathbb{P}_0(L) = \{P \in \mathbb{P}(L); InfP \neq 0\}$.

Remark 3.10

- The set $\mathbb{P}_0(L)$ can only be defined within inf-*-complete posets, where the existence and uniqueness of the infimum is guaranteed (see Remark 2.3).
- When all these sets do exist (i.e., when L is an inf-*-complete poset), the following subsethood chain is obviously verified:

$$\mathbb{P}_{iC}(L) \subseteq \mathbb{P}_{NC}(L) \subseteq \mathbb{P}_{SC}(L) \subseteq \mathbb{P}(L).$$

- If L is a finite poset equipped with an inf-operation, L is clearly inf-*-complete and $\mathbb{P}_{iC}(L) \subseteq \mathbb{P}_0(L)$.
- If L is an inf-*-complete poset and verifies the non-contradiction law ($x \cdot x' = 0$ for all $x \in L$), then $\mathbb{P}_0(L) \subseteq \mathbb{P}_{iC}(L)$. This is the case, in particular, of complete ortholattices.
- As a consequence of the two previous considerations, it appears that for finite ortholattices it is $\mathbb{P}_0(L) = \mathbb{P}_{iC}(L)$.

Remark 3.11. Let $\mathfrak{F} \subseteq \mathbb{P}(L)$ and let $C : \mathfrak{F} \to \mathfrak{F}$. If

1. $C^* : \mathbb{P}(L) \to \mathbb{P}(L)$ is such that $C^*(A) \subseteq C(A) \forall A \in \mathfrak{F}$;
2. $\mathfrak{F}$ verifies that for any P, Q such that $P \in \mathfrak{F}$ and $Q \subseteq P$ it is $Q \in \mathfrak{F}$;

then $C^*(P) \in \mathfrak{F}$ whenever $P \in \mathfrak{F}$, that is, $C^* : \mathfrak{F} \to \mathfrak{F}$ is well defined. Note also that all the families $\mathfrak{F}$ defined above fulfill the second condition.

In what follows we analyze the behavior of three operators, denoted as $C_\leqslant$, $C_\bullet$ and $C_\wedge$, with respect to the above families of subsets $\mathfrak{F} \subseteq \mathbb{P}(L)$. We exclude the case of $\mathbb{P}(L)$, since obviously any operator C is well defined in $\mathbb{P}(L)$, and, if $C : \mathbb{P}(L) \to \mathbb{P}(L)$ is an operator of consequences, it is clearly not consistent in $\mathbb{P}(L)$, since $L \in \mathbb{P}(L)$ and $C(L) = L$ contains all the contradictory pairs (x,x') for any $x \in L$.

3.2.1. The consequence operator $C_\leqslant$

Let $(L,\leqslant)$ be a preset. Then (see [3]) it is possible to define the operator $C_\leqslant : \mathbb{P}(L) \to \mathbb{P}(L)$, given by $C_\leqslant(P) = \{q \in L; \exists p \in P: p \leqslant q\}$.

Theorem 3.12. *Let $(L,\leqslant)$ be a preset and let $\mathfrak{F} \subseteq \mathbb{P}(L)$ be such that $C_\leqslant(P) \in \mathfrak{F}$ for all $P \in \mathfrak{F}$. Then $C_\leqslant : \mathfrak{F} \to \mathfrak{F}$ is an operator of consequences for $\mathfrak{F}$ in L.*

Proof. Since for all $p \in P$ it is $p \leqslant p$, then clearly $P \subseteq C_\leqslant(P)$. Now take $P, Q \in \mathfrak{F}$ such that $P \subseteq Q$. If $q \in C_\leqslant(P)$, there exists $p \in P$ such that $p \leqslant q$, but since it is also $p \in Q$, it follows $q \in C_\leqslant(Q)$. Hence $C_\leqslant(P) \subseteq C_\leqslant(Q)$. Finally, if $q \in C^2_\leqslant(P)$, there is $p_1 \in C_\leqslant(P)$ such that $p_1 \leqslant q$. But then there is some $p \in P$ such that $p \leqslant p_1$. Hence, $p \leqslant q$, or $q \in C_\leqslant(P)$, that is, $C^2_\leqslant(P) \subseteq C_\leqslant(P)$. □

Theorem 3.12 proves that $C_\leqslant$ is a consequence operator for any $\mathfrak{F}$ satisfying $C_\leqslant(P) \in \mathfrak{F}$ whenever $P \in \mathfrak{F}$. The next result establishes which families $\mathfrak{F}$ verify this latter property and analyzes the consistency of the resulting operators.

Theorem 3.13. *Let $(L,\leqslant,')$ be a preset endowed with a negation. Then:*

1. *$(L, \mathbb{P}_{SC}(L), C_\leqslant)$ is a structure of consequences, and $C_\leqslant$ is consistent in $\mathbb{P}_{SC}(L)$ if and only if for no $a,b \in L$ it is $[a \leqslant b,\ a \nleqslant a'$ and $b' \nleqslant (b')']$.*
2. *$(L, \mathbb{P}_{NC}(L), C_\leqslant)$ is a structure of consequences and $C_\leqslant$ is consistent in $\mathbb{P}_{NC}(L)$.*
3. *If $\cdot$ is an inf-operation in $(L,\leqslant)$, then $(L, \mathbb{P}_{iC}(L), C_\leqslant)$ is a structure of consequences and $C_\leqslant$ is consistent in $\mathbb{P}_{iC}(L)$.*
4. *If $(L,\leqslant)$ is an inf-*-complete poset, then $(L, \mathbb{P}_0(L), C_\leqslant)$ is a structure of consequences, and $C_\leqslant$ is consistent in $\mathbb{P}_0(L)$ if and only if L verifies the non-contradiction law.*

Proof

1. If $C_\leqslant(P) \notin \mathbb{P}_{SC}(L)$ for some $P \in \mathbb{P}_{SC}(L)$, it implies that exists $q \in C_\leqslant(P)$ such that $q \leqslant q'$ and there exists $p \in P$ such that $p \leqslant q$, or $q' \leqslant p'$. So, it would be $p \leqslant q \leqslant q' \leqslant p'$, which is contradictory with $P \notin \mathbb{P}_{SC}(L)$. Therefore $(L, \mathbb{P}_{SC}(L), C_\leqslant)$ is a structure of consequences. To prove the consistency characterization, let us suppose first that there exist $a,b \in L$ verifying $[a \leqslant b$, $a \nleqslant a'$ and $b' \nleqslant (b')']$. Then choosing $P = \{a,b'\}$, it is $P \in \mathbb{P}_{SC}(L)$ and $b,\ b' \in C_\leqslant(P)$, so, $C_\leqslant$ is not consistent in $\mathbb{P}_{SC}(L)$. Let us now suppose that $C_\leqslant$ is not consistent in $\mathbb{P}_{SC}(L)$, so, there exist $P \in \mathbb{P}_{SC}(L)$ and $q \in L$ such that $q,q' \in C_\leqslant(P)$ and $p_1,p_2 \in P$ such that $p_1 \leqslant q$ and $p_2 \leqslant q'$. Then, we have a couple of elements $p_1,q \in L$ verifying $p_1 \leqslant q$, $p_1 \nleqslant p_1'$ (since $p_1 \in P$ and $P \in \mathbb{P}_{SC}(L)$) and $q' \nleqslant (q')'$ (because $q' \leqslant (q')'$ implies $p_2 \leqslant p_2'$, which is contradictory with $p_2 \in P$ and $P \in \mathbb{P}_{SC}(L)$).
2. If it was $C_\leqslant(P) \notin \mathbb{P}_{NC}(L)$ for some $P \in \mathbb{P}_{NC}(L)$, there would be $q_1,q_2 \in C_\leqslant(P)$ such that $q_1 \leqslant q_2'$, with $p_1 \leqslant q_1, p_2 \leqslant q_2$, for some $p_1,p_2 \in P$. Then, since $q_2' \leqslant p_2'$, it would follow $p_1 \leqslant q_1 \leqslant q_2' \leqslant p_2'$, which is contradictory with $P \in \mathbb{P}_{NC}(L)$. Therefore, $(L, \mathbb{P}_{NC}(L), C_\leqslant)$ is a structure of consequences.
   To prove the consistency of $C_\leqslant$ is consistent in $\mathbb{P}_{NC}(L)$, let us now suppose that there exist $q',q \in C_\leqslant(P)$. Then there would exist $p^*,p \in P$ such that $p^* \leqslant q'$ and $p \leqslant q$ or $q' \leqslant p'$. So, by the transitivity of $\leqslant$, $p^* \leqslant p'$, which is contradictory with $P \in \mathbb{P}_{NC}(L)$.
3. $(L, \mathbb{P}_{iC}(L), C_\leqslant)$ is a structure of consequences (see Remark 3.11). The consistency of $C_\leqslant$in $\mathbb{P}_{iC}(L)$ easily follows from its consistency in $\mathbb{P}_{NC}(L)$ and the inclusion $\mathbb{P}_{iC}(L) \subseteq \mathbb{P}_{NC}(L)$ (see Remark 3.6).
4. $(L, \mathbb{P}_0(L), C_\leqslant)$ is a structure of consequences (see Remark 3.11). Consistency is verified whenever L is an inf-*-complete poset where the law $x \cdot x' = 0$ holds where it is $\mathbb{P}_0(L) \subseteq \mathbb{P}_{iC}(L)$, and by then Remark 3.6 and the consistency of $C_\leqslant$ in $\mathbb{P}_{iC}(L)$, it follows the consistency of $C_\leqslant$ in $\mathbb{P}_0(L)$. Consistency is impossible in the absence of the non-contradiction law, if $a \in L$ such that $a \cdot a' \neq 0$. It can be $a \cdot a' = a$ (or $a \cdot a' = a$), then $a \leqslant a'$ (or $a' \leqslant a$), and this means that $P = \{a\}$ (or $P = \{a'\}$) verifies $P \in \mathbb{P}_0(L)$ and $a,a' \in C_\leqslant(P)$, so $C_\leqslant$ is not consistent in $\mathbb{P}_0(L)$. If $a \cdot a' = b$, with $b \notin \{0,a,a'\}$, it is $[b \leqslant a'$ and $b \leqslant a]$, which entails $[b \leqslant a'$ and $a' \leqslant b']$, and hence, by transitivity, $b \leqslant b'$. Then it is $P = \{b\} \in \mathbb{P}_0(L)$ and $b,b' \in C_\leqslant(P)$, so, $C_\leqslant$ is not consistent in $\mathbb{P}_0(L)$. □

Corollary 3.14. *$C_\leqslant$ is never consistent in $\mathbb{P}_{SC}(L)$ when L is a non-trivial ortholattice.*

Proof. Indeed, if L is a non-trivial ortholattice, it suffices to choose $a = b = x$, with $x \in L - \{0,1\}$, in order to fulfill the characterization given in the first item of the above theorem. □

Example 3.15. As an illustration of the characterization given in the first item of Theorem 3.13, choose $P = \{b,e'\}$ in the poset of Fig. 1 (it is $b,b' \in C_{\leqslant}(P)$), or $P = \{c,g'\}$ in the ortholattice on the right of the same figure (where it is $g,g' \in C_{\leqslant}(P)$). An example of a preset where $C_{\leqslant}$ is consistent in $\mathbb{P}_{SC}(L)$ is the totally ordered set $x \leqslant y \leqslant y' \leqslant x'$.

Note also that if $\{p\} \in \mathfrak{F}$ for all $p \in P$, $C_{\leqslant}(P)$ is always "resoluble" by means of the consequences of all the premises in P:

Theorem 3.16. *Provided* $\{p\} \in \mathfrak{F}$ *for all* $p \in P \in \mathfrak{F}$, $C_{\leqslant}(P) = \cup_{p\in P} C_{\leqslant}(p)$.

Proof. Obviously, if $p \in P$, $C_{\leqslant}(p) \subseteq C_{\leqslant}(P)$, and then $\cup_{p\in P} C_{\leqslant}(p) \subseteq C_{\leqslant}(P)$. Reciprocally, if $q \in C_{\leqslant}(P)$, since there is $p \in P$ such that $p \leqslant q$, it results $q \in C_{\leqslant}(p)$, and also $q \in \cup_{p\in P} C_{\leqslant}(p)$, or $C_{\leqslant}(P) \subseteq \cup_{p\in P} C_{\leqslant}(p)$. □

Recall as well that consequence operators provide new preorders [3]:

Theorem 3.17. *Let* $(L, \mathfrak{F}, C)$ *be a structure of consequences. The binary relation* $\leqslant_C$*, defined in* $L_{\mathfrak{F}} = \{p \in L; \{p\} \in \mathfrak{F}\}$*, by*

$$p \leqslant_C q \iff q \in C(p), \quad \textit{where } C(p) = C(\{p\}),$$

verifies,

- $\leqslant_C$ *is a preorder in* $L_{\mathfrak{F}}$.
- *It is* $C_{\leqslant_C} \subseteq C|_{\mathfrak{F}^*}$*, with* $C|_{\mathfrak{F}^*}$ *the restriction of* C *to* $\mathfrak{F}^* \subseteq \mathfrak{F}$.
- $\leqslant_{C_{\leqslant_C}} = \leqslant_C$.

Proof. Obvious. □

This last result does obviously hold, for example, if $\mathfrak{F} = \mathbb{P}(L)$, in which case it is $L_{\mathfrak{F}} = L$ and $\mathfrak{F}^* = \mathfrak{F}$.

Regarding the concept of $\cdot$-compatibility introduced in Definition 3.7, and considering structures of consequences $(L, \mathfrak{F}, C_{\leqslant})$ where the preset L is endowed with an inf-operation $\cdot$, it appears that, in general, $C_{\leqslant}$ is not $\cdot$-compatible. For example, choosing $P = \{e,f\} \in \mathbb{P}_{NC}(L)$ in the ortholattice of Fig. 1, it is $e \cdot f = b$, but neither $e \leqslant b$, nor $f \leqslant b$, that is, $e \cdot f \notin C_{\leqslant}(\{e,f\})$. Note that this implies that the sets $C_{\leqslant}(P)$ are not necessarily filters (see Definition 3.9 and comment below). However, since $C_{\leqslant}(P)$ can be written as $C_{\leqslant}(P) = \cup_{p\in P} C_{\leqslant}(p)$ (Theorem 3.16), it results that $C_{\leqslant}(P)$ is the union of the principal filters generated by the individual premises.

3.2.2. The consequence operator $C_{\bullet}$

Let $(L, \leqslant, \cdot)$ be a preset endowed with an inf-operation. Then (see [3]) it is possible to define the operator $C_{\bullet} : \mathbb{P}(L) \to \mathbb{P}(L)$, given by

$$C_{\bullet}(P) = \{q \in L; \exists \{p_1, p_2, \ldots, p_n\} \subseteq P : p_1 \cdot p_2 \cdots p_n \leqslant q\}.$$

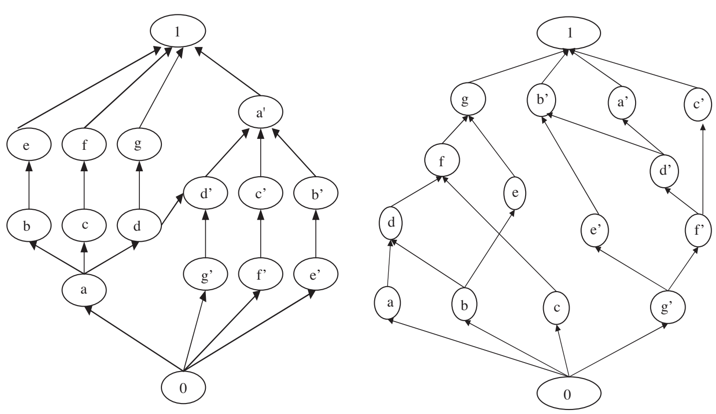

Fig. 1. Examples of a poset (left) and an ortholattice (right).

Remark 3.18

1. $C_{\leqslant} \subseteq C_{\bullet}$, i.e., $C_{\leqslant}(P) \subseteq C_{\bullet}(P)$ for any $P \in \mathbb{P}(L)$.
2. $C_{\bullet}(\{p_{\mathbf{1}}, \ldots, p_n\}) = C_{\leqslant}(p_{\mathbf{1}} \cdots p_n)$ for any $p_{\mathbf{1}}, \ldots, p_n \in L$.
3. If P is totally ordered, then $C_{\bullet}(P) = C_{\leqslant}(P)$.

Similarly to $C_{\leqslant}$, the operator $C_{\bullet}$ may be used to build structures of consequences:

Theorem 3.19. *Let $(L, \leqslant, \cdot)$ be a preset endowed with an inf-operation and let $\mathfrak{F} \subseteq \mathbb{P}(L)$ be such that $C_{\bullet}(P) \in \mathfrak{F}$ for all $P \in \mathfrak{F}$. Then $C_{\bullet} : \mathfrak{F} \to \mathfrak{F}$ is an operator of consequences for $\mathfrak{F}$ in L.*

Proof. If $p \in P$, it is $p \leqslant p$, so $p \in C_{\bullet}(P)$. Hence, $P \subseteq C_{\bullet}(P)$. If $P \subseteq Q$, it is obvious that $C_{\bullet}(P) \subseteq C_{\bullet}(Q)$. Finally, if $q \in C_{\bullet}^2(P)$, there exist $p_1, p_2, \ldots, p_n \in C_{\bullet}(P)$ such that $p_1 \cdot p_2 \cdots p_n \leqslant q$. Now, since $p_i \in C_{\bullet}(P)$ for $1 \leqslant i \leqslant n$, there exist $p_{i_1}, p_{i_2}, \ldots, p_{i_{k_i}} \in P$ such that $p_{i_1} \cdot p_{i_2} \cdots p_{i_{k_i}} \leqslant p_i$. Hence $(p_{1_1} \cdot p_{1_2} \cdots p_{1_{k_1}}) \cdots (p_{n_1} \cdot p_{n_2} \cdots p_{n_{k_n}}) \leqslant p_1 \cdots p_n \leqslant q$, and then $q \in C_{\bullet}(P)$. That is, $C_{\bullet}^2(P) \subseteq C_{\bullet}(P)$, and $C_{\bullet}^2(P) = C_{\bullet}(P)$. $\square$

Therefore, $C_{\bullet}$ is a consequence operator for any $\mathfrak{F}$ verifying $C_{\bullet}(P) \in \mathfrak{F}$ whenever $P \in \mathfrak{F}$. The next result establishes which families $\mathfrak{F}$ verify this property and analyzes the consistency of the resulting operators.

Theorem 3.20. *Let $(L, \leqslant, \cdot, ')$ be a preset endowed with an inf-operation and a negation. Then:*

1. *$(L, \mathbb{P}_{SC}(L), C_{\bullet})$ is a structure of consequences if and only if there does not exist $l_1, \ldots, l_k \in L$ verifying $[\forall i \in \{1, \ldots, k\}, l_i \nleqslant l_i']$ and $l_1 \cdots l_k \leqslant (l_1 \cdots l_k)'$. Whenever $(L, \mathbb{P}_{SC}(L), C_{\bullet})$ is a structure of consequences, then $C_{\bullet}$ is consistent in $\mathbb{P}_{SC}(L)$.*
2. *$(L, \mathbb{P}_{NC}(L), C_{\bullet})$ is a structures of consequences if and only if there does not exist $l_1, \ldots, l_k \in L$ verifying $[\forall i, j \in \{1, \ldots, k\}, l_i \nleqslant l_j']$ and $l_1 \cdots l_k \leqslant (l_1^* \cdots l_m^*)'$ for some $\{l_1^*, \ldots, l_m^*\} \subseteq \{l_1, \ldots, l_k\}$. Whenever $(L, \mathbb{P}_{NC}(L), C_{\bullet})$ is a structure of consequences, then $C_{\bullet}$ is consistent in $\mathbb{P}_{NC}(L)$.*
3. *$(L, \mathbb{P}_{iC}(L), C_{\bullet})$ is a structure of consequences and $C_{\bullet}$ is consistent in $\mathbb{P}_{iC}(L)$.*
4. *If $(L, \leqslant)$ is an inf-*-complete poset, then $(L, \mathbb{P}_0(L), C_{\bullet})$ is a structure of consequences, and $C_{\bullet}$ is consistent in $\mathbb{P}_0(L)$ if and only if L verifies the non-contradiction law.*

Proof

1. Let us first suppose that there exist $l_{\mathbf{1}}, \ldots, l_k \in L$ verifying the stated properties. Then $P = \{l_{\mathbf{1}}, \ldots, l_k\}$ is such that $P \in \mathbb{P}_{SC}(L)$ and $l_{\mathbf{1}} \cdots l_k \in C_{\bullet}(P)$, and then $l_{\mathbf{1}} \cdots l_k \leqslant (l_{\mathbf{1}} \cdots l_k)'$ means that $C_{\bullet}(P) \notin \mathbb{P}_{SC}(L)$, proving that $C_{\bullet}$ is not a structure of consequences in $\mathbb{P}_{SC}(L)$. Now, if $C_{\bullet}$ is not a structure of consequences in $\mathbb{P}_{SC}(L)$, it implies that for some $P \in \mathbb{P}_{SC}(L)$, it exists $q \in C_{\bullet}(P)$, such that $q \leqslant q'$. So, there exists $\{p_{\mathbf{1}}, \ldots, p_n\} \subseteq P$ such that $p_{\mathbf{1}} \cdots p_n \leqslant q$, or $q' \leqslant (p_{\mathbf{1}} \cdots p_n)'$, as $q \leqslant q'$, it is $p_{\mathbf{1}} \cdots p_n \leqslant (p_{\mathbf{1}} \ldots p_n)'$. Since $p_i \in P$ and $P \in \mathbb{P}_{SC}(L)$, it is $p_i \nleqslant p_i'$ for all $i \in \{1, \ldots, n\}$.
   Let us finally prove that $C_{\bullet}$ is consistent in $\mathbb{P}_{SC}(L)$, let us suppose that there exists $q, q' \in C_{\bullet}(P)$, so, there exist $p_{\mathbf{1}}, \ldots, p_r \in P$ such that $p_{\mathbf{1}} \cdots p_r \leqslant q$, or $q' \leqslant (p_{\mathbf{1}} \cdots p_r)'$. Similarly, there exist $p_1^*, \ldots, p_n^* \in P$ such that $p_1^* \cdots p_n^* \leqslant q'$. Then, by the transitivity of $\leqslant$, it would be $p_1^* \cdots p_n^* \leqslant (p_1 \cdots p_r)'$ and hence $p_1 \cdots p_r \cdot p_1^* \cdots p_n^* \leqslant (p_1 \cdots p_r \cdot p_1^* \cdots p_n^*)'$. The contradiction cames from $p_1 \cdots p_r \cdot p_1^* \cdots p_n^* \in C_{\bullet}(P)$, what implies $C_{\bullet}(P) \notin \mathbb{P}_{SC}(L)$.
2. If $l_{\mathbf{1}}, \ldots, l_k \in L$ verify the two stated properties, then $P = \{l_{\mathbf{1}}, \ldots, l_k\}$ is such that $P \in \mathbb{P}_{NC}(L)$ and $l_1 \cdots l_k, l_1^* \cdots l_m^* \in C_{\bullet}(P)$, and then $l_1 \cdots l_k \leqslant (l_1^* \cdots l_m^*)'$ entails that $C_{\bullet}(P) \notin \mathbb{P}_{NC}(L)$. Reciprocally, if there exists $P \in \mathbb{P}_{NC}(L)$ such that $C_{\bullet}(P) \notin \mathbb{P}_{NC}(L)$, then there exist $q, q^* \in C_{\bullet}(P)$ such that $q \leqslant (q^*)'$. But the fact $q, q^* \in C_{\bullet}(P)$ implies the existence of $p_1, \ldots, p_r, p_1^*, \ldots, p_s^* \in P$ such that $p_{\mathbf{1}} \cdots p_r \leqslant q$ and $p_1^* \cdots p_s^* \leqslant q^*$, or $(q^*)' \leqslant (p_1^* \cdot \ldots \cdot p_s^*)'$, and by transitivity, it is $p_1 \cdot \ldots \cdot p_r \leqslant (p_1^* \cdot \ldots \cdot p_s^*)'$. Then the set $\{p_1, \ldots, p_r, p_1^*, \ldots, p_s^*\} \subseteq L$ verifies the two conditions given in the characterization.
   Finally, the fact that $C_{\bullet}$ is consistent in $\mathbb{P}_{NC}(L)$ has already been proved in the first item, since $C_{\bullet}(P) \notin \mathbb{P}_{SC}(L)$ implies $C_{\bullet}(P) \notin \mathbb{P}_{NC}(L)$.
3. If $C_{\bullet}(P) \notin \mathbb{P}_{iC}(L)$, it means that there exists $\{q_1, q_2, \ldots, q_r\}, \{q_1^*, q_2^*, \ldots, q_n^*\} \in C_{\bullet}(P)$ such that $q_1^* \cdot q_2^* \cdots q_n^* \leqslant (q_1 \cdot q_2 \cdots q_r)'$. Since $q_i, q_j^* \in C_{\bullet}(P)$, there exist $p_{i,1}, \ldots, p_{i,k_i} \in P$ such that $p_{i,1} \cdots p_{i,k_i} \leqslant q_i$ for all $i \in \{1, \ldots, r\}$, and there exist $p_{j,1}^*, \ldots, p_{j,k_j}^* \in P$ such that $p_{j,1}^* \cdots p_{j,k_j}^* \leqslant q_j^*$ for all $j \in \{1, \ldots, n\}$. So, $p_{1,1} \cdots p_{1,k_1} \cdots p_{r,1} \cdots p_{r,k_r} \leqslant q_1 \cdots q_r$, and then, because $'$ is a negation, it is $(q_1 \cdots q_r)' \leqslant (p_{1,1} \cdots p_{1,k_1} \cdots p_{r,1} \cdots p_{r,k_r})'$. It is also $p_{1,1}^* \cdots p_{1,k_1}^* \cdots p_{n,1}^* \cdots p_{n,k_n}^* \leqslant q_1^* \cdots q_n^*$, so $p_{1,1}^* \cdots p_{1,k_1}^* \cdots p_{n,1}^* \cdots p_{n,k_n}^* \leqslant q_1^* \cdots q_n^* \leqslant (q_1 \cdots q_r)' \leqslant (p_{1,1} \cdots p_{1,k_1} \cdots p_{r,1} \cdots p_{r,k_r})'$, which is absurd since $P \in \mathbb{P}_{iC}(L)$.
   Finally, the consistency of $C_{\bullet}$ in $\mathbb{P}_{iC}(L)$ follows from its consistency in $\mathbb{P}_{NC}(L)$, since $\mathbb{P}_{iC}(L) \subseteq \mathbb{P}_{NC}(L)$.
4. $P \in \mathbb{P}_0(L)$ implies $C_{\bullet}(P) \in \mathbb{P}_0(L)$ (this easily follows from the fact that $(L, \mathbb{P}_0(L), C_{\wedge})$ is a structure of consequences and $C_{\bullet} \subseteq C_{\wedge}$; both things are proved in Section 3.2.3).
   Consistency is fulfilled whenever L is an inf-$*$-complete poset verifying the law $x \cdot x' = 0$ it is $\mathbb{P}_0(L) \subseteq \mathbb{P}_{iC}(L)$, and as the consistency of $C_{\bullet}$ in $\mathbb{P}_{iC}(L)$ has just been proved, the Remark 3.6 shows the consistency of $C_{\bullet}$ in $\mathbb{P}_0(L)$. To prove that $C_{\bullet}$ is not consistent in the absence of the non-contradiction law, it is equals to the proof of the last item of Theorem 3.13, since $C_{\bullet}(p) = C_{\leqslant}(p)$ for any $p \in L$. $\square$

Corollary 3.21. *$(L, \mathbb{P}_{SC}(L), C_\bullet)$ is never a structure of consequences when L is a non-trivial ortholattice. $(L, \mathbb{P}_{SC}(L), C_\bullet)$ and $(L, \mathbb{P}_{NC}(L), C_\bullet)$ are always consistent structures of consequences when L is a totally ordered set.*

Proof. If L is a non-trivial ortholattice, choosing $P = \{a, a'\}$ for any $a \in L - \{0,1\}$ provides $P \in \mathbb{P}_{SC}(L)$ and $C_\bullet(P) = L \notin \mathbb{P}_{SC}(L)$. If L is a totally ordered set, it is not possible to find elements $l_1, \ldots, l_k \in L$ simultaneously verifying the two conditions given in either item 1 or item 2 of Theorem 3.20, since it is clearly $l_1 \cdots l_j = l_i$ for some $i \in \{1, \ldots, j\}$. □

Example 3.22. Examples of the characterizations given in the two first points of Theorem 3.20 may be found in the presets of Fig. 1, choosing $P = \{b, c\}$ in the left-hand side poset or $P = \{a, b\}$ in the ortholattice depicted on the right.

Contrarily to $C_\leqslant$, the operator $C_\bullet$ is always $\cdot$-compatible (Definition 3.7):

Theorem 3.23. *Let $(L, \leqslant, \cdot)$ be a preset endowed with an inf-operation and let $(L, \mathfrak{F}, C_\bullet)$ be a structure of consequences. Then $C_\bullet$ is $\cdot$-compatible in $\mathfrak{F}$.*

Proof. If $\{q_1, q_2\} \subseteq C_\bullet(P)$, it is $p_1 \cdot p_2 \cdots p_n \leqslant q_1$, and $p_1^* \cdot p_2^* \cdots p_m^* \leqslant q_2$, that imply $p_1 \cdot p_2 \cdots p_n \cdot p_1^* \cdot p_2^* \cdots p_m^* \leqslant q_1 \cdot q_2$, or $q_1 \cdot q_2 \in C_\bullet(P)$. Since $p_1 \cdot p_2 \leqslant p_1, p_1 \cdot p_2 \leqslant p_2$, it is $\{p_1, p_2\} \in C_\bullet(p_1 \cdot p_2)$. □

Thanks to Theorem 3.8, the above result implies that $C_\bullet(\{p_1, p_2, \ldots, p_n\}) = C_\bullet(p_1 \cdot p_2 \cdots p_n)$ for any $p_1, \ldots, p_n$ such that $\{p_1, \ldots, p_n\}, \{p_1 \cdot p_2 \cdots p_n\} \in \mathfrak{F}$.

Note finally that the sets $C_\bullet(P)$ are clearly filters (see Definition 3.9).

3.2.3. The consequence operator $C_\wedge$

Let $(L, \leqslant, \cdot)$ be a poset endowed with an inf-operation which is inf-*-complete. Then it is possible to define the operator $C_\wedge : \mathbb{P}(L) \to \mathbb{P}(L)$, given by

$$C_\wedge(P) = \{q \in L; \mathit{Inf} P \leqslant q\}.$$

It is easy to verify that $C_\wedge(P)$ coincides with $C_\bullet(P)$ whenever P is finite, and that, otherwise, it is $C_\bullet(P) \subseteq C_\wedge(P)$ for any $P \in \mathbb{P}(L)$. Also, clearly $C_\wedge(P) = C_\wedge(\mathit{Inf}\, P)$.

Similarly to $C_\leqslant$ and $C_\bullet$, the operator $C_\wedge$ is a consequence operator for any $\mathfrak{F} \subseteq \mathbb{P}(L)$ where it is well defined:

Theorem 3.24. *Let $(L, \leqslant, \cdot)$ be a poset endowed with an inf-operation which is inf-*-complete, and let $\mathfrak{F} \subseteq \mathbb{P}(L)$ be such that $C_\wedge(P) \in \mathfrak{F}$ for all $P \in \mathfrak{F}$. Then $C_\wedge : \mathfrak{F} \to \mathfrak{F}$ is an operator of consequences for $\mathfrak{F}$ in L.*

Proof. If $p \in P$, it is clearly $\mathit{Inf}\, P \leqslant p$, so $p \in C_\wedge(P)$ and hence $P \subseteq C_\wedge(P)$. If $P \subseteq Q$, it is obvious that $C_\wedge(P) \subseteq C_\wedge(Q)$ since $\mathit{Inf}\, Q \leqslant \mathit{Inf}\, P$. Finally, if $q \in C_\wedge^2(P)$, it is $\mathit{Inf}(C_\wedge(P)) \leqslant q$, but $\mathit{Inf}(C_\wedge(P)) = \mathit{Inf}\, P$, so $q \in C_\wedge(P)$. □

The next Theorem establishes the conditions under which $C_\wedge$ is well defined and consistent. Of course, when L is finite the characterizations appear to be equivalent to those given in Theorem 3.20:

Theorem 3.25. *Let $(L, \leqslant, \cdot, ')$ be a poset endowed with an inf-operation $\cdot$ which is inf-*-complete, and with a negation $'$. Then:*

1. *$(L, \mathbb{P}_{SC}(L), C_\wedge)$ is a structure of consequences if and only if there does not exist $P \subseteq L$ such that $[\forall p \in P, p \nleqslant p']$ and $\mathit{Inf}\, P \leqslant (\mathit{Inf} P)'$. Whenever $(L, \mathbb{P}_{SC}(L), C_\wedge)$ is a structure of consequences, then $C_\wedge$ is consistent in $\mathbb{P}_{SC}(L)$.*
2. *$(L, \mathbb{P}_{NC}(L), C_\wedge)$ is a structure of consequences if and only if there does not exist $P \subseteq L$ such that $[\forall p_1, p_2 \in P, p_1 \nleqslant p_2']$ and $\mathit{Inf}\, P \leqslant (\mathit{Inf}\, Q)'$ for some $Q \subseteq P$. Whenever $(L, \mathbb{P}_{NC}(L), C_\wedge)$ is a structure of consequences, then $C_\wedge$ is consistent in $\mathbb{P}_{NC}(L)$.*
3. *$(L, \mathbb{P}_{iC}(L), C_\wedge)$ is a structure of consequences if and only if there does not exist $P \subseteq L$ such that $[\forall \{p_1, \ldots, p_r\}, \{p_1^*, \ldots, p_n^*\} \subseteq P, p_1^* \cdot \ldots \cdot p_n^* \nleqslant (p_1 \cdot \ldots \cdot p_r)']$ and $\mathit{Inf}\, P \leqslant (\mathit{Inf}\, Q)'$ for some $Q \subseteq P$. Whenever $(L, \mathbb{P}_{iC}(L), C_\wedge)$ is a structure of consequences, then $C_\wedge$ is consistent in $\mathbb{P}_{iC}(L)$.*
4. *$(L, \mathbb{P}_0(L), C_\wedge)$ is a structure of consequences, and $C_\wedge$ is consistent in $\mathbb{P}_0(L)$ if and only if L verifies the non-contradiction law.*

Proof

1. If there exists $P \subseteq L$ verifying the stated conditions, then it is clearly $P \in \mathbb{P}_{SC}(L)$ and $C_\wedge(P) \notin \mathbb{P}_{SC}(L)$, and hence $(L, \mathbb{P}_{SC}(L), C_\wedge)$ is not a structure of consequences. Reciprocally, if there exists $P \subseteq L$ such that $P \in \mathbb{P}_{SC}(L)$ and $C_\wedge(P) \notin \mathbb{P}_{SC}(L)$, it implies the existence of $q \in C_\wedge(P)$ such that $q \leqslant q'$, but then $\mathit{Inf}\, P \leqslant q$, or $q' \leqslant (\mathit{Inf}\, P)'$, so, $\mathit{Inf}\, P \leqslant (\mathit{Inf}\, P)'$.
   Finally, let us suppose that $q, q' \in C_\wedge(P)$ for some $P \in \mathbb{P}_{SC}(L)$ such that $C_\wedge(P) \in \mathbb{P}_{SC}(L)$. Then it would be $\mathit{Inf}\, P \leqslant (\mathit{Inf}\, P)'$, and since $\mathit{Inf}\, P \in C_\wedge(P)$, we would have $C_\wedge(P) \notin \mathbb{P}_{SC}(L)$, which is contradictory with the hypothesis.
2. If there exists $P \subseteq L$ verifying the stated conditions, then it is clearly $P \in \mathbb{P}_{NC}(L)$ and $\mathit{Inf}\, P, \mathit{Inf}\, Q \in C_\wedge(P)$, so $C_\wedge(P) \notin \mathbb{P}_{NC}(L)$, and hence $(L, \mathbb{P}_{NC}(L), C_\wedge)$ is not a structure of consequences. Now, if there exists $P \subseteq L$ such that $P \in \mathbb{P}_{NC}(L)$ and $C_\wedge(P) \notin \mathbb{P}_{NC}(L)$, then there exist $q, q^* \in C_\wedge(P)$ such that $q \leqslant (q^*)'$. But $q, q^* \in C_\wedge(P)$ provides $\mathit{Inf}\, P \leqslant q$ and $\mathit{Inf}\, P \leqslant q^*$ or $(q^*)' \leqslant \mathit{Inf}\, P$, that along with $q \leqslant (q^*)'$ entail $\mathit{Inf}\, P \leqslant (\mathit{Inf}\, P)'$, so it suffices to choose $P = Q$ in order to get the characterization expected.

Note finally that $C_\wedge$ is clearly consistent in $\mathbb{P}_{NC}(L)$ since $\mathbb{P}_{NC}(L) \subseteq \mathbb{P}_{SC}(L)$ and it has just been proved that $C_\wedge$ is consistent in $\mathbb{P}_{SC}(L)$.

3. The proof is analogous to the one given in the previous item. Evidently, if L is finite $(L, \mathbb{P}_{iC}(L), C_\wedge)$ is a structure of consequences, since it is impossible to find any $P \subseteq L$ satisfying the given conditions.
4. To prove that $C_\wedge(P) \in \mathbb{P}_0(L)$ whenever $P \in \mathbb{P}_0(L)$, it suffices to notice that $Inf\,(C_\wedge(P)) \leqslant Inf\,P$ and $Inf\,P \leqslant Inf(C_\wedge(P))$, and then the antisymmetry yields $Inf(C_\wedge(P)) = Inf\,P$ which entails $Inf(C_\wedge(P)) \neq 0$. To prove the consistency of $C_\wedge$ when L verifies the non-contradiction law, it suffices to take $P \in \mathbb{P}_0(L)$ and note that $q, q' \in C_\wedge(P)$ would entail $Inf\,P \leqslant q \cdot q' = 0$, i.e., $Inf\,P = 0$, which is contradictory with $P \in \mathbb{P}_0(L)$. $\square$

Remark 3.26

- $(L, \mathbb{P}_{SC}(L), C_\wedge)$ is never a structure of consequences when L is a non-trivial ortholattice (this is a corollary of the previous theorem, or a trivial consequence of Corollary 3.21).
- The last item in the above Theorem includes the case of the complete ortholattice introduced in [12]. In addition to $C_\wedge$, [12] introduced, also in $\mathbb{P}_0(L)$, the operator $C_{\wedge\vee}(P) = \{q \in L;\ p_\wedge \leqslant q \leqslant p_\vee\}$, with $p_\wedge = Inf\,P$ and $p_\vee = Sup\,P$. $C_{\wedge\vee}$ is also a consistent consequence operator in $\mathbb{P}_0(L)$ and obviously it is $C_{\wedge\vee}(P) \subseteq C_\wedge(P)$.
- If L is a Boolean algebra and $\mathfrak{F} = \mathbb{P}_0(L)$, $C_\wedge$ is the greatest operator of consequences (see [4]).
- Considering $InfP \neq 0$, as it is done in $\mathbb{P}_0(L)$, is interesting in order to avoid contradictions in P, i.e., to avoid the existence of incompatible subsets $\{p_1, p_2, \ldots, p_n\} \subseteq P$ such that $p_1 \cdot p_2 \cdots p_n = 0$, that would entail $C_\wedge(P) = L$.

Example 3.27. The same examples as the ones given for $C_\bullet$ are valid here, since when L is finite it is $C_\wedge = C_\bullet$, and all the sets considered for $C_\bullet$ were finite posets. An example of a poset L (necessarily infinite) such that $(L, \mathbb{P}_{iC}(L), C_\wedge)$ is not a structure of consequences is given by $L = [0,1]$, choosing $x' = 1 - x$ and $P = (0.5, 1]$, which verifies $P \in \mathbb{P}_{iC}(L)$ and $C_\wedge(P) = [0.5, 1] \notin \mathbb{P}_{iC}(L)$, since $0.5 \leqslant (0.5)'$. Note that the same example illustrates that neither $(L, \mathbb{P}_{NC}(L), C_\wedge)$ nor $(L, \mathbb{P}_{SC}(L), C_\wedge)$ are structures of consequences, since clearly $C_\wedge(P) \notin \mathbb{P}_{SC}(L)$.

Similarly to the case of $C_\bullet$, it is easy to prove that $C_\wedge$ is $\cdot$-compatible (according to the compatibility concept given in Definition 3.7), it is clear that if $(L, \leqslant, \cdot)$ is a poset endowed with an inf-operation which is inf-*-complete, and $(L, \mathfrak{F}, C_\wedge)$ is a structure of consequences, then $C_\wedge$ is $\cdot$ -compatible in $\mathfrak{F}$.

Then, following Theorem 3.8, it is $C_\wedge(\{p_1, p_2, \ldots, p_n\}) = C_\wedge(p_1 \cdot p_2 \cdots p_n)$ for any $p_1, \ldots, p_n$ such that $\{p_1, \ldots, p_n\}, \{p_1 \cdot p_2 \cdots p_n\} \in \mathfrak{F}$. Note also that, as it happens with $C_\bullet(P)$, the set $C_\wedge(P)$ is always a filter. In addition, it coincides with the principal filter generated by $InfP$ (recall that for any P it is $C_\wedge(P) = C_\wedge(InfP)$).

Table 1 summarizes the main results obtained in this section regarding the definition of the operators $C_\leqslant \subseteq C_\bullet \subseteq C_\wedge$ in the different families $\mathfrak{F} \subseteq \mathbb{P}(L)$, as well as their consistency.

Let us finish this section briefly analyzing what happens when dealing with singleton premises:

- Obviously, if $\{p\} \in \mathfrak{F}$, $C_\wedge(p) = C_\leqslant(p) = C_\bullet(p)$ and $C_{\wedge\vee}(p) = \{p\}$.
- The above equalities imply $\cup_{p\in P} C_\bullet(p) = \cup_{p\in P} C_\wedge(p) = \cup_{p\in P} C_\leqslant(p)$, and since it is $\cup_{p\in P} C_\leqslant(p) = C_\leqslant(P)$ (Theorem 3.16) and $C_\leqslant \subseteq C_\bullet \subseteq C_\wedge$, it results $\cup_{p\in P} C_\bullet(p) \subseteq C_\bullet(P)$ and $\cup_{p\in P} C_\wedge(p) \subseteq C_\wedge(P)$. Nevertheless, contrary to what happens with $C_\leqslant$, it is not necessarily $\cup_{p\in P} C_\bullet(p) = C_\bullet(P)$ or $\cup_{p\in P} C_\wedge(p) = C_\wedge(P)$, i.e., neither $C_\bullet(P)$ nor $C_\wedge(P)$ are, in general, reducible to the union of the consequences of the elements in P. Indeed, consider the ortholattice to the right of Fig. 1 and take $P = \{d, e\}$. It is $C_\bullet(d) = \{d, f, g, 1\}$, $C_\bullet(e) = \{e, g, 1\}$ and $C_\bullet(P) = C_\bullet(d) \cup C_\bullet(e) \cup \{b\}$. Exactly the same example proves that $C_\wedge(P)$ is not always reducible.
- Regarding the relation $\leqslant_C$ defined in Theorem 3.17, it is clear that $\leqslant_{C_\leqslant} = \leqslant_{C_\bullet} = \leqslant_{C_\wedge} = \leqslant$.

4. Conjectures and refutations in presets

The algebraic models proposed in [12,13] within the framework of ortholattices dealt not only with consequences but also with a broader set, the set of *conjectures*, made of, in addition to consequences, *hypotheses* and *speculations* (or speculative conjectures as they were called in [12]). In [14] the model was enlarged with the so-called *refutations*, defined as the set made of all elements which are not conjectures.

Table 1
Definition and consistency of $C_\leqslant$, $C_\bullet$ and $C_\wedge$.

	$\mathbb{P}_0(L)$	$\mathbb{P}_{iC}(L)$	$\mathbb{P}_{NC}(L)$	$\mathbb{P}_{SC}(L)$	$\mathbb{P}(L)$
$C_\leqslant$	ø	✔	✔	$\otimes$	$\odot$
$C_\bullet$	ø	✔	$\times$	$\times$	$\odot$
$C_\wedge$	ø	$\times$	$\times$	$\times$	$\odot$

$\times$: C is not necessarily a consequence operator in $\mathfrak{F}$; ✔: consistent; ø : consistent in inf-*-complete posets verifying $x \cdot x' = 0$; $\otimes$: not necessarily consistent and $\odot$: not consistent.

Provided the information available on something is given by a set P of non-contradiction statements, the conjectures from P are understood as the statements that can not deductively refute P, that is, those whose negation can not follow from P. Since deduction is modeled by operators of consequence (C), let's state that q refutes P ($q \in Ref(P)$) when its negation q' is in $C(P)$, and that conjectures are in the set $\{q \in L; q' \in C(P)\}^c = Ref(P)^c = \{q \in L; q' \notin C(P)\}$. Notwithstanding, this idea does not always guarantee that consequences are conjectures, for instance, in the case C is not consistent in P. Then, the definition:

$$Conj_C(P) = C(P) \cup \{q \in L; q' \notin C(P)\}$$

allows to have $C(P) \subset Conj_C(P)$. Hence, in the set $Conj_C(P) - C(P)$ is possible to identify the elements from which the premises in P can be deduced (hypotheses), and those (speculations) that are neither consequences, nor hypotheses. In this way, consequences appear as safe or necessary conjectures, hypotheses and speculations as unsafe or contingent ones, and if C is consistent in P is $Conj_C(P) = \{q \in L; q' \notin C(P)\}$.

In the following we propose a generalization of all these concepts to the case of presets, starting from an abstract consequence operator.

4.1. Definitions and main properties

In this section $(L, \leqslant, ')$ will be a preset endowed with a negation and $(L, \mathfrak{F}, C)$ will be a structure of consequences.

Definition 4.1 (*Conjectures*). Given $P \in \mathfrak{F}$, *the set of C-conjectures of P*, denoted by $Conj_C(P)$, is defined as

$$Conj_C(P) = \{q \in L; q \in C(P) \text{ or } q' \notin C(P)\}.$$

Obviously, $C(P) \subseteq Conj_C(P)$, and since $P \subseteq C(P)$, it results $P \subseteq C(P) \subseteq Conj_C(P)$.

Remark 4.2. In [13] L was taken as a complete ortholattice, $\mathfrak{F}$ was taken as $\mathbb{P}_0(L)$, and the set of conjectures associated to a consequence operator $C : \mathbb{P}_0(L) \to \mathbb{P}_0(L)$ was defined as $Conj_C(P) = \{q \in L; q' \notin C(P)\}$. Definition 4.1 coincides with the latter when dealing with complete ortholattices and $\mathfrak{F} = \mathbb{P}_0(L)$. Indeed, in such cases it is always $C(P) \subseteq \{q \in L; q' \notin C(P)\}$, because if it was $q \in C(P)$ such that $q' \in C(P)$, it would be $Inf\ C(P) = 0$ (since $q \cdot q' = 0$ is true in any ortholattice), and this would be contradictory with $C(P) \in \mathbb{P}_0(L)$.

If C is a consistent consequence operator (Definition 3.4), the set $Conj_C(P)$ clearly reduces to $Conj_C(P) = \{q \in L; q' \notin C(P)\}$. Note also that consistency ensures anti-monotonicity of conjectures, as the two following results show:

Theorem 4.3. *Let $(L, \mathfrak{F}, C)$ be a structure of consequences where C is consistent.*

- *Then for any $P, Q \in \mathfrak{F}$ such that $P \subseteq Q$ it is $Conj_C(Q) \subseteq Conj_C(P)$. That is, the operator $Conj_C$ is anti-monotonic.*
- *Let $(L, \mathfrak{F}, C^*)$ be another structure of consequences where C^* is consistent and verifying $C \subseteq C^*$. Then $Conj_{C^*} \subseteq Conj_C$.*

Proof. Obvious. □

Corollary 4.4. $Conj_{C_\wedge}(P) \subseteq Conj_{C_\bullet}(P) \subseteq Conj_{C_\leqslant}(P) \subseteq Conj_{C_0}(P)$, *provided the consequence operators are consistent for P.*

Proof. Follows from the chain $C_0 \subseteq C_\leqslant \subseteq C_\bullet \subseteq C_\wedge$. □

Within the set $Conj_C(P)$, and apart from the consequences $C(P)$, the set of *C-hypotheses* of P can be defined in the following way:

Definition 4.5 (*Hypotheses*). Given $P \in \mathfrak{F}$, *the set of C-hypotheses of P*, denoted by $Hyp_C(P)$, is defined as

$$Hyp_C(P) = \{q \in Conj_C(P) - C(P); \{q\} \in \mathfrak{F} \quad \text{and} \quad P \subseteq C(q)\},$$

with $C(q) = C(\{q\})$.

Remark 4.6. In [13], choosing L as a complete ortholattice and taking $\mathfrak{F} = \mathbb{P}_0(L)$, the set of hypotheses associated to a consequence operator $C : \mathbb{P}_0(L) \to \mathbb{P}_0(L)$ was defined as $Hyp^*_C(P) = \{q \in L - (\{0\} \cup P \cup \{InfP\}); P \subseteq C(q)\}$. It is also seen from [13] that $Hyp^*_C(P) \subseteq Conj_C(P)$ and $Hyp^*_C(P) \cap C(P) = \emptyset$, so the set $Hyp^*_C(P)$ could have equivalently been defined as $Hyp^*_C(P) = \{q \in Conj_C(P) - (\{0\} \cup C(P)); P \subseteq C(q)\}$. Since in addition it is clearly $\{q\} \in \mathbb{P}_0(L)$ if and only if $q \neq 0$, it appears that Definition 4.5 coincides with the one given in [13] when dealing with complete ortholattices and $\mathfrak{F} = \mathbb{P}_0(L)$.

Obviously, it results $Hyp_C(P) \cap C(P) = \emptyset$, and $C(P) \subseteq C^2(q) = C(q)$, that is, the consequences of P are also consequences of all $q \in Hyp_C(P)$. Since in addition it is $Hyp_C(P) \subseteq Conj_C(P)$, the hypotheses are those conjectures whose consequences contain all the consequences of P, that is, those from which all the consequences of P deductively 'follow'. In this sense, the hypotheses are conjectures 'explaining' both P and what 'follows' from P (the available information on something).

Similarly to what happens with conjectures, the operator Hyp_C is anti-monotonic, but this is obviously true even in the absence of consistency, since for any $P, Q \in \mathfrak{F}$ such that $P \subseteq Q$ it is $Hyp_C(Q) \subseteq Hyp_C(P)$. That is, the operator Hyp_C is anti-monotonic.

Let us now define the so-called *speculations*, which are those conjectures that are neither consequences nor hypotheses [12,13]:

Definition 4.7 (*Speculations*). Given $P \in \mathfrak{F}$, *the set of C-speculations of P*, denoted by $Sp_C(P)$, is defined as

$$Sp_C(P) = Conj_C(P) - [C(P) \cup Hyp_C(P)]$$

In general, the operator Sp_C is *non-monotonic*, that is, it is neither monotonic nor anti-monotonic (see [13] where this is proved in the ortholattice case).

To summarize, the set $Conj_C(P)$ is partitioned in the form

$$Conj_C(P) = C(P) \cup Hyp_C(P) \cup Sp_C(P)$$

Remark 4.8. Although any consequence operator is defined such that $C(P) \in \mathfrak{F}$ whenever $P \in \mathfrak{F}$, this is not always the case for the sets $Conj_C(P)$, $Hyp_C(P)$ or $Sp_C(P)$, that may not belong to $\mathfrak{F}$ even if P does. As an example, consider L as the Boolean algebra $(\mathbb{P}(E), \cap, \cup, \emptyset, E)$, where E is a non-empty set, with the set complement acting as a negation. Choose $\mathfrak{F} = \mathbb{P}_0(L) = \{P \in \mathbb{P}(L); Inf P \neq \emptyset\}$ and the consequence operator $C_\wedge(P) = \{q \in L; Inf\ P \subseteq q\}$. Now, take $P = \{p_1, p_2\}$ from Fig. 2. Clearly, it is $P \in \mathbb{P}_0(L)$, since $Inf\ P = p_1 \cap p_2 \neq \emptyset$. On the other hand, it is $r_1, r_2 \in Hyp_C(P)$ and $q_1, q_2 \in Sp_C(P)$, but $r_1 \cap r_2 = q_1 \cap q_2 = \emptyset$ implies $Inf\ Hyp_C(P) = Inf\ Sp_C(P) = Inf\ Conj_C(P) = \emptyset$, or $Hyp_C(P), Sp_C(P), Conj_C(P) \notin \mathbb{P}_0(L)$.

Finally, following what was done in [14] in the context of ortholattices, it is possible to define the set of *C-refutations*, made of those elements whose negations "follow" from P:

Definition 4.9 (*Refutations*). Given $P \in \mathfrak{F}$, *the set of C-refutations of P*, denoted by $Ref_C(P)$, is defined as

$$Ref_C(P) = \{q \in L; q' \in C(P)\}$$

Clearly, for any $P \in \mathfrak{F}$ it is

$$Conj_C(P) \cup Ref_C(P) = L$$

and

$$Conj_C(P) \cap Ref_C(P) = \{q \in C(P); q' \in C(P)\}$$

If C is a consequence operator which is consistent for P, then it is $Conj_C(P) \cap Ref_C(P) = \emptyset$, and hence $L = Conj_C(P) \cup Ref_C(P)$ is a partition, i.e., it verifies the nice property $Ref_C(P) = Conj_C(P)^c$. Fig. 3 illustrates the classification of the set L when the consequence operator C is not consistent for P (left) and when it is consistent for P (right).

Remark 4.10. As it was pointed out in Remark 3.3, when dealing with isomorphic structures of consequences $(L, \mathfrak{F}, C)$ and $(L^*, \mathfrak{F}^*, C^*)$, the set of consequences $C^*(P^*)$ can be calculated from $C(P)$, i.e., if $\mathfrak{f}$ denotes the isomorphism between L and L^* and $\mathfrak{f}_e$ denotes its extension to $\mathbb{P}(L)$, for any $P \in \mathfrak{F}$ it is $C^*(\mathfrak{f}_e(P)) = \mathfrak{f}_e(C(P))$. Furthermore if N is the negation in L, and the negation in L^*, N^*, is taken as $N^* = \mathfrak{f} \circ N \circ \mathfrak{f}^{-1}$, then the sets of conjectures, hypotheses and speculations are also isomorphic. Indeed, $Conj_{C^*}(\mathfrak{f}_e(P))$ can be calculated from $Conj_C(P)$ as follows:

$$\begin{aligned} Conj_{C^*}(\mathfrak{f}_e(P)) &= C^*(\mathfrak{f}_e(P)) \cup \{\mathfrak{f}(q) \in L^*; N^*(\mathfrak{f}(q)) \notin C^*(\mathfrak{f}_e(P))\} = \mathfrak{f}_e(C(P)) \cup \{\mathfrak{f}(q) \in L^*; \mathfrak{f}(N(q)) \notin \mathfrak{f}_e(C(P))\} \\ &= \mathfrak{f}_e(C(P)) \cup \mathfrak{f}_e(\{q \in L; N(q) \notin C(P)\}) = \mathfrak{f}_e(Conj_C(P)) \end{aligned}$$

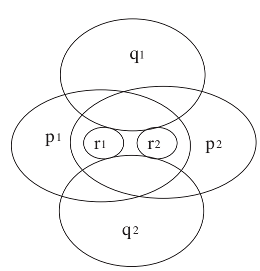

Fig. 2. The family of subsets used in Remark 4.8.

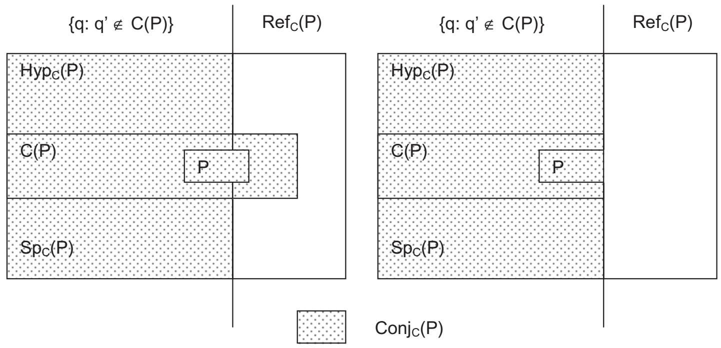

Fig. 3. Classification of L when the consequence operator C is not consistent for P (left) and when it is (right).

Hypotheses and speculations follow an analogous behavior: it is

$$Hyp_{C^*}(\mathfrak{f}_e(P)) = \mathfrak{f}_e(Hyp_C(P)) \quad \text{and} \quad Sp_{C^*}(\mathfrak{f}_e(P)) = \mathfrak{f}_e(Sp_C(P).$$

Referring to consistency, if C is consistent in $\mathfrak{F}$, C^* is also consistent in $\mathfrak{F}^*$.

Therefore, *for isomorphic structures of consequences with isomorphic negations, conjectures, hypotheses and speculations are also isomorphic.*

4.2. Examples

Three different consequence operators, $C_\leqslant$, $C_\bullet$ and $C_\wedge$, have been introduced in Section 3.2. In the following we calculate the sets of conjectures, hypotheses, speculations and refutations associated to them, and point out some of their main properties.

4.2.1. Conjectures and refutations associated to $C_\leqslant$

The following results are easily obtained by applying to $C_\leqslant$ the definitions established in Section 4.1.

Let $(L, \leqslant, ')$ be a preset endowed with a negation and let $(L, \mathfrak{F}, C_\leqslant)$ be a structure of consequences. Then, for any $P \in \mathfrak{F}$:

- $Conj_{C_\leqslant}(P) = \{q \in L; \exists p \in P : p \leqslant q\} \cup \{q \in L; \forall p \in P : p \not\leqslant q'\}$
- $Hyp_{C_\leqslant}(P) = \{q \in L; \{q\} \in \mathfrak{F}, \forall p \in P : (p \not\leqslant q, p \not\leqslant q', q \leqslant p)\}$
- $Sp_{C_\leqslant}(P) = \{q \in L; \forall p \in P : (p \not\leqslant q, p \not\leqslant q'), [\{q\} \notin \mathfrak{F}$ or $\exists p \in P : q \not\leqslant p]\}$
- $Ref_{C_\leqslant}(P) = \{q \in L; \exists p \in P : p \leqslant q'\}$

Remark 4.11

- If $C_\leqslant$ is consistent for P, then:
  - $Conj_{C_\leqslant}(P) = \{q \in L; \forall p \in P : p \not\leqslant q'\}$
  - If L is a Boolean algebra, then $Conj_{C_\leqslant}(P) = \{q \in L; \forall p \in P : p \cdot q \neq 0\}$ (recall that in Boolean algebras $a \leqslant b'$ and $a \cdot b = 0$ are equivalent).
  - For all $P \in \mathfrak{F}$, such that if $p \in P$, then $\{p\} \in \mathfrak{F}$, $Conj_{C_\leqslant}(P) = \cap_{p \in P} Conj_{C_\leqslant}(p)$.
- If L is a poset, the statement $[p \not\leqslant q]$ is equivalent to $[q < p$ or $p\mathrm{NC}q]$, where $p\mathrm{NC}q$ indicates that p and q are not comparable. This entails:
  - $Hyp_{C_\leqslant}(P) = \{q \in L; \{q\} \in \mathfrak{F}, \forall p \in P : (q < p, p \not\leqslant q')\}$
  - If the poset L has first element 0 and verifies the non-contradiction law, then clearly $0 \notin Hyp_{C_\leqslant}(P)$ and $[q \neq 0, q < p]$ implies $p \not\leqslant q'$, so finally

    $$Hyp_{C_\leqslant}(P) = \{q \in L; \{q\} \in \mathfrak{F}, \forall p \in P : 0 < q < p\}$$
- If L has first element 0 and last element 1, then $1 \in C_\leqslant(P)$ and $0 \in Ref_{C_\leqslant}(P)$.

Example 4.12. Let us consider the totally ordered set $([0,1], \leqslant)$ with $' = 1 - id_{[0,1]}$ and $P = \{0.7, 0.9\}$. It is easy to verify that $P \in \mathbb{P}_{iC}([0,1])$, and then, thanks to Theorem 3.13, we know that $C_\leqslant$ is consistent for P.

- $C_\leqslant(P) = [0.7, 1] \cup [0.9, 1] = [0.7, 1]$.
- $Conj_{C_\leqslant}(P) = (0.3, 1] \cap (0.1, 1] = (0.3, 1]$. Note that $Conj_{C_\leqslant}(P) \notin \mathbb{P}_{SC}([0,1])$, since $0.4 \leqslant 0.4' = 0.6$. Then $Conj_{C_\leqslant}(P) \notin \mathbb{P}_{iC}([0,1])$, but it belongs to $\mathbb{P}_0([0,1])$.

- $Hyp_{C_{\leqslant}}(P) = \{q; q \in (0.3, 1] - [0.7, 1]; 1 - q \notin [0.7, 1]\} = (0.3, 0.7)$. Again, $Hyp_{C_{\leqslant}}(P) \notin \mathbb{P}_{SC}([0,1])$, and hence $Hyp_{C_{\leqslant}}(P) \notin \mathbb{P}_{iC}([0,1])$, but $Hyp_{C_{\leqslant}}(P) \in \mathbb{P}_0([0,1])$.
- $Sp_{C_{\leqslant}}(P) = \emptyset$.
- $Ref_{C_{\leqslant}}(P) = Conj_{C_{\leqslant}}(P)^c = [0, 0.3]$ (since $C_{\leqslant}$ is consistent for P).

4.2.2. Conjectures and Refutations associated to $C_\bullet$

Let $(L, \leqslant, \cdot, ')$ be a preset endowed with a negation and an inf-operation and let $(L, \mathfrak{F}, C_\bullet)$ be a structure of consequences. Then, for any $P \in \mathfrak{F}$:

- $Conj_{C_\bullet}(P) = \{q \in L; \exists\{p_1, \ldots, p_n\} \subseteq P : p_1 \cdots p_n \leqslant q\} \cup \{q \in L; \forall\{p_1, \ldots, p_n\} \subseteq P : p_1 \cdots p_n \nleqslant q'\}$
- $Hyp_{C_\bullet}(P) = \{q \in L; \{q\} \in \mathfrak{F}, \forall\{p_1, \ldots, p_n\} \subseteq P : (p_1 \cdots p_n \nleqslant q', p_1 \cdots p_n \nleqslant q), \forall p \in P : q \leqslant p\}$
- $Sp_{C_\bullet}(P) = \{q \in L; \forall\{p_1, \ldots, p_n\} \subseteq P : (p_1 \cdots p_n \nleqslant q', p_1 \cdots p_n \nleqslant q), [\{q\} \in \mathfrak{F}$ or $\exists p \in P : q \nleqslant p]\}$
- $Ref_{C_\bullet}(P) = \{q \in L; \exists\{p_1, \ldots, p_n\} \subseteq P : p_1 \cdots p_n \leqslant q'\}$

Remark 4.13. If $C_\bullet$ is consistent for P, then:

- $Conj_{C_\bullet}(P) = \{q \in L; \forall\{p_1, \ldots, p_n\} \subseteq P : p_1 \cdot \ldots \cdot p_n \nleqslant q'\}$
- If L is a Boolean algebra, then $Conj_{C_\bullet}(P) = \{q \in L; \forall\{p_1, \ldots, p_n\} \subseteq P : p_1 \cdots p_n \cdot q \neq 0\}$.

Remark 4.14. All the sets calculated in Example 4.12 for $C_{\leqslant}$ are valid for $C_\bullet$, since the set $P = \{0.7, 0.9\}$ is totally ordered (see Remark 3.18).

4.2.3. Conjectures and refutations associated to $C_\wedge$

Let $(L, \leqslant, \cdot, ')$ be a poset endowed with an inf-operation which is inf-$*$-complete and with a negation, and let $(L, \mathfrak{F}, C_\wedge)$ be a structure of consequences. Then, for any $P \in \mathfrak{F}$:

- $Conj_{C_\wedge}(P) = \{q \in L; InfP \leqslant q\} \cup \{q \in L; InfP \nleqslant q'\}$
- $Hyp_{C_\wedge}(P) = \{q \in L; \{q\} \in \mathfrak{F}, q < InfP, InfP \nleqslant q'\}$
- $Sp_{C_\wedge}(P) = \{q \in L; InfP \nleqslant q, InfP \nleqslant q', [\{q\} \notin \mathfrak{F}$ or $q \nleqslant InfP]\}$
- $Ref_{C_\wedge}(P) = \{q \in L; InfP \leqslant q'\}$

To calculate $Hyp_{C_\wedge}(P)$ and $Sp_{C_\wedge}(P)$ it suffices to take into account that $[\forall p \in P{:}q \leqslant p]$ is equivalent to $q \leqslant Inf\ P$, or $[\exists p \in P{:}q \nleqslant p]$ is equivalent to $q \nleqslant Inf\ P$.

Remark 4.15

- If $C_\wedge$ is consistent for P, then:
  - $Conj_{C_\wedge}(P) = \{q \in L; InfP \nleqslant q'\}$.
  - If L is a Boolean algebra, then $Conj_{C_\wedge}(P) = \{q \in L; InfP \cdot q \neq 0\}$.
- If L verifies the non-contradiction law, then:

$$Hyp_{C_\wedge}(P) = \{q \in L; \{q\} \in \mathfrak{F}, 0 < q < InfP\},$$

and hence $Hyp_{C_\wedge} = Hyp_{C_{\leqslant}}$.

- If L is a complete ortholattice and $\mathfrak{F} = \mathbb{P}_0(L)$, then all the sets given in Section 4.2.3 coincide with those defined in [12].

Example 4.16. Let us use the results obtained in this Section in order to calculate the conjectures (consequences, hypotheses and speculations) and the refutations of the set of premises $P = \{e, f\}$ in the ortholattice (and hence, a poset verifying the non-contradiction law) at the right of Fig. 1. It is easy to check that $P \in \mathbb{P}_{iC}(L)$, and then, according to Section 3, the three operators $C_{\leqslant}, C_\bullet$ and $C_\wedge$ are consistent for P. Also, since L is finite, it is $C_\bullet = C_\wedge$, so we only need to deal with one of them. For the consequences operator $C_{\leqslant}$, it is:

- $C_{\leqslant}(P) = \{q \in L; \exists p \in P{:}p \leqslant q\} = \{e, f, g, 1\}$
- $Conj_{C_{\leqslant}}(P) = \{q \in L; \forall p \in P : p \nleqslant q'\} = \{q \in L; \exists p \in P : p \leqslant q'\}^c = \{g', 0\}^c = \{a, b, c, d, e, f, g, a', b', c', d', e', f', 1\}$
- $Hyp_{C_{\leqslant}}(P) = \{q \in L; \{q\} \in \mathbb{P}_{iC}(L), \forall p \in P : 0 < q < p\} = \{b\}$
- $Sp_{C_{\leqslant}}(P) = Conj_{C_{\leqslant}}(P) - [C_{\leqslant}(P) \cup Hyp_{C_{\leqslant}}(P)] = \{a, c, d, a', b', c', d', e', f'\}$
- $Ref_{C_{\leqslant}}(P) = \{q \in L; \exists p \in P : p \leqslant q'\} = \{0, g'\}$

For the consequences operator $C_\wedge$, it is:

- $C_\wedge(P) = \{q \in L; b \leqslant q\} = \{b, d, e, f, g, 1\}$
- $Conj_{C_\wedge}(P) = \{q \in L; b \nleqslant q'\} = \{a, b, c, d, e, f, g, a', b', c', 1\}$

- $Hyp_{C_\wedge}(P) = \{q \in L; \{q\} \in \mathbb{P}_{iC}(L), 0 < q < b\} = \emptyset$
- $Sp_{C_\wedge}(P) = Conj_{C_\wedge}(P) - [C_\wedge(P) \cup Hyp_{C_\wedge}(P)] = \{a, c, a', b', c'\}$
- $Ref_{C_\wedge}(P) = \{q \in L; b \leqslant q'\} = \{0, d', e', f', g'\}$

5. The case of fuzzy sets

The simplest algebra of fuzzy sets, the one introduced by Zadeh [15] in 1965, corresponds to the set $L = [0,1]^X = \{\mu; \mu{:}X \to [0,1]\}$ (where X is the universe of discourse), endowed with the pointwise partial order

$$\mu \leqslant \sigma \iff \mu(x) \leqslant \sigma(x) \quad \forall x \in X$$

whose first element is μ_0 ($\mu_0(x) = 0, \forall x \in X$), or $\mu_0 = \mu_\emptyset$, and whose last element is μ_1 ($\mu_1(x) = 1, \forall x \in X$), or $\mu_1 = \mu_X$. The only inf-operation on the poset $([0,1]^X, \leqslant)$ is $\cdot = \min$, defined as

$$(\mu \cdot \sigma)(x) = \min(\mu(x), \sigma(x)) \quad \forall x \in X$$

The set is inf-$*$-complete, and among the negations $'$ on $[0,1]^X$, the most considered (see e.g. [6]) are those functionally given by

$$\mu'(x) = \varphi^{-1}(1 - \varphi(\mu(x))) \quad \forall x \in X$$

with φ an order-automorphism of the totally ordered unit interval $([0,1], \leqslant)$, that is, a function $\varphi{:}[0,1] \to [0,1]$, strictly increasing verifying the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$. All these negations, known as *strong negations*, verify the involutive property $\mu'' = \mu$, for all $\mu \in [0,1]^X$, but none of them verifies the non-contradiction law along with the given inf-operation. The simplest is the so-called *standard negation*, defined as $\mu'(x) = 1 - \mu(x)$ for all $x \in X$, obtained by means of the identity function $\varphi(x) = x$. Similarly to min, it is also possible to equip $[0,1]^X$ with the sup-operation max, and it appears that $([0,1]^X, \min, \max, ')$ is a De Morgan algebra (see e.g. [6]).

CHC models with fuzzy sets are considered in [2,9], where the case of residuated lattices, $(L, \vee, \wedge, \otimes, \to; 0, 1)$, is taken into account and it is proven that in complete residuated lattices, with an involutive negation, consequences for the operator $C_\wedge$ in $\mathfrak{F} = \mathbb{P}_0(L)$ can be written as those elements, q, for which $q' \otimes \inf P = 0$, which implies the consistency of the consequences structure $(L, \mathbb{P}_0(L), C_\wedge)$.

Focussing on the aim of the present paper, what is relevant is that $([0,1]^X, \leqslant, \min, ')$ is an infinite bounded poset which is inf-$*$-complete, so the results proposed in this paper provide a framework for conjecturing from fuzzy sets. In the following we apply the main results of Sections 3 and 4 to $([0,1]^X, \leqslant, \min, ')$ where, for simplicity reasons, $'$ is taken as the standard negation (the results obtained can be easily generalized to the case of arbitrary strong negations).

Let us first discuss which families $\mathfrak{F} \subseteq \mathbb{P}([0,1]^X)$ may be considered. Since $([0,1]^X, \leqslant, \min, ')$ is equipped with an inf-operation and is inf-$*$-complete, any of the families $\mathfrak{F}$ introduced at the beginning of Section 3.2 may be used when dealing with fuzzy sets. They may be written as follows, taking into account that for any $\mu \in [0,1]^X$ $[\mu \leqslant \mu']$ is equivalent to $[\forall x \in X{:}\mu(x) \leqslant 0.5]$.

1. $\mathbb{P}_{SC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu \in P, \exists x \in X : \mu(x) > 0.5\}$
2. $\mathbb{P}_{NC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu, \sigma \in P, \exists x \in X : \mu(x) + \sigma(x) > 1\}$
3. $\mathbb{P}_{iC}([0,1]^X) = \{P \subseteq [0,1]^X; \forall \{\mu_1, \ldots, \mu_r\}, \{\sigma_1, \ldots, \sigma_n\} \in P, \exists x \in X : \min(\mu_1(x), \ldots, \mu_r(x)) + \min(\sigma_1(x), \ldots, \sigma_n(x)) > 1\}$
4. $\mathbb{P}_0([0,1]^X) = \{P \subseteq [0,1]^X; \exists x \in X : (\mathit{Inf}P)(x) \neq 0\}$

It could also be interesting to consider the following important family $\mathfrak{F}$ of fuzzy sets:

$$\mathfrak{F} = \mathbb{P}_n([0,1]^X) = \{P \subseteq [0,1]^X; \forall \mu \in P, \mu \text{ is normalized}\} = \{P \subseteq [0,1]^X; \forall \mu \in P, \exists x \in X, \mu(x) = 1\},$$

made of those fuzzy sets that are not only non-selfcontradictory for any strong negation (i.e., they verify $\mu \nleqslant \mu'$ for any strong negation $'$), but also have at least one prototype in X (the $x \in X$ such that $\mu(x) = 1$). Clearly, $\mathbb{P}_n([0,1]^X) \subseteq \mathbb{P}_{SC}([0,1]^X)$ but $\mathbb{P}_n([0,1]^X) \nsubseteq \mathbb{P}_{NC}([0,1]^X)$ (and hence $\mathbb{P}_n([0,1]^X) \nsubseteq \mathbb{P}_{iC}([0,1]^X)$).

Regarding the three consequences operators analyzed in Section 3.2, any of them could be used in $([0,1]^X, \leqslant, \min, ')$, and their behavior easily follows from the results obtained for the general case. Let us, as an example, examine in detail the case of $C_\leqslant$, which is defined as follows for any $P \subseteq [0,1]^X$:

$$C_\leqslant(P) = \{\sigma \in [0,1]^X; \exists \mu \in P : \mu \leqslant \sigma\}.$$

Section 3.2.1, in particular Theorem 3.13, shows that $C_\leqslant$ verifies the following properties:

- $C_\leqslant(P) = [0,1]^X$ for any P such that $\mu_0 \in P$, $C_\leqslant(\mu_1) = \{\mu_1\}$.
- For all $P \in \mathfrak{F}$, $\mu_1 \in C_\leqslant(P)$.
- Since the pointwise order $\leqslant$ is a partial order: $\sigma \in C_\leqslant(\mu)$ and $\mu \in C_\leqslant(\sigma) \iff \mu = \sigma$.
- $C_\leqslant(P) = \cup_{\mu \in P} C_\leqslant(\mu)$, provided $\{\mu\} \in \mathfrak{F}$ for all $\mu \in P$.
- $([0,1]^X, \mathbb{P}_{NC}([0,1]^X), C_\leqslant)$ is a structure of consequences and $C_\leqslant$ is consistent in $\mathbb{P}_{NC}([0,1]^X)$.

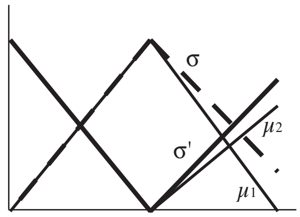

Fig. 4. Fuzzy sets in $\mathbb{P}_n([0,1]^X)$ (proof of Theorem 5.1).

- $([0,1]^X, \mathbb{P}_{SC}([0,1]^X), C_\leqslant)$ and $([0,1]^X, \mathbb{P}([0,1]^X), C_\leqslant)$ are structures of consequences, but $C_\leqslant$ is neither consistent in $\mathbb{P}([0,1]^X)$ nor in $\mathbb{P}_{SC}([0,1]^X)$ (since it is easy to find $\sigma_1, \sigma_2 \in [0,1]^X$ such that $\sigma_1 \leqslant \sigma_2, \sigma_1 \nleqslant \sigma_1'$ and $\sigma_2 \nleqslant \sigma_2'$).
- $([0,1]^X, \mathbb{P}_{iC}([0,1]^X), C_\leqslant)$ is a structure of consequences and $C_\leqslant$ is consistent in $\mathbb{P}_{iC}([0,1]^X)$.
- $([0,1]^X, \mathbb{P}_0([0,1]^X), C_\leqslant)$ is a structure of consequences, but $C_\leqslant$ is not consistent in $\mathbb{P}_0([0,1]^X)$ (since $[0,1]^X$ is a poset that does not verify the non-contradiction law).

Only the following aspect needs to be investigated, because it does not follow from the general results of Section 3.2.1: whether $([0,1]^X, \mathbb{P}_n([0,1]^X), C_\leqslant)$ is a (consistent) structure of consequences:

Theorem 5.1. $([0,1]^X, \mathbb{P}_n([0,1]^X), C_\leqslant)$ *is a structure of consequences, but* $C_\leqslant$ *is not consistent in* $\mathbb{P}_n([0,1]^X)$.

Proof. If $P \in \mathbb{P}_n([0,1]^X)$, then $C_\leqslant(P) \in \mathbb{P}_n([0,1]^X)$. Indeed, if $\sigma \in C_\leqslant(P)$, then there exists $\mu \in P$ such that $\mu \leqslant \sigma$, but since $P \in \mathbb{P}_n([0,1]^X)$, there exists $x \in X$ such that $\mu(x) = 1$, so for that x, it is $\sigma(x) = 1$, and $C_\leqslant(P) \in \mathbb{P}_n([0,1]^X)$. But $C_\leqslant : \mathbb{P}_n([0,1]^X) \to \mathbb{P}_n([0,1]^X)$ is not consistent: for example, taking $P = \{\mu_1, \mu_2\}$ in Fig. 4, it is $\sigma, \sigma' \in C_\leqslant(P)$. $\square$

Let us now use the results of Section 4 in order to calculate the sets of conjectures and refutations associated to $C_\leqslant$ in $[0,1]^X$:

- $Conj_{C_\leqslant}(P) = \{\sigma \in [0,1]^X; \exists \mu \in P : \mu \leqslant \sigma\} \cup \{\sigma \in [0,1]^X; \forall \mu \in P : \mu \nleqslant \sigma'\} = \{\sigma \in [0,1]^X; \exists \mu \in P : \mu \leqslant \sigma\} \cup \{\sigma \in [0,1]^X; \forall \mu \in P : \exists x \in X, \mu(x) + \sigma(x) > 1\}$. Of course, if $C_\leqslant$ is consistent for P, then $Conj_{C_\leqslant}(P) = \{\sigma \in [0,1]^X; \forall \mu \in P : \exists x \in X, \mu(x) + \sigma(x) > 1\}$
- $Hyp_{C_\leqslant}(P) = \{\sigma \in [0,1]^X; \{\sigma\} \in \mathfrak{F}, \forall \mu \in P : (\sigma < \mu, \mu \nleqslant \sigma')\} = \{\sigma \in [0,1]^X; \{\sigma\} \in \mathfrak{F}, \forall \mu \in P : (\sigma < \mu, \exists x \in X : \mu(x) + \sigma(x) > 1)\}$
- $Sp_{C_\leqslant}(P) = \{\sigma \in [0,1]^X; \forall \mu \in P : (\mu \nleqslant \sigma, \mu \nleqslant \sigma'), [\{\sigma\} \notin \mathfrak{F} \text{ or } \exists \mu \in P : \sigma \nleqslant \mu]\}$
- $Ref_{C_\leqslant}(P) = \{\sigma \in [0,1]^X; \exists \mu \in P : \mu \leqslant \sigma'\}$

The following example illustrates all the above results:

Example 5.2. Let $X = \{a, b\}$, and denote by $\mu_{i,j}$ those fuzzy sets such that $\mu_{i,j}(a) = i$, and $\mu_{i,j}(b) = j$, with $i,j \in [0,1]$. Take the set of premises $P = \{\mu_{0.5,1}, \mu_{1,0.3}\} \in \mathbb{P}_{NC}([0,1]^X)$ and the consistent consequence operator $C_\leqslant$ defined in $\mathfrak{F} = \mathbb{P}_{NC}([0,1]^X)$. It is:

- $C_\leqslant(P) = \{\mu_{i,j} \in [0,1]^X;\ \exists \mu \in P : \mu \leqslant \mu_{i,j}\} = \{\mu_{i,1}; i \geqslant 0.5\} \cup \{\mu_{1,j}; j \geqslant 0.3\}$
- $Conj_{C_\leqslant}(P) = \{\mu_{i,j} \in [0,1]^X; \forall \mu \in P : \mu \nleqslant \mu'_{i,j}\} = \{\mu_{i,j} \in [0,1]^X; \mu_{0.5,1} \nleqslant \mu_{1-i,1-j}\} \cap \{\mu_{i,j} \in [0,1]^X; \mu_{1,0.3} \nleqslant \mu_{1-i,1-j}\} = [0,1]^X - [\{\mu_{i,0} : i \leqslant 0.5\} \cup \{\mu_{0,j} : j \leqslant 0.7\}$
- $Hyp_{C_\leqslant}(P) = \{\mu_{i,j}; \{\mu_{i,j}\} \in \mathbb{P}_{NC}([0,1]^X), \forall \mu \in P : (\mu_{i,j} < \mu, \exists x \in X : \mu_{i,j}(x) + \mu(x) > 1)\} = \emptyset$
- $Sp_{C_\leqslant}(P) = Conj_{C_\leqslant}(P) - [C_\leqslant(P) \cup Hyp_{C_\leqslant}(P)] = [0,1]^X - [\{\mu_{i,0} : i \leqslant 0.5\} \cup \{\mu_{0,j} : j \leqslant 0.7\} \cup \{\mu_{i,1}; i \geqslant 0.5\} \cup \{\mu_{1,j}; j \geqslant 0.3\}]$

6. Conclusions

As it was said in the introduction, algebras as the De Morgan ones do not fit in the working hypotheses made in [12,13], since they do not verify the non-contradiction and excluded-middle laws. This lack is overcome in this paper: now the only necessary underlaying structure is a preordered set endowed with a negation, that can be enriched with an inf-operation or upgraded to an inf-*-complete poset. So, this paper studies some properties of CHC models built on preordered sets that are weaker structures than the others where CHC models had been studied before. Furthermore, in order to keep some properties that hold in stronger structures, this paper considers *consistent* operators of consequences. This is the case, for instance, of the non-contradiction principle that in both ortholattices and residuated lattices, allows that the intersection between the negation of a consequence and the available joint information is null or empty.

In addition, three different consequence operators have been analyzed in detail, defining them on different families of subsets:

- $C_\leqslant$, which only provides as consequences those elements 'following' *from some premise*;
- $C_\bullet$, which provides as consequences those elements 'following' *from the conjunction of any finite number of premises*;
- $C_\wedge$, which considers the elements 'following' *from the conjunction of all the premises*.

The operators $C_{\leqslant}$ and $C_{\bullet}$ actually define *partial consequences* of the set of premises.

Conjectures, which are classified in consequences, hypotheses and speculations, have been defined starting from an abstract consequence operator. Although any operator of consequences $C : \mathfrak{F} \to \mathfrak{F}$ allows to take the sets $C(P)$ as new sets of premises, this is not the case with the operators of conjectures, hypotheses, and speculations, that map $\mathfrak{F}$ to $\mathbb{P}(L)$, but not necessarily to $\mathfrak{F}$. Hence, in general, *the sets* $Conj_C(P)$, $Hyp_C(P)$, *and* $Sp_C(P)$, *cannot be taken as new sets of premises*. The sets $\mathfrak{F}$ are also useful to control the consistency of the consequence operator.

Note finally that in the world of formal sciences, it is not surprising that the concept of consequence does precede that of conjecture, or that the formalization of the concepts underlying 'guessing' do come from 'deduction'. Nevertheless, the non-existence of a formal framework in which the particular concept of consequence does follow from that of conjecture as a 'safe' one, is not clear enough. Where and how to define conjectures in such a way that consequences could be derived from them as a particular case, is an open question. This problem can be shortly stated as follows: what comes first, the idea of conjecture or that of consequence?

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References

[1] G. Birkhoff, Lattice Theory, Amer. Math. Society, Colloq. Publs, third ed. (seventh print), 1993.
[2] E. Castiñeira, S. Cubillo, A. Pradera, E. Trillas, On conjectures and consecuences in $F(X)$, in: 19th IEEE-NAFIPS'2000, 2000, pp. 101–106.
[3] J.L. Castro, E. Trillas, Sobre preordenes y operadores de consecuencias de Tarski, Theoria 4 (11) (1989) 419–425 (in Spanish).
[4] A. Fernández Pineda, E. Trillas, C. Vaucheret, Additional comments on conjectures, hypotheses and consequences in orthocomplemented lattices, Artificial Intelligence and Symbolic Computation, Lecture Notes in Artificial Intelligence, Springer, Berlin, 2000, pp. 107–114.
[5] Y. Mingsheng, W. Huaiqing, Lattice-theoretic models of conjectures, hypotheses and consequences, Artificial Intelligence 139 (2002) 253–267.
[6] A. Pradera, E. Trillas, E. Renedo, An overview on the construction of fuzzy set theories, New Mathematics and Natural Computation 1 (3) (2005) 329–358.
[7] D. Qiu, Automata theory based on quantum logic: some characterizations, Information and Computation 190 (2004) 179–195.
[8] D. Qiu, Automata theory based on quantum logic: reversibilities and pushdown automata, Theoretical Computer Science 386 (2007) 38–56.
[9] D. Qiu, A note on Trillas' CHC models, Artificial Intelligence 171 (2007) 239–254.
[10] E. Trillas, A. Álvarez, A.R. de Soto, Short note: counting conjectures, Mathware and Soft Computing 14 (2) (2007) 165–170.
[11] E. Trillas, E. Castiñeira, S. Cubillo, Averaging premises, Mathware and Soft Computing 8 (2001) 83–91.
[12] E. Trillas, S. Cubillo, E. Castiñeira, On conjectures in orthocomplemented lattices, Artificial Intelligence 117 (2) (2000) 255–275.
[13] E. Trillas, M. Mas, M. Monserrat, J. Torrens, Conjecturing from consequences, International Journal of General Systems 38 (5) (2009) 567–578.
[14] E. Trillas, A. Pradera, A. Álvarez, On the reducibility of Hypotheses and Consequences, Information Sciences, forthcoming, doi:10.1016/j.ins.2009.01.008.
[15] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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Mieres, *29 de Junio de 2011*

Itziar García-Honrado, declara haber contribuido tanto en el enunciado y demostración de los teoremas que aparecen en el artículo "Consequences and conjectures in preordered sets", como también en la redacción y las revisiones del mismo.

Y para que así conste lo firma en Mieres a fecha de 29 de Junio de 2011.

Fdo. Itziar García-Honrado

Enric Trillas, como primer firmante del artículo "Consequences and conjectures in preordered sets" y director de la presente Tesis, certifica lo declarado anteriormente.

Y para que así conste lo firma en Mieres a fecha de 29 de Junio de 2011.

Fdo. Enric Trillas

7.2. On an Attempt to Formalize Guessing

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1

On an Attempt to Formalize Guessing

Itziar García-Honrado and Enric Trillas

Summary. Guessing from a piece of information is what humans do in their reasoning processes, and that is why to some extent reasoning and obtaining conjectures can be considered equivalent. Most of these reasoning processes consist in posing new questions whose possible answers are non contradictory with the previous information. This is the idea that allows to introduce different mathematical models by means of conjecture operators, built up depending on how the concept of non-contradiction is understood. The relevant contribution of this chapter is to show that there exist conjecture operators not coming from Traski's operators of consequences, a new result untying the concept of conjecture, as it is introduced, from a previous way of deduction. The concept of a conjecture proves to include those of logical consequences, hypotheses and speculations.

Deduction is a necessary part of Induction
William Whewell [21]

1.1 Introduction

In Science, the method of reasoning is the so called empirical method, based in experiments and different kind of proofs. It allows to build theories, or mathematical models, always subjected to test its provisory validity, or to refuse them. So, this method allows the sequential development of theoretical models in order to get the one that currently better explains the reality. The empirical method at least englobes the following tree general types of reasoning, going from a given body of knowledge to some conclusions,

- *Deduction*, allowing to go from a general to a particular case, by applying known laws, models or theories. So, the conclusions can be called logical consequences, in the sense that they necessarily follow from the available information.
  Therefore, deduction does not allow to get "new" information, but to clearly deploy the known information. Deduction is typical of formal theories.

- *Abduction*, allowing to find contingent explanations for the information. Is a kind of reasoning in which one chooses the hypothesis that could best explain the evidence. It is used to look for hypotheses of a given information that, then, can be deduced from them.
  For instance, in a medical diagnose's problem in which the available information consists in the symptoms: 'fever' and 'sore throat', our hypothesis could be 'anginas', since some symptoms of having anginas are *fever* and *sore throat.*

- *Induction*, which takes us beyond our current evidence, or knowledge, to contingent conclusions about the unknown.
  From particular observations, induction allows us to provisionally establish a 'law' that can explain these observations, and that is a (contingent) conjecture in the sense that it is not contradictory with the observations. This kind of reasoning is typical of Experimental Science.

Once a single hypothesis is selected as the explanation of some evidence, following Popper (see [11], [10]), it is only a provisional explanation that should be submitted to the strongest than possible tests trying to refute it. Before, Popper it was C.S. Peirce who described (see, for instance, [9]) the processes of science as a combination of induction, abduction and deduction.

Hence, *the real process to built up science's models consists on working with conjectures.* That is, building up possible explanations (conjectures called hypotheses) from observations, that can change with new observations. Then, after deducing some necessary consequences of the hypothesis, they must be checked by repeated experiments to test its suitability.

1.1.1

Most of ordinary, everyday, or commonsense reasoning is nothing else than conjecturing or guessing. Often, human reasoning consists in either conjecturing or refuting hypotheses to explain something, or in conjecturing speculations towards some goal. Adding to guessing the reasoning done by similarity, or analogical reasoning, a very big part of ordinary reasoning is obtained.

Only a little part of everyday reasoning could be typified as deductive reasoning, that is typical of formal sciences in the context of proof, like in the case of mathematical proof. Can deductive reasoning be seen as a particular type of conjecturing?

Since human evolution is in debt with the people's capacity for conjecturing and, even more, scientific and technological research is based on systematic processes of guessing and of doing analogies, it seems relevant to study what is a conjecture. How the concept of a conjecture can be described and where and how can it be formalized? Is deduction actually a pre-requisite for the formalization of conjectures?

This paper deals with these questions, and to begin with let us pose a very simple but typical example of an everyday life decision taken on the base of conjecturing. *Why each year many people decide to buy a ticket of the Christmas Lottery?* Since this lottery counts with more than $90,000$ different numbers, the probability of winning the first award is smaller than $1/90,000 = 111{\cdot}10^{-7}$. Such actually small probability does not seem what conducts to the decision of buying a ticket. Instead, it comes from the fact that what is known on the lottery (the previous information) is *not incompatible* with the statement 'I can win the first award'. Hence, this statement is the conjecture on which the decision of buying a ticket is primarily based and that, as the small probability of winning shows, has a big risk.

Human ordinary reasoning and scientific reasoning can be considered as a sum of different kinds of reasoning, induction, deduction, abduction, reasoning by similarities, and also by some intrinsic characteristics of humans [21] such as imagination, inspiration,... In order to show how the model of formalizing guessing can work, it follows an example collected in [5].

Let L be an ortholattice with the elements, **m** for *midday*, **e** for *eclipse*, and **s** for *sunny*, and its corresponding negations, conjunctions and disjunctions. It is known that it is *midday, midday and not sunny*, and *neither is an eclipse nor it is sunny.*

Representing *and* by product, $\cdot$, *or* by sum, $+$, and *not* by $'$, the set of premises is $P = \{m, m{\cdot}s', (e{\cdot}s)'\}$. So, the *résumé* of this information can be identified with $p_\wedge = m{\cdot}m{\cdot}s'{\cdot}(e{\cdot}s)' = m{\cdot}m{\cdot}s'{\cdot}(s' + e') = m{\cdot}s'$.

Among conjectures we can distinguish consequences, hypothesis and speculations. Then, once understood that $a \leq b$ means that b is a logical consequence of a (see [4] for the equivalence of this two notions), it follows,

- *It is not sunny*, s', is a consequence of P, since $p_\wedge = m{\cdot}s' \leq s'$.
- The statements "*it is midday and not sunny and there is an eclipse*", $m{\cdot}s'{\cdot}e$, and "*there is an eclipse*", e, are conjectures of P. Then, since they are not contradictory with $m{\cdot}s'$.
  - $m{\cdot}s'{\cdot}e$ is a hypothesis of P, since $m{\cdot}s'{\cdot}e < m{\cdot}s'$. Therefore, if it is known $m{\cdot}s'{\cdot}e$, it can be deduced all the given information, since: $m{\cdot}s'{\cdot}e \leq m$, $m{\cdot}s'{\cdot}e \leq m{\cdot}s'$, and $m{\cdot}s'{\cdot}e \leq s' \leq s' + e' = (e{\cdot}s)'$.
  - e is a speculation of P, since neither e follows from $m{\cdot}s'$ ($m{\cdot}s' \not\leq e$), nor $m{\cdot}s'$ follows from e ($e \not\leq m{\cdot}s'$). However, asserting that there is an eclipse is non contradictory with the information, P.

- The statements *sunny* and *not midday* are refutations of P, since $m{\cdot}s' \leq s' = (s)'$, and $m{\cdot}s' \leq m = (m')'$. The elements s and m', contradictory with the résumé $p_\wedge$, *refute* the information given by P.

So, and of course in a very restricted and closed framework, this could be a formalization in an ortholattice of a human reasoning.

1.2 Towards the problem: Where can knowledge be represented?

Without representation it cannot be done any formalization process. Although the precise classic and quantum reasoning can be formalized through representations in boolean algebras and orthomodular lattices, respectively, everyday reasoning is neither totally formalizable in these algebraic structures, nor in De Morgan algebras. A reason for this is that the big number of properties they enjoy give a too rigid framework for a type of reasoning in which context, purpose, time, imprecision, uncertainty, and analogy, often play jointly an important role. For instance, when interpreting the linguistic connective *and* by the operation *meet* of these lattices, it is needed a big amount of (not always available) information on the two components of the conjunctive statement to be sure that *and* is its infimum. In addition, the *meet* is commutative but the Natural Language *and* is not always so, since, when 'time' intervenes this property is not always preserved. Hence, for representing everyday reasoning, usually expressed in terms of natural language, more flexible algebraic structures are needed. Standard algebras of fuzzy sets (see [12], [16]) are a good instance of such flexible structures, of which the following abstract definition of a *Basic Flexible Algebra* seems to be a good enough algebraic structure.

Definition 1. *A Basic Flexible Algebra (BFA) is a seven-tuple* $\mathcal{L} = (L, \leq, 0, 1; \cdot, +, ')$*, where* L *is a non-empty set, and*

1. $(L, \leq)$ *is a poset with minimum* 0*, and maximum* 1*.*
2. $\cdot$ *and* $+$ *are mappings (binary operations)* $L \times L \to L$*, such that:*
   a) $a{\cdot}1 = 1{\cdot}a = a$*,* $a{\cdot}0 = 0{\cdot}a = 0$*, for all* $a \in L$
   b) $a + 1 = 1 + a = 1$*,* $a + 0 = 0 + a = a$*, for all* $a \in L$
   c) *If* $a \leq b$*, then* $a{\cdot}c \leq b{\cdot}c$*,* $c{\cdot}a \leq c{\cdot}b$*, for all* $a, b, c \in L$
   d) *If* $a \leq b$*, then* $a + c \leq b + c$*,* $c + a \leq c + b$*, for all* $a, b, c \in L$
3. $' : L \to L$ *verifies*
   a) $0' = 1$*,* $1' = 0$
   b) *If* $a \leq b$*, then* $b' \leq a'$
4. *It exists* $L_0, \{0, 1\} \subset L_0 \subsetneq L$*, such that with the restriction of the order and the three operations* $\cdot, +,$ *and* $'$ *of* $\mathcal{L}$*,* $\mathcal{L}_0 = (L_0, \leq, 0, 1; \cdot, +, ')$ *is a boolean algebra*

It is immediate to prove that in any BFA it holds: $a{\cdot}b \leq a \leq a+b$, and $a{\cdot}b \leq b \leq a+b$,

for all $a, b \in L$. Hence, provided the poset $(L, \leq)$ were itself a lattice with operations *min* and *max*, it follows $a{\cdot}b \leq min(a,b) \leq max(a,b) \leq a+b$.

Lattices with negation and, in particular, ortholattices and De Morgan algebras are instances of BFAs. Also the *standard algebras* of fuzzy sets $([0,1]^X, T, S, N)$ are particular BFAs if taking, for μ, σ in $[0,1]^X$, $\mu{\cdot}\sigma = T \circ (\mu \times \sigma)$, $\mu + \sigma = S \circ (\mu \times \sigma)$, $\mu' = N \circ \mu$, with $0 = \mu_0$, $1 = \mu_1$ (the functions constantly zero and one, respectively), the partial pointwise order, $\mu \leq \sigma \Leftrightarrow \mu(x) \leq \sigma(x)$ for all $x \in X$, T a continuous t-norm, S a continuous t-conorm, and N a strong negation (see [12], [16], [2]).

Notice that although neither idempotency, nor commutativity, nor associativity, nor distributivity, nor duality, nor double-negation, etc., are supposed, ortholattices ([16]) (and in particular orthomodular lattices and boolean algebras), De Morgan algebras, and algebras of fuzzy sets (and in particular the standard ones), are particular cases of BFA. Nevertheless, it should be newly recalled that, for what concerns the representation of Natural Language and Commonsense Reasoning, their too big number of properties imply a too rigid representation's framework. Notwithstanding, this paper will only work in the case the BFA is an ortholattice (see Appendix).

1.3 Towards the concept of a conjecture

The skeleton of the examples in 1.1.1 helps to pose the following definition and questions, relatively to a given problem on which some information constituted by a non-empty set, $P = \{p_1, p_2, ..., p_n\}$ of n premises p_i is known.

- *Definition:* q is a conjecture from P, provided q is not incompatible with the information on the given problem once it is conveyed throughout all p_i in P.
- *Questions*
  a) Where do the objects ('represented' statements) p_i and q belong to? That is, which is L such that $P \subset L$ and $q \in L$?
  b) With which algebraic structure is endowed L?
  c) How can the information on the current problem that is conveyed by P be translated into L? How to state that P is consistent?
  d) How to translate that q is *not incompatible* with such information?

On the possible answers to these four questions depend the 'formalization' of the concept of a conjecture. Of course, the answer to question (a) is in strict dependence of the context and characteristics of the current problem, for instance would this problem deserve a 'body of information' given by imprecise statements, the set L could be a subset of fuzzy sets in the corresponding universe of discourse X, that is, $L \subset [0,1]^X$. Consequently, and once L is choosen, it will be endowed with an algebraic structure that could respond to the current problem's context, purpose and characteristics, for instance, would the problem concern a probabilistic reasoning, the structure of L will be either a boolean algebra, or an orthomodular lattice, provided the problem is a classical or a quantum one, respectively. Although questions (c) and (d) deserve some discussion, let us previously consider the information's set P and its consequences.

Let us point out that with $P \neq \emptyset$, the authors adhere to the statement 'Nothing comes from nothing', attributed to Parmenides.

1.3.1 Bodies of information

We will always deal with reasonings made from some previous information given by a finite set of statements, and once they are represented in a BFA $\mathcal{L}$ (suitable for the corresponding problem), by elements $p_1, ..., p_n$ in L. Each p_i is a *premise* for the reasoning, and $P = \{p_1, ..., p_n\} \subset L$ is the *set of premises*. In what follows, it will be supposed that P is free of incompatible elements, that is, for instance, there are not elements p_i, p_j in P such that $p_i \leq p_j'$, or $p_i \cdot p_j = 0$. Provided there were $p_i \leq p_j'$, it would be $p_1 \cdot ... \cdot p_n = 0$, and to avoid that possibility we will suppose that the résumé $r(P)$ of the information contained in P is different from zero: $r(P) \neq 0$. Analogously, provided this information is given by what follows deductively from P, and C is an operator of consequences, we will suppose that $C(P) \neq L$.

A set P of premises that is free from incompatibility is a *body of information*, and it will be taken in a concrete family $\mathfrak{F}$ of subsets in L like, for example (see [18]),

1. $\mathfrak{F}_1 = \mathcal{P}(L)$
2. $\mathfrak{F}_2 = \{P \in \mathcal{P}(L);$ for no $p \in P: \ p \leq p'\}$
3. $\mathfrak{F}_3 = \{P \in \mathcal{P}(L);$ for no $p_i, p_j \in P: \ p_i \leq p_j'\}$
4. $\mathfrak{F}_4 = \{P \in \mathcal{P}(L);$ for no finite subsets $\{p_1, ..., p_r\}, \{p_1^*, ..., p_m^*\} \subseteq P$: $p_1^* \cdot ... \cdot p_m^* \leq (p_1 \cdot ... \cdot p_r)'\}$.
5. $\mathfrak{F}_5 = \{P \in \mathcal{P}(L); p_1 \cdot ... \cdot p_n \nleq (p_1 \cdot ... \cdot p_n)'\}$
6. $\mathfrak{F}_6 = \mathbb{P}_0(L) = \{P \in \mathcal{P}(L); p_1 \cdot ... \cdot p_n \neq 0\}$

Obviously, $\mathfrak{F}_4 \subset \mathfrak{F}_3 \subset \mathfrak{F}_2 \subset \mathfrak{F}_1$, $\mathfrak{F}_5 \subset \mathfrak{F}_6 \subset \mathfrak{F}_1$, and if L is finite $\mathfrak{F}_4 \subset \mathfrak{F}_6$. If elements in L verify the non contradiction law, it is $\mathfrak{F}_6 \subset \mathfrak{F}_4$. If $\mathcal{L}$ is a boolean algebra, it is $\mathfrak{F}_3 = \mathfrak{F}_4 = \mathfrak{F}_5 = \mathfrak{F}_6 \subset \mathfrak{F}_1$

Once the family $\mathfrak{F}$ is selected in agreement with the kind of incompatibility that is the one suitable for the current problem, a consequence's operator in the sense of Tarski (see [18]) is a mapping $C : \mathfrak{F} \to \mathfrak{F}$, such that,

- $P \subset C(P)$, C is *extensive*
- If $P \subset Q$, then $C(P) \subset C(Q)$, C is *monotonic*
- $C(C(P)) = C(P)$, or $C^2 = C$, C is a *closure*,

for all P, Q in $\mathfrak{F}$. In addition, only *consistent* operators of consequences will be considered, that is, those verifying

- If $q \in C(P)$, then $q' \notin C(P)$.

Operators of consequences are abstractions of 'deductive' processes.

1.4 The discussion

The discussion will be done under the supposition that L is endowed with an ortholattice structure $\mathcal{L} = (L, \cdot, +, '; 0, 1)$.

1.4.1

The information conveyed by the body of information P can be described, at least, by:

1. The logical consequences that follow from P, each time a consequence operator C is fixed. By the set $C(P)$, deploying what is in P.
2. By a suitable résumé of P in some set. Let us call $r(P)$ such a résumé.

What is in (2) is not clear enough without knowing what is to be understood by $r(P)$ or, at least, which properties is $r(P)$ submitted to verify, as well as to which set $r(P)$ does belong to. Three instances for $r(P)$ are:

- $r(P) = p_\wedge = p_1 \cdot \ldots \cdot p_n \in L$
- $r(P) = p_\vee = p_1 + \ldots + p_n \in L$
- $r(P) = [p_\wedge, p_\vee] = \{x \in L; p_\wedge \leq p \leq p_\vee\}$, with $r(P) \in \mathbb{P}(L)$

Anyway, and to state the consistency of P, in case (2) it is reasonable to take $r(P)$ not self-contradictory, for instance $r(P) \nleq r(P)'$, $(r(P) \nsubseteq r(P)')$, for what it should be $r(P) \neq 0$ $(r(P) \neq \emptyset)$, since $r(P) = 0 \leq 1 = 0' = r(P)'$. In case (1) the consistency of P can be stated by supposing $C(P) \neq L$.

1.4.2

In the case (1), the non incompatibility between the information conveyed by P and a 'conjecture' q is given by $q' \notin C(P)$. In the case (2), and provided it is $r(P) \in L$, there are three different forms of expressing such non incompatibility: $r(P) \cdot q \neq 0$, $r(P) \cdot q \nleq (r(P) \cdot q)'$, and $r(P) \nleq q'$ (see [7]). All that conducts to the following four possible definitions of the set of conjectures from P:

- $Conj_C(P) = \{q \in L; q' \notin C(P)\}$, provided $C(P) \neq L$.

- $Conj_1(P) = \{q \in L; r(P) \cdot q \neq 0\}$
- $Conj_2(P) = \{q \in L; r(P) \cdot q \nleq (r(P) \cdot q)'\}$
- $Conj_3(P) = \{q \in L; r(P) \nleq q'\}$

With the last three definitions *a problem arises*: *Are they coming from some operator of consequences in the form of the definition of $Conj_C$?* For instance, to have $Conj_3(P) = \{q \in L; q' \notin C_r(P)\}$, it is necessary that $C_r(P) = \{q \in L; r(P) \leq q\}$, and provided $r(P)$ verifies

$$r(P) \leq p_\wedge,\ P \subset Q \text{ implies } r(Q) \leq r(P), \text{ and } r(C_r(P)) = r(P),$$

C_r is an operator of consequences, since:

- $r(P) \leq p_\wedge \leq p_i$ $(1 \leq i \leq n)$, means $P \subset C_r(P)$.
- If $P \subset Q$, if $q \in C_r(P)$, from $r(P) \leq q$ and $r(Q) \leq r(P)$, follows $q \in C_r(Q)$. Hence, $C_r(P) \subset C_r(Q)$
- Obviously, $C_r(P) \subset C_r(C_r(P))$. If $q \in C_r(C_r(P))$, from $r(C_r(P)) \leq q$ and $r(C_r(P)) = r(P)$ follows $r(P) \leq q$, or $q \in C(P)$. Hence, $C_r(C_r(P)) = C_r(P)$.

In addition, C_r is *consistent* since if $q \in C_r(P)$ and $q' \in C(P)$, from $r(P) \leq q$ and $r(P) \leq q'$, follows $r(P) \leq q \cdot q' = 0$, or $r(P) = 0$, that is absurd. Hence, $q \in C_r(P) \Rightarrow q' \notin C_r(P)$. In particular, if $r(P) = p_\wedge$, $Conj_3$ comes from the consistent operator of consequences $C_\wedge(P) = \{q \in L; p_\wedge \leq q\}$, that is the greatest one if L is a boolean algebra and $\mathfrak{F} = \mathbb{P}_0(L)$ (see [1]).

Remark 1. To have $C(P) \subset Conj_C(P)$, it is sufficient that C is a consistent operator of consequences, since then $q \in C(P)$ implies $q' \notin C(P)$, and $q \in Conj_C(P)$. This condition is also necessary since, if $C(P) \subset Conj_C(P)$, $q \in C(P)$ implies $q \in Conj_C(P)$, or $q' \notin C(P)$. Hence, the consistency of C is what characterizes the inclusion of $C(P)$ in $Conj_C(P)$, that consequences are a particular type of conjectures. For instance, it is $C_\wedge(P) \subset Conj_3(P)$, and $Conj_3(P) = \{q \in L; q' \notin C_\wedge(P)\}$.

1.4.3

Concerning $Conj_1(P) = \{q \in L; r(P) \cdot q \neq 0\}$, it is $Conj_1(P) = \{q \in L; q' \notin C_1(P)\}$ provided $C_1(P) = \{q \in L; r(P) \cdot q' = 0\}$. Let us only consider the case in which $r(P) = p_\wedge \neq 0$.
It is $P \subset C_1(P)$, since $p_\wedge \cdot p_i' = 0$ $(1 \leq i \leq n)$. If $P \subset Q$, $q \in C_1(P)$, or $p_\wedge \cdot q' = 0$, with $q_\wedge \leq p_\wedge$ implies $q_\wedge \cdot q' = 0$, and $q \in C_1(Q)$, thus, $C_1(P) \subset C_1(Q)$. Nevertheless, C_1 can not be always applicable to $C_1(P)$ since it easily can be $r(C_1(P)) = 0$, due to the fail of the consistency of C_1. For instance, if $\mathcal{L}$ is the ortholattice in figure 1.1, with $P = \{f, e\}$ for which $p_\wedge = b$, it is

$$C_1(P) = \{1, a, b, c, d, e, f, g, a', c'\}, \text{ and } r(C_1(P)) = 0.$$

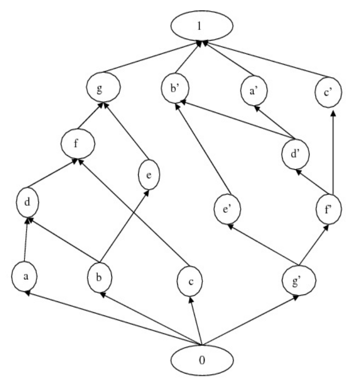

Fig. 1.1. Ortholattice

Hence, $Conj_1$ *is not coming from an operator of consequences*, but only from the extensive and monotonic one C_1, for which the closure property $C_1(C_1(P)) = C_1(P)$ has no sense, since $C_1(P)$ can not be taken as a body of information.

Notice that if $\mathcal{L}$ is a boolean algebra, and $q \in C_1(P)$, from $p_\wedge \cdot q' = 0$ follows $p_\wedge = p_\wedge \cdot q + p_\wedge \cdot q' = p_\wedge \cdot q \neq 0$, that means $q \in Conj_1(P)$: $C_1(P) \subset Conj_1(P)$. In addition, $p_\wedge = p_\wedge \cdot q$ is equivalent to $p_\wedge \leq q$, that is, $q \in C_\wedge(P)$: $C_1(P) \subset C_\wedge(P)$. Even more, in this case, if $q \in C_\wedge(P)$, or $p_\wedge \leq q$, it follows $p_\wedge \cdot q' \leq q \cdot q' = 0$, and $q \in C_1(P)$. Thus, if $\mathcal{L}$ is a boolean algebra, $C_\wedge = C_1$, and $Conj_1 = Conj_{C_\wedge}$.

1.4.4

Concerning $Conj_2(P) = \{q \in L; p_\wedge \cdot q \not\leq (p_\wedge \cdot q)'\}$, to have $Conj_2(P) = \{q \in L; q' \notin C_2(P)\}$, it should be $C_2(P) = \{q \in L; p_\wedge \cdot q' \leq (p_\wedge \cdot q')'\}$.

Of course, if $\mathcal{L}$ is a boolean algebra, it is $C_2(P) = \{q \in L; p_\wedge \cdot q' = 0\} = \{q \in L; p_\wedge \leq q\} = C_\wedge(P)$, $Conj_2(P) = Conj_{C_\wedge}(P)$, and $Conj_2$ comes from the operator of consequences $C_\wedge$. In the general case in which $\mathcal{L}$ is an ortholattice, it is $P \subset C_2(P)$ since $p_\wedge \cdot p_i = 0 \leq 0' = 1$. If $P \subset Q$, and $q \in C_2(P)$, or $p_\wedge \cdot q' \leq (p_\wedge \cdot q')'$, with $q_\wedge \leq p_\wedge$ implies $q_\wedge \cdot q' \leq p_\wedge q'$ and $(p_\wedge \cdot q')' \leq (q_\wedge \cdot q')'$, that is, $q_\wedge \cdot q' \leq q_\wedge \cdot q' \leq (p_\wedge \cdot q')' \leq (q_\wedge \cdot q')'$, or $q_\wedge \cdot q' \leq (q_\wedge \cdot q')'$, and $q \in C_2(Q)$. Hence, $C_2(P) \subset C_2(Q)$, and C_2 is expansive and monotonic.

Notwithstanding, in the ortholattice in figure 1.1, with $P = \{g\}$ ($p_\wedge = g$) is $C_2(P) = \{q \in L; g{\cdot}q \nleq (g{\cdot}q)'\} = \{a, b, c, d, e, f, g, b', a', c', d', e', f', g', 1\}$ with $r(C_2(P)) = 0$. Hence, C_2 is not applicable to $C_2(P)$, since $C_2(P)$ is not a body of information, and the closure property has no sense (notice that C_2 is not consistent). Thus, unless $\mathcal{L}$ is a boolean algebra, $Conj_2$ is not a conjectures operator coming from a consequences one.

1.4.5

Concerning $Conj_3(P) = \{q \in L; p_\wedge \nleq q'\}$, to have $Conj_3(P) = \{q \in L; q' \notin C_3(P)\}$, it should be $C_3(P) = \{q \in L; p_\wedge \leq q\} = C_\wedge(P)$, as it is said in Remark 1.

1.4.6

Let us consider again the operators

- $C_4(P) = \{q \in L; q \leq p_\vee\}$
- $C_5(P) = \{q \in L; p_\wedge \leq q \leq p_\vee\}$

As it is easy to check, only the second is an operator of consequences that is consistent unless $p_\wedge = 0$ and $p_\vee = 1$. With it, it is $Conj_5(P) = \{q \in L; q' \notin C_5(P)\} = \{q \in L; p_\wedge \nleq q' \textit{ or } q' \nleq p_\vee\}$.

1.4.7

When it is $Conj(P) = \emptyset$? It is clear that

- $Conj_C(P) = \{q \in L; q' \notin C(P)\} = \emptyset$, if and only if $C(P) = L$
- $Conj_i(P) = \emptyset$ $(1 \leq i \leq 3)$, if $r(P) = 0$, in which case $C_\wedge(P) = L$, and also $C_1(P) = C_2(P) = L$.

Notice that $C(P) = L$ implies that also C is not consistent. These cases are limiting ones, and facilitate a reason for supposing that C is consistent and $r(P) \neq 0$. Concerning the operator $Conj_5$, it is empty provided $p_\wedge = 0$ and $p_\vee = 1$, that is when $C_5(P)$ is not consistent.

1.4.8

It is easy to check that, if $r(P) \neq 0$, it is $Conj_1(P) \subset Conj_2(P) \subset Conj_3(P)$, and that if $\mathcal{L}$ is a boolean algebra the three operators do coincide.

1.4.9

What happens if the information conveyed by P can be given by two different résumés $r_1(P)$ and $r_2(P)$?

Let us denote by $Conj_j^{(i)}$ $(1 \leq i \leq 2,\ 1 \leq j \leq 3)$ the corresponding operators of conjectures. If $r_1(P) \leq r_2(P)$, it is easy to check that

$$Conj_j^{(1)} \subset Conj_j^{(2)}, \text{ for } 1 \leq j \leq 3.$$

Analogously, if $C_1 \subset C_2$ (that is, $C_1(P) \subset C_2(P)$ for all $P \in \mathfrak{F}$), then $Conj_{C_2}(P) \subset Conj_{C_1}(P)$.

Remark 2. In the case in which the information conveyed by P is what can deductively follow from P, there can be more than one single consistent operator of consequences to reflect such deductive processes. If $\mathcal{C}$ is the set of such operators, it can be defined

$$Conj_{\mathcal{C}}(P) = \bigcap_{C \in \mathcal{C}} Conj_C(P),$$

but a possible problem with this operator is that it can easily be a too small set.

1.4.10 The Goldbach's conjecture

Let $\mathbb{N}$ be the set of positive integers as characterized by the five Peano's axioms, namely:

p_1. 1 is in $\mathbb{N}$.
p_2. If n is in $\mathbb{N}$, also its successor, s(n), is in $\mathbb{N}$.
p_3. It is not any $n \in \mathbb{N}$ such that $s(n) = 1$.
p_4. If $s(n) = s(m)$, then $n = m$.
p_5. If a binary property concerning positive integers holds for 1, and provided it holds for n it is proven it also holds for $n + 1$, then such property holds for all numbers in $\mathbb{N}$.

The proof of a single 'not p_i' $(1 \leq i \leq 5)$ will mean *a refutation* of the Peano's characterization of $\mathbb{N}$.

The majority of mathematicians believe (supported by the 1936 Gentzen's proof on the consistency of P, based on transfinite induction up to some ordinal number), that the set $P = \{p_1, p_2, p_3, p_4, p_5\}$ is consistent. The elementary theory of numbers consist in *all that is deductively derivable in finitistic form from P*. Let us represent by C (operator of consequences) such a form of deduction.

Given the statement q: *All even number larger than 2 is the sum of two prime numbers*, it can be supposed $q \in Conj_C(P)$, since no a single instance of an even number that is not the sum of two primes (not q) has been found after an extensive search for it. This statement q is the *The Goldbach's conjecture*, that will be solved once proven $q \in C(P)$.

1.5 The properties of the operators of conjectures

1.5.1

In all the cases in which there is a expansive operator C such that $Conj(P) = \{q \in L; q' \notin C(P)\}$ and $C(P) \subset Conj(P)$, since $P \subset C(P)$, it is $P \subset Conj(P)$, and also $Conj$ is an expansive operator. This is what happens always when $Conj$ is given by one of such operators, as are the cases $Conj_C$, $Conj_3$, and also that of $Conj_4$.
In the case of $Conj_2$, it is $p_\wedge \cdot p_i = p_\wedge \nleq p'_\wedge$, and also $P \subset Conj_2(P)$. In the case of $Conj_1$, it is $p_\wedge \cdot p_i = p_\wedge \neq 0$, and also $P \subset Conj_1(P)$. With $Conj_5$, provided C_5 is consistent, it is also $P \subset Conj_5(P)$.

1.5.2

If $P \subset Q$, since $C(P) \subset C(Q)$, provided $q \in Conj_C(Q)$, or $q' \notin C(Q)$, it is $q' \notin C(P)$, and $q \in Conj_C(P)$. Then $Conj_C(Q) \subset Conj_C(P)$, and the operators $Conj_C$ are anti-monotonic. Hence, $Conj_3$ is also anti-monotonic.

With $Conj_1$, if $P \subset Q$ and $q \in Conj_1(Q)$, or $q_\wedge \cdot q \nleq (q_\wedge \cdot q)'$, from $q_\wedge \leq p_\wedge$ follows $q_\wedge \cdot q \leq p_\wedge \cdot q$ and $0 < p_\wedge \cdot q$, or $q \in Conj_1(P)$. Thus, $Conj_1(Q) \subset Conj_1(P)$, and the operator $Conj_1$ is anti-monotonic.

With $Conj_2$, if $P \subset Q$ and $q \in Conj_2(Q)$, or $q_\wedge \cdot q \nleq (q_\wedge \cdot q)'$, from $q_\wedge \leq p_\wedge$ follow $q_\wedge \cdot q \leq p_\wedge \cdot q$ and $(p_\wedge \cdot q)' \leq (q_\wedge \cdot q)'$. Hence, provided $p_\wedge \cdot q \leq (p_\wedge \cdot q)'$ ($q \notin Conj_2(P)$), will follow $q_\wedge \cdot q \leq p_\wedge \cdot q \leq (p_\wedge \cdot q)' \leq (q_\wedge \cdot q)'$, that is absurd. Thus, $q \in Conj_2(P)$, or $Conj_2(Q) \subset Conj_2(P)$, and the operator $Conj_2$ is anti-monotonic.

1.5.3

For what concerns $r(P)$, it should be noticed that the idea behind it is to reach a 'compactification' of the information conveyed by the p_i in P. Of course, how to express and represent $r(P)$ depends on the current problem that, in some cases, offers no doubts on how to represent $r(P)$. For instance, if the problem consists in doing a backwards reasoning with scheme

If p, then q, and not q: not p,

$r(P)$ does represent the statement $(p \to q){\cdot}q'$, with which it must follow $(p \to q){\cdot}q' \leq p'$, to be sure that p' follows deductively from $P = \{p \to q, q'\}$, under $C_\wedge$ and provided $r(P) \neq 0$. That is, to have

$$p' \in C_\wedge(\{p \to q, q'\}) = \{x \in L; (p \to q){\cdot}q' \leq x\} = [(p \to q){\cdot}q', 1] = [r(P), 1].$$

Provided it were $r(P) = 0$, it will follow the non-informative conclusion

$$p' \in \{x \in L; 0 \leq x\} = L.$$

1.5.4

After what has been said, it seems that any operator of conjectures does verify some of the following five properties,

1. $Conj(P) \neq \emptyset$
2. $0 \notin Conj(P)$
3. It exists an operator C such that $Conj(P) = \{q \in L; q' \notin C(P)\}$
4. $Conj$ is expansive: $P \subset Conj(P)$
5. $Conj$ is anti-monotonic: If $P \subset Q$, then $Conj(Q) \subset Conj(P)$

Let us reflect on properties 3, 4, and 5, in the hypothesis that C is consistent, that is, it verifies '$q \in C(P) \Rightarrow q' \notin C(P)$'. Obviously, $C(P) \subset Conj(P)$.

- If $Conj$ is anti-monotonic, C is monotonic.
  `Proof.` Provided $P \subset Q$, if $q \in C(P)$ follows $q' \notin Conj(P)$ and, since $Conj(Q) \subset Conj(P)$ it is $q' \notin Conj(Q)$, or $q \in C(Q)$. Hence, $C(P) \subset C(Q)$ $\square$
- If C is extensive and monotonic, $Conj$ is extensive and anti-monotonic.
  `Proof.` It is obvious that $P \subset Conj(P)$, since from $C(P) \subset Conj(P)$ follows $P \subset C(P) \subset Conj(P)$. Provided $P \subset Q$, follows that if $q \in Conj(Q)$, or $q' \notin C(Q)$, it is also $q' \notin C(P)$, or $q \in Conj(P)$. $\square$

Thus, provided C is consistent, *a sufficient condition* to have $Conj_C$ expansive and anti-monotonic is that C is expansive and monotonic. In addition $Conj_C$ is anti-monotonic *if and only if* C is monotonic.

What, if C is also a closure? $C^2(P) = C(P)$ implies $Conj_C(P) = Conj(C(P))$. Thus $q \in Conj_C(P) \Leftrightarrow q' \notin C(P) \Leftrightarrow q' \notin C(C(P)) \Leftrightarrow q \in Conj_C(C(P))$: $Conj_C(P) = Conj_C(C(P))$. Hence, if C is a consistent consequences operator the associated operator $Conj_C$ is extensive, anti-monotonic, and verifies $Conj_C \circ C = Conj_C$.

1.6 Kinds of conjectures

1.6.1

It is clear that with $Conj_3(P)$ it is $C_3(P) = C_\wedge(P) \subset Conj_3(P)$, as it was said before, but what with $Conj_1(P)$ and $Conj_2(P)$ that are not coming from an operator of consequences? Is there any subset of $Conj_1(P)$ and $Conj_2(P)$ that consists of logical consequences of P? *Are always logical consequences a particular case of conjectures?*

Namely, given $Conj_2(P) = \{q \in L; p_\wedge \cdot q \not\leq (p_\wedge \cdot q)'\}$, exists $C(P) \subset Conj_2(P)$ such that C is a Tarski's operator? Of course, such C is not $C_2(P) = \{q \in L; p_\wedge \cdot q' \leq (p_\wedge \cdot q')'\}$, but is there any subset of $Conj_2(P)$ that consists of logical consequence of P? The answer is affirmative, since $q \in C_\wedge(P)$, or $p_\wedge \leq q$, is equivalent to $p_\wedge \cdot q = p_\wedge$, and as it cannot be $p_\wedge \leq p'_\wedge$, it is $p_\wedge \cdot q \leq (p_\wedge \cdot q)'$, or $q \in Conj_2(P)$. Then, $C_\wedge(P) \subset Conj_2(P)$. Thus, what can be said on the difference $Conj_2(P) - C_\wedge(P)$?

Concerning $Conj_1(P) = \{q \in L; p_\wedge \cdot q \neq 0\}$, it is known that $C(P)$ is not $C_1(P) = \{q \in L; p_\wedge \cdot q' = 0\}$, but it is also $C_\wedge(P) \subset C_1(P)$, that newly allows to ask on the difference $Conj_1(P) - C_\wedge(P)$?

The idea behind the two former questions is to classify the conjectures in $Conj_i(P) - C_\wedge(P)$, $i = 1, 2$, that is, those conjectures that are not 'safe' or 'necessary' ones, but *contingent* in the sense that it could be simultaneously $q \in Conj_i(P) - C_\wedge(P)$ and $q' \in Conj_i(P) - C_\wedge(P)$. What is clear is that

$$Conj_i(P) - C_\wedge(P) = \{q \in Conj_i(P); q < p_\wedge\} \cup \{q \in Conj_i(P); q\ NC\ p_\wedge\},$$

with the sign NC instead of non '*order comparable*'. Let's call as follows these two subsets,

- $Hyp_i(P) = \{q \in Conj_i(P); q < p_\wedge\}$, and its elements '*hypotheses* for P'.
- $Sp_i(P) = \{q \in Conj_i(P); q\ NC\ p_\wedge\}$, and its elements '*speculations* from P'.

Notice that since $0 \notin Conj_i(P)$ $(i = 1, 2)$, it is actually $Hyp_i(P) = \{q \in Conj_i(P); 0 < q < p_\wedge\}$. Obviously, the decomposition

$$Conj_i(P) = C_\wedge(P) \cup Hyp_i(P) \cup Sp_i(P)$$

is a partition of $Conj_i(P)$, and defining

$$Ref_i(P) = L - Conj_i(P), \text{ as the set of } \textit{refutations} \text{ of P},$$

the following partition of L is obtained,

$$L = Ref_i(P) \cup Conj_i(P) = Ref_i(P) \cup Hyp_i(P) \cup Sp_i(P) \cup C_\wedge(P).$$

1.6.2

If $P \subset Q$, from $q_\wedge \leq p_\wedge$, if $0 < q < q_\wedge$, follows $0 < q < p_\wedge$, that is $Hyp_i(Q) \subset Hyp_i(P)$. That is, the operators Hyp_i are, like $Conj_i$, anti-monotonic.

Concerning Sp_i, some examples (see [6]) show it is neither monotonic, nor anti-monotonic, that is, if $P \subset Q$ there is no any fixed law concerning how can $Sp_i(P)$ and $Sp_i(Q)$ be compared: they cannot be comparable by set inclusion. In fact, coming back to figure 1.1, and taking $P_1 = \{f\} \subset P_2 = \{e, f\}$, it is $Sp_3(P_1) = \{a, e, a', b'c', d', e'\}$ and $Sp_3(P_2) = \{a, c, a', b', c'\}$, which are non-comparable. We can say that Sp_i are *purely non-monotonic* operators. Notice that it is

$$Sp_1(P) = \{q \in L;\ p_\wedge{\cdot}q \neq 0 \ \&\ p_\wedge NCq\}, \text{ and}$$

$$Sp_2(P) = \{q \in L;\ p_\wedge{\cdot}q \nleq (p_\wedge{\cdot}q)' \ \&\ p_\wedge NCq\},$$

that, if $\mathcal{L}$ is a boolean algebra, are coincidental, since in such case '$p_\wedge{\cdot}q \nleq (p_\wedge{\cdot}q)' \Leftrightarrow p_\wedge{\cdot}q \neq 0$.

Remark 3. Provided C is a consistent operator of consequences, and in a similar vein to Gödel's First Incompleteness Theorem, let us call C-decidable those elements in $C(P)$, and consider the set

$$U_C(P) = \{q \in L; q \notin C(P) \& q' \notin C(P)\}.$$

$U_C(P)$ consists in the C-undecidable elements in L given P, those that neither follow deductively from P (under C), nor their negation follows deductively from P (under C). It is,

$$U_C(P) = C(P)^c \cap Conj_C(P) = [Sp(P) \cup Hyp(P) \cup Ref(P)] \cap Conj_C(P) =$$

$$Sp(P) \cup Hyp(P)$$

Thus, given a consistent set of premises ($C(P) \neq L$, or $p_\wedge \nleq 0$), reasonably the C-undecidable elements in L are either the speculations or the hypotheses: $C-$undecidability coincides with contingency.

If C^* is a consistent and more powerful operator of consequences than C ($C(P) \subset C^*(P)$), it is obvious that it holds $U_{C^*}(P) \subset U_C(P)$, but not that $U_C(P) \subset U_{C^*}(P)$: What is C-undecidable is not necessarily C^*-undecidable. It happens analogously if C and C^* are not comparable, but $C(P) \cap C^*(P) \neq \emptyset$. The undecidability under C does not imply the undecidability under a C^* that is not less powerful than C.

1.6.3 Explaining the experiment of throwing a dice

Which are the *significative results* that can be obtained when throwing a dice? The reasonable questions that can be posed relatively to the results of the experiment are, for instance,

- Will an even number be obtained?, with answer representable by $\{2, 4, 6\}$
- Will an odd number be obtained?, with answer representable by $\{1, 3, 5\}$
- Will a six be obtained?, with answer representable by $\{6\}$
- Will fail the throw?, with answer representable by $\emptyset$
- Will any number be obtained?, with answer representable by $\{1, 2, 3, 4, 5, 6\}$, etc.

Thus, the questions can be answered by the subsets of the 'universe of discourse' $X = \{1, 2, ..., 6\}$, with which the boolean algebra of events is $\mathbb{P}(X)$, and the body of information for the experiment of throwing a dice is $P = \{X\}$, with $p_\wedge = X \neq \emptyset$. Since $L = P(X)$ is a boolean algebra, it can be taken the consistent operator of consequences $C_\wedge(P) = \{Q \in \mathbb{P}(X); X \subset Q\} = \{X\}$. Hence,

$$Conj_{C_\wedge}(P) = \{Q \in P(X); X \not\subseteq Q^c\} = \{Q \in P(X); Q \neq \emptyset\}, \text{ and}$$

$$Ref_{C_\wedge}(P) = \{\emptyset\},$$

that is, the conjectures on the experiments are all the non-empty subsets of X. In addition,

$$Hyp(P) = \{Q \in \mathbb{P}(X); \emptyset \subset Q \subset X\}, \text{ and}$$

$$Sp(P) = \{Q \in \mathbb{P}(X); Q \neq \emptyset \ \& \ Q \ NC \ X\} = \emptyset,$$

show that $Conj_{C_\wedge} = C_\wedge \cup Hyp(P) = \{X\} \cup \{Q \in \mathbb{P}(X); \emptyset \neq Q \neq X\}$. That is, the significative results of the experiment are those $Q \subset X$ *that are neither empty, nor coincidental with the 'sure event'* X: those that are contingent. In fact, in the case of betting on the result of throwing a dice, nobody will bet on 'failing', and nobody will be allowed to bet on 'any number'.

Hence, the presented theory of conjectures explains well the experiment, and the risk of betting for an event can be controlled by means of a probability $\mathfrak{p} : \mathbb{P}(X) \to \mathbb{P}(X)$, defined by $\mathfrak{p}(\{i\}) = \mathfrak{p}_i$ $(1 \leq i \leq 6)$ such that $0 \leq \mathfrak{p}_i \leq 1$, and $\sum_{i \in \{1,...,6\}} \mathfrak{p}_i = 1$, with the values $\mathfrak{p}_i$ depending on the physical characteristics of the dice. For instance, the probability of 'obtaining even', is

$$\mathfrak{p}(\{2, 4, 6\}) = \mathfrak{p}_2 + \mathfrak{p}_4 + \mathfrak{p}_6.$$

1.7 On refutation and falsification

Let it be $P = \{p_1, ..., p_n\}$, with $p_\wedge \neq 0$, and a conjecture's operator $Conj_C(P) = \{q \in L; q' \notin C(P)\}$ with C an, at least, expansive and monotonic operator. The corresponding operator of refutations is $Ref(P) = L - Conj(P) = \{r \in L; r' \in C(P)\}$. When it can be specifically said that $r \in L$:

1) refutes P?, and 2) refutes $q \in Conj(P)$?

The answer to these two questions depends on the chosen operator C, provided $r \in Ref(P)$, $r' \in C(P)$, and that r is incompatible with the totality of the given information. In particular, and supposed $r \in Ref(P)$:

- For $Conj_3(P)$, $C = C_\wedge$
  1. r refutes P, if r is contradictory will all $p_i \in P$, that is, $p_1 \leq r'$,..., $p_n \leq r'$. Notice that this chain of inequalities implies $p_\wedge \leq r'$, or simply $r' \in C_\wedge(P)$.
  2. r refutes q, if $r' \in C_\wedge(\{q\})$ and $q \leq r'$, that is simply if $r' \in C_\wedge(\{q\})$.
- For $Conj_2(P)$, $C = C_2$ $(C_2(P) = \{q \in L; p_\wedge \cdot q' \leq (p_\wedge q')'\})$
  1. r refutes P, if all $p_i \cdot r$ are self-contradictory, that is, $p_1 \cdot r \leq (p_1 \cdot r)'$,..., $p_n \cdot r \leq (p_n \cdot r)'$, implying $p_\wedge \cdot r \leq (p_i \cdot r)'$, for $1 \leq i \leq n$. From $p_\wedge \leq p_i$, follows $(p_i \cdot r)' \leq (p_\wedge \cdot r)'$, and $p_\wedge \cdot r \leq (p_\wedge \cdot r)'$, or simply $r' \in C_2(P)$.
  2. r refutes q, if $r' \in C_2(\{q\})$ and $q \cdot r \leq (q \cdot r)'$, that is simply if $r' \in C_2(\{q\})$.
- For $Conj_1(P)$, $C = C_1$ $(C_2(P) = \{q \in L; p_\wedge \cdot q' = 0\})$
  1. r refutes P, if $p_1 \cdot r = 0$,..., $p_n \cdot r = 0$, implying $p_\wedge \cdot r = 0$, that implies $p_\wedge \cdot r = 0$, or simply $r' \in C_1(P)$.
  2. r refutes q, if $r' \in C_1(\{q\})$ and $q \cdot r = 0$, that is simply if $r' \in C_1(\{q\})$.

Hence, for these three cases

1. r refutes P, if $r' \in C_1(P)$: r' follows deductively from P
2. r refutes q, provided $r \in Ref(P)$, and $r' \in C(\{q\})$: r is a refutation whose negation follows deductively from $\{q\}$.

1.7.1

In the particular case in which $h \in Hyp(P)$ $(0 < h < p_\wedge)$, and in addition to the former answers in agreement with Popper's ideas on the falsification of theories $(C(P) = P)$ and hypotheses (see [11] [10]), it can be said what follows

- If $h \in Hyp_C(P)$, then $C(P) \subset C(\{h\}) \subset Conj_C(P)$, proven by the following sequences: 1) $p_\wedge \le q$ & $h \le p_\wedge \Rightarrow h < q$. 2)$h \le q$ & $h \le q' \Rightarrow h = 0$. 3)$h \nleq q'$ & $p_\wedge \le q'$ & $h \le p_\wedge \Rightarrow h \le q'$ which is absurd: $p_\wedge \nleq q'$.

Hence, in order to ascertain that some $h \in L$ is not a hypothesis for P (*falsification* of h), it suffices to find $q \in C(P)$ such that $q \notin C(\{h\})$, or $r \in C(\{h\})$ such that $r \notin Conj_C(P)$. In these cases it is $q \in Ref(P)$, and $r \in Ref(P)$: both refute h. Thus:

- Something that follows deductively from P, but not from $\{h\}$, makes h be false.
- Something that follows deductively from $\{h\}$, but is not conjecturable from P, makes h be false.

Remark 4. From $p_\wedge \le p_i$ and $0 < h < p_\wedge$, it follows $P \cup \{p_\wedge\} \subset C(\{h\})$, hence

$$P \cup \{p_\wedge\} \subset C(P) \subset C(\{h\}),$$

that, although only in part, remembers the statement in [21], '*Deduction justifies by calculation what Induction has happily guessed*'.

1.8 The relevance of speculations

It is $Sp_3(P) = \{q \in Conj_3; p_\wedge NCq\} = \{q \in L; p_\wedge \nleq q' \text{ \& } p_\wedge NCq\}$, hence, if $q \in Sp_i(P)$ $(i = 1, 2, 3)$ it is not $p_\wedge \cdot q = p_\wedge$ (equivalent to $p_\wedge \le q$, or $q \in C_\wedge(P)$).

Thus, if $q \in Sp_i(P)$, it is $0 < p_\wedge \cdot q \cdot q < p_\wedge$, that is $p_\wedge \cdot q \in Hyp_i(P)$. This result shows a way of reaching hypotheses from speculations, and *in the case the ortholattice $\mathcal{L}$ is an orthomodular one*, for any $h \in Hyp_i(P)$, it exists $q \in Sp_3(P)$ such that $h = p_\wedge \cdot q$ (see [19]), there are no other hypotheses, and it is

$$Hyp(P) = p_\wedge \cdot Sp_3(P).$$

Of course, this result also holds if $\mathcal{L}$ is a boolean algebra.

Analogously, since $p_\wedge \le p_\wedge + q$, it is $p_\wedge + q \in C_\wedge(P)$, that shows a way of reaching logical consequences from speculations, and if $\mathcal{L}$ is an orthomodular lattice (and *a fortiori* if it is a boolean algebra), there are not other consequences (see[19]), that is, $C_\wedge(P) = p_\wedge + Sp_3(P)$.

Remarks 1

- *$p_\wedge \cdot Sp_3(P) \subset Hyp(P)$, could remember a way in which humans search for how to explain something. Once P and $p_\wedge$ are known, a $q \in L$ such that $p_\wedge \nleq q'$ and $p_\wedge NCq$, that is, neither incompatible, nor comparable with*

$p_\wedge$, gives the explanation or hypotheses $p_\wedge \cdot q$ for P, provided $p_\wedge \cdot q \neq 0$, and $p_\wedge \cdot q \neq p_\wedge$. Of course, an interesting question is how to find such $q \in Sp_3(P)$. In some cases, may be q is found by similarity with a former case in which a more or less similar problem was solved, and plays the role of a metaphor for the current one.

- *Out of orthomodular lattices, there are hypotheses and consequences that are not reducible, that is, belonging to $Hyp(P) - p_\wedge \cdot Sp_3(P)$, or to $C_\wedge(P) - (p_\wedge + Sp_3(P))$ (see [19]).*

1.9 Conclusion

This paper represents a conceptual upgrading of a series of papers on the subject of conjectures, a subject christianized in [13] as 'CHC Models'.

1.9.1

In the course of millennia the brain's capability of conjecturing resulted extremely important for the evolution of the species Homo. Such capability helped members in Homo to escape from predators, to reach adequate food, to protect themselves from some natural events, or even catastrophes, as well as to produce fire, to make artifacts, and to travel through high mountains, deserts, forests, rivers and seas. Without articulate language and partially articulate guessing, possibly Homo would have neither prevailed over the rest of animals, nor constituted the social, religious and economic organizations typical of humankind. And one of the most distinguishing features of Homo Sapiens is the act, and especially the art, of reasoning, or goal-oriented managing conjectures. Even more, scientific and technological research is a human activity that manages guessing in a highly articulated way. Actually, reasoning and conjecturing are joint brain activities very difficult to separate one from the other.

Although consequences and hypotheses, as well as several types of nonmonotonic reasoning, deserved a good deal of attention by logicians, philosophers, computer scientists, and probabilists, no attempt at formalizing the concept of conjecture appeared before [5] was published. In the framework of an ortholattice, conjectures were defined in [5] as those elements non-incompatible with a given set of (non-incompatible) premises reflecting the available information. That is, conjectures are those elements in the ortholattice that are "possible", once a résumé of the information given by the premises is known. This is the basic definition of which consequences (or safe, necessary conjectures), hypotheses (or explicative contingent conjectures), and speculations (or lucubrative, speculative contingent conjectures) are particular cases. It should also be pointed out that neither the set of hypotheses, nor that of speculations, can be taken as bodies of information. Processes to obtain consequences perform deductive reasoning, or deduction. Those to obtain hypotheses perform abductive reasoning, or abduction, and those to obtain speculations perform inductive reasoning, or induction, a term that is also more generally applied to obtaining either hypotheses or speculations, and then results close to the term “reasoning". Of course, in Formal Sciences and in the context of proof, the king of reasoning processes is deduction.

1.9.2

Defining the operators of conjectures only by means of consistent consequences ones (see [6]) has the drawback of placing deduction before guessing, when it can be supposed that guessing is more common and general than deduction, and this is a particular (and safe) type of the former. After the publication of some papers ([5], [7], [6], [19], [17], [18], [1]) on the subject it yet remained the doubt on the existence of conjecture's operators obtained without consequences' operators, and this paper liberates from such doubt by showing that to keep some properties that seem to be typical of the concept of conjecture, it suffices to only consider operators that are extensive and monotonic, but without enjoying the closure property. These operators are reached by considering (like it was done in [7]), three different ways of defining non-incompatibility by means of non-self-contradiction. Of these three ways, only one of them conducts to reach conjectures directly through logical consequences that is just the one considered in [5]. Of course, the existence of operators of conjectures not coming from extensive and monotonic operators remains an open problem.

Appendix

Although basic flexible algebras are very general structures, it is desirable that they verify the principles of Non-contradiction and Excluded-middle, to ground what is represented in a ‘solid’ basement. For that goal it will be posed some definitions on the incompatibility concept of contradictory and self-contradictory elements in a BFA. In the first place, (see [3], [14])

- Two elements a, b in a BFA are said to be *contradictory* with respect to the negation $'$, if $a \leq b'$.
- An element a in a BFA is said to be *self-contradictory* with respect to the negation $'$, if $a \leq a'$.

The classical principles of Non-contradiction (NC) and Excluded-Middle (EM) can be defined in the way that is typical of modern logic,

- NC: $a \cdot a' = 0$
- EM: $a + a' = 1$

Any lattice with a strong negation (i.e. $(a')' = a$, for all $a \in L$) verifying these last principles, is an ortholattice. So, a Boolean Algebra verifies these principles. But, if dealing with fuzzy sets, for instance with the standard algebra of fuzzy sets $([0,1]^X, \min, \max, N)$, that is a De Morgan algebra, these principles, formulated in the previous way, do not hold. Nevertheless, if the Aristotle's formulation of the first principle: "an element and its negation is impossible" is translated by "an element and its negation are self-contradictory", the mathematical representation of these principles changes in the form,

- NC: $a{\cdot}a' \leq (a{\cdot}a')'$
- EM: $(a + a')' \leq ((a + a')')'$

With these new formulation, De Morgan algebras and functionally expressible BFA of fuzzy sets also verify those principles (see [8]), if dealing with a strong negation N_φ for fuzzy sets, that is, $N_\varphi(x) = \varphi^{-1}(1 - \varphi(x))$, with φ an order-automorphism of the unit interval. In fact, it can be used as intersection any function T that verifies $T(a, N_\varphi(a)) \leq \varphi^{-1}(\frac{1}{2})$ in order to satisfy the principle of NC. In the case of EM, it is enough any function S, that satisfies $\varphi^{-1}(\frac{1}{2}) \leq S(a, N_\varphi(a))$. Notice that all t-norms are in the condition of T, and all t-conorms are in the condition of S (see [15]).

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References

1. C. Vaucheret A. Fernandez Pineda, E. Trillas. Additional comments on conjectures, hypotheses, and consequences in orthocomplemented lattices. *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, pages 107–114, 2001.
2. C. Alsina, M. J. Frank, and B. Schweizer. *Associative Functions. Triangular Norms and Copulas*. World Scientific, Singapore, 2006.
3. G. Bodiou. *Théorie dialectique des probabilités (englobant leurs calculs classique et quantique)*. Gauthier-Villars, 1964.
4. J. L. Castro and E. Trillas. Sobre preordenes y operadores de consecuencias de Tarski. *Theoria*, 4(11):419–425, 1989. (In Spanish).
5. E. Castiñeira E. Trillas, S. Cubillo. On conjectures in orthocomplemented lattices. *Artificial Intelligence*, 117:255–275, 2000.
6. M. Monserrat E. Trillas, M. Mas. Conjecturing from consequences. *International Journal of General Systems*, 38:567–578, 2009.
7. A. Pradera E. Trillas. A reflection on rationality, guessing and measuring. In *Proceedings IPMU*, pages 777–784, Annecy, 2002.
8. I. García-Honrado and E. Trillas. Characterizing the principles of non contradiction and excluded middle in [0,1]. *Internat. J. Uncertainty Fuzz. Knowledge-Based Syst.*, 2:113–122, 2010.
9. C. S. Peirce. Deduction, induction, and hypothesis. *Popular Science Monthly*, (13):470–482, 1878.
10. K. Popper. *The logic of Scientific Discovery*. Hutchinson & Co. Ltd, London, 1959.
11. K. Popper. *Conjectures and Refutations*. Rutledge & Kegan Paul, London, 1963.
12. A. Pradera, E. Trillas, and E. Renedo. An overview on the construction of fuzzy set theories. *New Mathematics and Natural Computation*, 1(3):329–358, 2005.
13. D. Qiu. A note on trillas' chc models. *Artificial Intelligence*, 171:239–254, 2007.
14. E. Trillas. Non contradiction, excluded middle, and fuzzy sets. In Pal PiGesù and Petrosino, editors, *International Workshop of Fuzzy Logic and Applications*, pages 1–11, Palermo, Italy, 2009. Springer (LNAI5571).
15. E. Trillas, C. Alsina, and A. Pradera. Searching for the roots of non-contradiction and excluded-middle. *International Journal of General Systems*, 31(5):499–513, 2002.

16. E. Trillas, C. Alsina, and A. Pradera. On a class of Fuzzy Set Theories. In *Proc. FUZZ-IEEE 2007*, pages 1–5, London, 2007.
17. E. Trillas, E. Castiñeira, and S. Cubillo. Averaging premises. *Mathware and Soft Computing*, 8:83–91, 2001.
18. E. Trillas, I. García-Honrado, and A. Pradera. Consequences and conjectures in preordered sets. *Information Sciences*, 180(19):3573–3588, 2010.
19. E. Trillas, A. Pradera, and A. Álvarez. On the reducibility of Hypotheses and Consequences. *Information Sciences*, 179(23):3957–3963, 2009.
20. S. Watanabe. *Knowing and guessing. A Quantitative Study of Inference and Information.* John Wiley and sons, New York, 1969.
21. W. Whewell. *Novum Organon Renovatum: Being The Second Part of The Philosophy of The Inductive Sciences.* John Parker and Son, London, 1858.

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June 22nd, 2011

Certification of acceptance to book contribution

Dear Dr. Itziar García-Honrado ,

I am glad to certify the acceptance of your contribution paper together with your co-author Prof. Dr. Enric Trillas

"On an Attempt to Formalize Guessing"

to our book in print

Seising, Rudolf, Veronica Sanz (eds.): *Soft Computing in Humanities and Social Sciences* (Studies in Fuzziness and Soft Computing) Berlin, New York, [u.a.]: Springer-Verlag (2011 or 2012).

Sincerely,

Dr. Rudolf Seising

Parte III

Otras publicaciones

Capítulo 8

Publicaciones en congresos, revistas y libros

Nuestro conocimiento es necesariamente finito, mientras que nuestra ignorancia es necesariamente infinita.
Karl Raimund Popper (1902-1994)

8.1. Congresos:

- Actas ESTYLF'08 (Mieres-Langreo, Septiembre 2008):
  - Trillas, E., García-Honrado, I. "La regla composicional de Zadeh: Una lección para principiantes", pp. 323-329.
  - Trillas, E., García-Honrado, I., Renedo E. "On the fuzzy law $(\mu \rightarrow \mu') \rightarrow \mu' = \mu_1$", pp. 323-329.
  - I. García-Honrado, E. Trillas, E. Renedo, Modelling conjunctions by ordinal sums", pp. 229-232.
- Proceedings IFSA-EUSFLAT'09 (Lisboa, Julio 2009): García-Honrado, I., Trillas, E., Guadarrama, S., Renedo E. "Evaluating premises, partial

consequences and partial hypotheses". 13th IFSA World congress and 6th EUSFLAT conference - IFSA-EUSFLAT´09. 2009. n. 0, pp. 897-902.

- Workshop Soft Computing in Humanities and Social Sciences. (Mieres) Ponencia oral del trabajo "Conjectural Reasoning" (Mieres, Septiembre 2009)

- Actas ESTYLF'10 (Punta Umbría, Febrero 2010): García-Honrado, I., Trillas, E., Guadarrama, S. "Grado de parentesco entre predicados". Actas del XV Congreso Español sobre Tecnologías y Lógica Fuzzy, ESTYLF 2010. Huelva, Spain: 2010. n. 0, pp. 163-168.

- Proceedings IPMU'10 (Dortmund, Junio 2010): Trillas, E., Nakama, T., García-Honrado, I. "Fuzzy Probabilities: Tentative Discussions on the Mathematical Concepts" . Proceedings of the 13th International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU2010). Dortmund, Germany: Springer, 2010. n. 0, pp. 139-148.

- NAFIPS'10 (Toronto, Julio 2010): Alsina C., Trillas, E., García-Honrado, I. "On the coincidence of conditional functions" . Proceedings of the Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS 2010). Toronto, Canada: 2010. n. 0.

- Proceedings WCCI'10 (Barcelona, Julio 2010): Trillas, E., Alsina C., García-Honrado, I. "On two properties of the conditional in fuzzy logic". Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2010): IEEE World Congress on Computational Intelligence (WCCI 2010) . Barcelona, Spain: IEEE Computer Society, 2010. n. 0, pp. 2477-2482.

- International Symposium "Fuzziness, Philosophy and Medicine". (Mieres) Ponencia oral del trabajo "Non-Contradiction and Excluded Middle in Fuzzy Logic. A look to the medicine field" (Mieres, Marzo 2011)

- Proceedings World conference on Soft Computing 2011 (San Francisco, Mayo 2011): I. García-Honrado, A. R. de Soto, E. Trillas, "Some (Unended) Queries on Conjecturing" (proceeding 150).
- Proceedings ISMVL'11 (Tuusula, Finlandia, Mayo 2011): I. García-Honrado, E. Trillas, "Notes on the Exclusive Disjunction". 41st IEEE International Symposium on Multiple-Valued Logic. 2011. pp. 73-77.
- Proceedings EUSFLAT-LFA'11 (Aix-les-Bains, Francia, Julio 2011): I. García-Honrado, E. Trillas, "Unended Reflections on Family Resemblance and Predicates Linguistic Migration". Aceptado.

8.2. Publicaciones en revistas científicas internacionales:

- Trillas, E, García-Honrado, I., Guadarrama, S., Renedo, E. "Crisp sets as classes of discontinuous fuzzy sets" International Journal of Approximate Reasoning (IJAR) Volume 50, Issue 8, Septiembre 2009, pp. 1298-1305.
- García-Honrado, I., Trillas, E. "Characterizing the principles of non contradiction and excluded middle in [0,1]". International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems (IJUFKS) . Singapore; New Jersey: World Scientific, 2010. n. 2, pp. 113-122.
- Trillas, E., García-Honrado, I., Pradera A. "Consequences and conjectures in preordered sets". Information Sciences . Amsterdam, Netherlands: Elsevier, 2010. V. 180, n. 19, pp. 3573-3588.
- I. García-Honrado y E. Trillas, "An essay on the linguistic roots of fuzzy sets". Information Sciences 181 4061-4074 (2011).

- I. García-Honrado, E. Trillas, C. Alsina, “Note: On a conditional conjunctive law with fuzzy sets”, Aceptado en Internat. J. Uncertainty Fuzz. Knowledge-Based Syst. .

- E. Trillas, I. García-Honrado,“Note: On a Dummetts deductive scheme in classical and fuzzy logics”. Aceptado en Fuzzy Sets and Systems.

8.3. Capítulos de libro

- I. García-Honrado, E. Trillas, “On an attempt to formalize guessing”, Tech. Rep. FSC-2010-11, European Centre for Soft Computing, aceptado en Soft Computing in Humanities and Social Sciences (Eds. R. Seising and V. Sanz) Springer- Verlag, 2010.

- E. Trillas, I. García-Honrado, “A reflection on the design of fuzzy conditionals”, Tech. Rep. FSC-2010-12, European Centre for Soft Computing, (aceptado en “Experimentation and Theory — A Homage to Abe Mamdani”), 2011.

- T. Nakama, E. Trillas, I. García-Honrado, “Axiomatic Investigation of Fuzzy Probabilities”, Tech. Rep. FSC-2010-22, European Centre for Soft Computing, aceptado en Soft Computing in Humanities and Social Sciences (Eds. R. Seising and V. Sanz) Springer-Verlag, 2010.

Capítulo 9

Anexo con algunas de ellas

Aquel que duda y no investiga, se torna no solo infeliz, sino también injusto.
Blaise Pascal (1623-1662)

1. Estudio de los conjuntos clásicos a través de una relación de equivalencia sobre los conjuntos *fuzzy*. Se añade por su relación con las conjeturas y el diseño de sistemas fuzzy.

   Trillas, E, García-Honrado, I., Guadarrama, S., Renedo, E. "Crisp sets as classes of discontinuous fuzzy sets" International Journal of Approximate Reasoning (IJAR) Volume 50, Issue 8, Septiembre 2009, pp. 1298-1305.

2. Un estudio de posibles medidas dentro del conjunto de conjeturas obtenido a partir del operador $C_{\leq}$, introducido en la publicación [TGHP10] recogida en la tesis.

   García-Honrado, I., Trillas, E., Guadarrama, S., Renedo E. "Evaluating premises, partial consequences and partial hypotheses". 13th IFSA World congress and 6th EUSFLAT conference - IFSA-EUSFLAT´09. 2009. n. 0, pp. 897-902.

3. Estudio del razonamiento disjuntivo en distintas estructuras algebraicas. Se añade por su relación con la deducción fuzzy.

   I. García-Honrado, E. Trillas, "Notes on the Exclusive Disjunction". 41st IEEE International Symposium on Multiple-Valued Logic. 2011. pp. 73-77.

4. Estudio de los Modelos CHC entre conjuntos clásicos y fuzzy. Se añade por su relación con las conjeturas y por estar citado en la memoria.

   I. García-Honrado, A. R. de Soto, E. Trillas, "Some (Unended) Queries on Conjecturing". Proceedings World Conference on Soft Computing 2011 (San Francisco, Mayo 2011), proceeding nº 150.

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Crisp sets as classes of discontinuous fuzzy sets ☆

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ABSTRACT

This paper aims to show how, by using a threshold-based approach, a path from imprecise information to a crisp 'decision' can be developed. It deals with the problem of the logical transformation of a fuzzy set into a crisp set. Such threshold arises from the ideas of contradiction and separation, and allows us to prove that crisp sets can be structurally considered as classes of discontinuous fuzzy sets. It is also shown that continuous fuzzy sets are computationally indistinguishable from some kind of discontinuous fuzzy sets.

1. Introduction

In several decision processes the final result has to be stated as a precise one, even though it is usually based on imprecise information [2]. In order to transform an imprecise conclusion into a precise one, we must reduce fuzzy sets to crisp sets, and sometimes to crisp singletons. For example, in fuzzy control applications, the fuzzy output is usually reduced to the center of area [4].

In this paper we deal with the concept of contradiction, and how it is related to decision processes in the sense that for taking a proper decision based on a fuzzy set, we should take into consideration not only the fuzzy set, but also its negation. For other approaches see [8,10].

2. Previous concepts

2.1. Crisp sets, fuzzy sets and discontinuity

Let us consider the poset $([0,1]^X, \leqslant)$ where X is a set, $[0,1]^X = \{\mu; \mu : X \to [0,1]\}$, and $\leqslant$ is the pointwise partial order, i.e. $\sigma \leqslant \mu \iff \sigma(x) \leqslant \mu(x)$ for $\mu, \sigma \in [0,1]^X$, and all $x \in X$.

It is well known that the set $\{0,1\}^X = \{\mu; \mu : X \to \{0,1\}\}$ of crisp sets, with the operations (min, max and $1 - id$) is isomorphic to the power set of $X, \mathscr{P}(X)$ with the operations ($\cap, \cup$ and ^{c}).

The crisp sets in the universe X can be viewed as discontinuous bivalent fuzzy ('classical preservation principle' [6]). In addition, by the 'resolution theorem' (see [5]), fuzzy sets $\mu \in [0,1]^X$ can also be viewed as unions of a special family of 'indexed' fuzzy sets,

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$$\mu(x) = \sup_{r\in[0,1]} f_r(x), \quad \textit{with } f_r(x) = \min(r, \mu^{(r)}(x)) \quad \forall x \in X,$$

where $\mu^{(r)} \in \{0,1\}^X$ is the characteristic function[1] of the strong r-cut $\{x \in X; r < \mu(x)\}$. Of course, functions f_r do represent discontinuous fuzzy sets.

These facts reflect a deep presence of discontinuity in the relationship between fuzzy and crisp sets, and in this paper it is shown how crisp sets can be understood as classes of a particular type of discontinuous fuzzy sets. In the jump from imprecision to precision, discontinuity plays – as it is intuitive – an important role.

2.2. Threshold of the contradiction of a fuzzy set

Let us consider the set of functions $[0,1]^X$ endowed with a negation, $\prime$, functionally expressed by means of a strong negation function N, that is,

$$\mu' = N \circ \mu, \quad \text{or} \quad \mu'(x) = N(\mu(x)), \quad \text{for all } x \in X.$$

From the properties of N it follows that there always exists a unique $n \in (0,1)$ such that $N(n) = n$ (n is the so-called fix-point of N), therefore, the functional equation $\mu = N(\mu)$ has a unique solution $\mu = \mu_n$, with μ_n being the function constantly equal to the number n.

As it was proved in [7], any strong negation N can be expressed as $N(x) = \varphi^{-1}(1-\varphi(x))$ for some order-automorphism φ of the unit interval. Hence, from $n = \varphi^{-1}(1-\varphi(n))$ it follows $n = \varphi^{-1}(\frac{1}{2})$.

Two functions μ and σ are contradictory if $\mu \leqslant \sigma\prime$. If $\mu \leqslant \mu\prime$, μ is *self-contradictory*. According to the above paragraph, $\mu \leqslant \mu'$ is equivalent to $\mu(x) \leqslant \varphi^{-1}(\frac{1}{2}) = \mu_n(x)$ for all $x \in X$. Therefore, we can say that the *threshold of self-contradiction* of fuzzy sets is $\varphi^{-1}(\frac{1}{2})$, provided N is a strong negation given by the order-automorphism φ.

3. First results

Let $n \in (0,1)$ be the fix-point of the strong negation N. Let $[0,1]^X_n$ (see [10]) be the set of all functions in $[0,1]^X$ except those that take the value n for some $x \in X$,

$$[0,1]^X_n = \{\mu; \mu: X \to [0,1]; \quad \mu(x) \neq n \quad \forall x \in X\}.$$

Of course, this set contains the set of the crisp sets, $\{0,1\}^X$ as a subset and it is $\mu \in [0,1]^X_n$ if and only if $\mu^{-1}(n) = \{x \in X; \mu(x) = n\} = \emptyset$.

Let us introduce a new mapping $\alpha : [0,1]^X_n \to \mathscr{P}(X)$, defined by

$$\alpha(\mu) = \{x \in X; n < \mu(x)\}.$$

This mapping fulfills the following properties:

1. α is non-decreasing: $\mu \leqslant \sigma \Rightarrow \alpha(\mu) \subseteq \alpha(\sigma)$.
   Indeed, $x \in \alpha(\mu) \Rightarrow n < \mu(x) \leqslant \sigma(x) \Rightarrow n < \sigma(x) \Rightarrow x \in \alpha(\sigma)$.
   Thus α is an order-homomorphism between $([0,1]^X_n, \leqslant)$ and $(\mathscr{P}(X), \subseteq)$.
2. $\alpha(\mu\prime) = (\alpha(\mu))^c$.
   Since N is a strong negation, it is $\alpha(\mu\prime) = \{x \in X; n < \mu\prime(x)\} = \{x \in X; n < N(\mu(x))\} = \{x \in X; N(n) > N(N(\mu(x))\} = \{x \in X; n > \mu(x)\} = (\alpha(\mu))^c$. In consequence, α establishes a partition on X: $\alpha(\mu) \cup \alpha(\mu\prime) = \alpha(\mu) \cup (\alpha(\mu))^c = X$, and $\alpha(\mu) \cap \alpha(\mu\prime) = \alpha(\mu) \cap (\alpha(\mu))^c = \emptyset$.
3. α is surjective: for all $A \in \mathscr{P}(X)$ there is $\mu_A \in [0,1]^X_n$ and $\alpha(\mu_A) = A$. Indeed, $\alpha(\mu_A) = \{x \in X; n < \mu_A(x)\} = \{x \in X; \mu_A(x) = 1\} = \mu_A^{-1}(1) = A$.
   Obviously, $\alpha(\mu_0) = \emptyset$ and $\alpha(\mu_1) = X$, with $\mu_1(x) = 1$ and $\mu_0(x) = 0$ for all x in X.

Remark 1. α is not injective. There is $\alpha(\mu) = \{x \in X; n < \mu(x)\} = \emptyset \iff \mu(x) < n$ for all $x \in X$.

Therefore, if $r < n, \alpha(\mu_r) = \{x \in X; n < \mu_r(x) = r\} = \emptyset$, where μ_r the function constantly equal to r. For $r_1, r_2 < n$, with $r_1 \neq r_2$ we have $\alpha(\mu_{r_1}) = \emptyset = \alpha(\mu_{r_2})$, but $\mu_{r_1} \neq \mu_{r_2}$. Analogously, $\alpha(\mu) = X \iff \{x \in X; n < \mu(x)\} = X \iff n < \mu(x)$ for all $x \in X$. Thus if $n < r, \alpha(\mu_r) = \{x \in X; r = \mu_r(x) < n\} = X$, where μ_r is the function constantly equal to r. Then for $n < r_1, r_2$, with $r_1 \neq r_2$, we have $\alpha(\mu_{r_1}) = X = \alpha(\mu_{r_2})$, but $\mu_{r_1} \neq \mu_{r_2}$.

4. Morphisms

4.1. Epimorphism between the distributive lattices $([0,1]^X_n, \min, \max)$ and $(\mathscr{P}(X), \cap, \cup)$

Let us consider the conjunction $(\cdot)$ and the disjunction $(+)$ operators on $[0,1]^X_n$ functionally expressed by

$$(\mu \cdot \sigma)(x) = \min\{\mu(x), \sigma(x)\}, \quad (\mu + \sigma)(x) = \max\{\mu(x), \sigma(x)\} \quad \text{for all } x \in X.$$

[1] For any subset $A \subset X$, the function defined as $\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is the characteristic function of A.

Obviously, $([0,1]^X, \cdot, +)$ is a distributive lattice, and:

- $\alpha(\mu\cdot\sigma) = \{x \in X; n < \min\{\mu(x), \sigma(x)\}\} = \{x \in X; n < \mu(x) \textit{ and } n < \sigma(x)\} = \alpha(\mu) \cap \alpha(\sigma)$.
- $\alpha(\mu+\sigma) = \{x \in X; n < \max\{\mu(x), \sigma(x)\}\} = \{x \in X; n < \mu(x) \textit{ or } n < \sigma(x)\} = \alpha(\mu) \cup \alpha(\sigma)$.

Notice that *these results do not hold* when $\cdot$ is a continuous t-norm $T < \min$, or $+$ is a continuous t-conorm $S > \max$ (see [1]).

Consequently, α *is an epimorphism* (see 3, Section 3) *between the distributive lattices* $([0,1]_n^X, \min, \max)$ *and* $(\mathscr{P}(X), \cap, \cup)$.

4.2. Isomorphism between $([0,1]_n^X/\alpha, \odot, \oplus)$ and $(\mathscr{P}(X), \cap, \cup)$

Consider the quotient set, $[0,1]_n^X/\alpha$, with the classes $[\mu] = \{\sigma \in [0,1]_n^X;\ \alpha(\sigma) = \alpha(\mu)\}$ for μ in $[0,1]_n^X$.

Let us define the two mappings,

$$\odot : [0,1]_n^X/\alpha \times [0,1]_n^X/\alpha \to [0,1]_n^X/\alpha, \quad [\mu]\odot[\sigma] = [\mu\cdot\sigma]$$
$$\oplus : [0,1]_n^X/\alpha \times [0,1]_n^X/\alpha \to [0,1]_n^X/\alpha, \quad [\mu]\oplus[\sigma] = [\mu+\sigma]$$

These mappings are operations in $[0,1]_n^X/\alpha$ since they do not depend on the elements representing the classes:

- If $[\sigma_1] = [\sigma_2], \quad [\phi_1] = [\phi_2] \Rightarrow \alpha(\sigma_1) = \alpha(\sigma_2), \quad \alpha(\phi_1) = \alpha(\phi_2)$
  $\Rightarrow \alpha(\sigma_1\cdot\phi_1) = \alpha(\sigma_1)\cap\alpha(\phi_1) = \alpha(\sigma_2)\cap\alpha(\phi_2) = \alpha(\sigma_2\cdot\phi_2)$
  $\Rightarrow [\sigma_1\cdot\phi_1] = [\sigma_2\cdot\phi_2]$
- If $[\sigma_1] = [\sigma_2], \quad [\phi_1] = [\phi_2] \Rightarrow \alpha(\sigma_1) = \alpha(\sigma_2), \quad \alpha(\phi_1) = \alpha(\phi_2)$
  $\Rightarrow \alpha(\sigma_1+\phi_1) = \alpha(\sigma_1)\cup\alpha(\phi_1) = \alpha(\sigma_2)\cup\alpha(\phi_2) = \alpha(\sigma_2+\phi_2)$
  $\Rightarrow [\sigma_1+\phi_1] = [\sigma_2+\phi_2]$.

Thus $([0,1]_n^X/\alpha, \odot, \oplus)$ is a distributive lattice whose minimum and maximum elements are, respectively, $[\mu_0] = \{\sigma \in [0,1]_n^X; \alpha(\sigma) = \emptyset\}$, $[\mu_1] = \{\sigma \in [0,1]_n^X; \alpha(\sigma) = X\}$.

Theorem 1. *The distributive lattices* $([0,1]_n^X/\alpha, \odot, \oplus)$ and $(\mathscr{P}(X), \cap, \cup)$, *are isomorphic.*

Proof. The mapping, $\beta : [0,1]_n^X/\alpha \to \mathscr{P}(X)$, given by $\beta([\mu]) = \alpha(\mu)$, verifies:

1. β is independent of the elements representing the classes:$[\mu] = [\sigma] \Rightarrow \alpha(\mu) = \alpha(\sigma) \Rightarrow \beta([\mu]) = \beta([\sigma])$.
2. β is a morphism:$\beta([\mu]\odot[\sigma]) = \beta([\mu\cdot\sigma]) = \alpha(\mu\cdot\sigma) = \alpha(\mu)\cap\alpha(\sigma) = \beta([\mu])\cap\beta([\sigma])$,$\beta([\mu]\oplus[\sigma]) = \beta([\mu+\sigma]) = \alpha(\mu+\sigma) = \alpha(\mu)\cup\alpha(\sigma) = \beta([\mu])\cup\beta([\sigma])$.
3. β is injective: $\alpha(\mu) = \alpha(\sigma) \Rightarrow [\mu] = [\sigma]$.
4. β is surjective: $\forall A \in \mathscr{P}(X) \Rightarrow \beta([\mu_A]) = A$.

Hence, β is an isomorphism. □

4.3. Isomorphism between $([0,1]_n^X/\alpha, \odot, \oplus, \prime)$ and $(\mathscr{P}(X), \cap, \cup, ^c)$

Let us define the unary operation $\prime : [0,1]_n^X/\alpha \to [0,1]_n^X/\alpha$ by

$$[\mu]\prime = [\mu'] = \{\sigma; \alpha(\sigma) = \{x \in X; n < \mu'(x)\}\} = \{\sigma; \alpha(\sigma) = \{x \in X; \mu(x) < n\}\},$$

which satisfies the following properties:

1a. This operation does not depend on the elements representing the classes. Indeed, if $[\mu] = [\sigma] \Rightarrow \alpha(\mu) = \alpha(\sigma) \Rightarrow (\alpha(\mu))^c = (\alpha(\sigma))^c \Rightarrow \alpha(\mu') = \alpha(\sigma') \Rightarrow [\mu'] = [\sigma']$.

2a. $[\mu]\oplus[\mu]' = [\mu+\mu'] = \{\sigma; \alpha(\sigma) = \{x \in X; n < \max\{\mu(x), N\circ\mu(x)\}\} = \{\sigma; \alpha(\sigma) = \{x \in X; n \leqslant \mu(x) \textit{ or } n < (N\circ\mu)(x)\} = \{\sigma; \alpha(\sigma) = \{x \in X; n < \mu(x) \textit{ or } \mu(x) < n\} = \{\sigma; \alpha(\sigma) = X\} = [\mu_1]$.

3a. $([\mu]\oplus[\mu]')' = [\mu_1]' = [\mu_1'] = [\mu_0]$. Hence, $[\mu]'\odot[\mu]'' = [\mu]'\odot[\mu] = [\mu]\odot[\mu]' = [\mu_0]$.

Theorem 2. $([0,1]_n^X/\alpha, \odot, \oplus, \prime)$ *is a boolean algebra isomorphic to* $(\mathscr{P}(X), \cap, \cup, ^c)$

Proof. Follows from Theorem 1 and properties 2a. and 3a., as well as from $\beta([\mu]\prime) = \beta([\mu\prime]) = \alpha(\mu\prime) = \alpha(\mu)^c = (\beta[\mu])^c$.

There is, of course, the same number of classes in $[0,1]_n^X/\alpha$ as sets in $\mathscr{P}(X)$. Even more, each class $[\mu]$ such that $\alpha(\mu) = A \in \mathscr{P}(X)$, contains one and only one crisp set, namely the set given by the characteristic function of the set A (see Section 2.1). Hence, we can use the crisp set belonging to each class as the representative of the class.

In this way we can state that *the crisp sets in X are nothing else than classes of some discontinuous fuzzy sets in X, once these are endowed with the Zadeh's algebra, given by a triplet* $(\min, \max, N)$[12]. □

Remark 2. In the case that instead of $[0,1]^X_n$ the whole set $[0,1]^X$ is taken, then for

$$\alpha(\mu) = \{x \in X; n < \mu(x)\}, \mu \in [0,1]^X,$$

it is $\alpha(\mu) \cup \alpha(\mu') = \{x \in X; n < \mu(x)\} \cup \{x \in X; \mu(x) < n\}$, what is not X, in general. Hence, $[0,1]^X/\alpha$ would not be a boolean algebra.

In addition, changing α to

$$\alpha_1(x) = \{x \in X; n \leqslant \mu(x)\},$$

a boolean algebra is not reached either. Now $\alpha_1(\mu) \cap \alpha_1(\mu\prime) = \{x \in X; n \leqslant \mu(x)\} \cap \{x \in X; \mu(x) \leqslant n\} = \{x \in X; \mu(x) = n\}$, what is not the empty-set, in general.

4.4. On Entemann's 'clarifications'

In order to prove that fuzzy logic is a *Proof Theory*, what Entemann does in [3], is to remove the propositions whose truth value is 0.5, that is, to restrict to fuzzy propositions A such that $t(A) \neq 0.5$ (he cannot decide what to do when $t(A) = 0.5$).

Consider, as Entemann does, a set $\mathscr{A}$ of fuzzy propositions $A, B\ldots$ with truth values $t(A), t(B) \in [0,1]$ that fulfills the axioms:

1. $0 \leqslant t(A), \quad t(B) \leqslant 1$
2. $t(A \wedge B) = \min(t(A), t(B))$
3. $t(A \vee B) = \max(t(A), t(B))$
4. $t(\neg A) = 1 - t(A)$

In this case is $n = 0.5$. Provided that there are no propositions A in $\mathscr{A}$ such that $t(A) = 0.5$, Theorem 2 gives the set of classes $[0,1]^{\mathscr{A}}_{0.5}/\alpha$, isomorphic to the boolean algebra $(\{0,1\}, \min, \max, 1 - id)$. Actually, these facts are behind Entemann's reasoning.

5. Functions in $[0,1]^X$ are computationally indistinguishable to those in $[0,1]^X_n$

In many of the application fields, only continuous membership functions μ on a closed interval $X = [a,b]$ of the real line are considered. Such fact seems to imply that Theorem 2 is not relevant for applications.

Nevertheless, in the current practice everything is done with numerical values $\mu(x)$ approaching the theoretical ones as much as the *computational precision threshold* (precision granularity) allows to do.

Let $\varepsilon > 0$ be the computational precision threshold in $[0,1]$,

- y_1 and y_2 in $[0,1]$ are *ε-computationally indistinguishable values* if $|y_1 - y_2| < \varepsilon$.
- μ and σ in $[0,1]^X$ are *ε-computationally indistinguishable membership functions* if $|\mu(x) - \sigma(x)| < \varepsilon$ for all $x \in X$. That is, if $\mu(x)$ and $\sigma(x)$ are always ε-computationally indistinguishable values.

Theorem 3. *For any* $\mu \in [0,1]^X$*, there exists a* $\widehat{\mu} \in [0,1]^X_n$ *that is ε-computationally indistinguishable from* μ.

Proof. Let $\varepsilon > 0$ be the computational precision threshold in $[0,1]$. Let us denote by $\mu^{-1}(n)$ the subset of the elements in X that are inverse image of $n \in (0,1)$ for the function μ, that is, $\mu^{-1}(n) = \{x \in X; \mu(x) = n\}$, define $\widehat{\mu}_\delta$ for any $\delta \leqslant \varepsilon$, by (see Fig. 1)

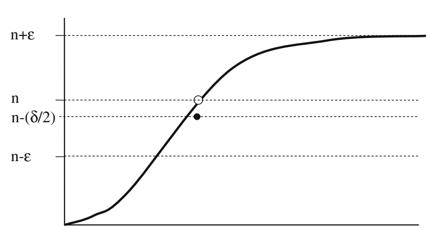

Fig. 1. Function $\widehat{\mu}_\delta$.

$$\widehat{\mu}_\delta(x) = \begin{cases} \mu(x), & \text{if } x \notin \mu^{-1}(n) \\ \mu(x) - \frac{\delta}{2}, & \text{if } x \in \mu^{-1}(n). \end{cases}$$

Obviously, it is $\widehat{\mu}_\delta^{-1}(n) = \emptyset$ and then $\widehat{\mu}_\delta \in [0,1]_n^X$. We get $|\mu(x) - (\mu(x) - \frac{\delta}{2})| = \frac{\delta}{2} < \varepsilon$. Hence, once given the computational precision threshold $\varepsilon > 0$, it follows

$$|\mu(x) - \widehat{\mu}_\delta(x)| < \varepsilon \quad \text{forall } x \in [a,b].$$

That is, μ and $\widehat{\mu}_\delta$ are ε-computationally indistinguishable. Notice that $\widehat{\mu}$ in Theorem 3 is not unique.

Remark 3. If $\mu \in [0,1]^X$ is continuous, by Theorem 3 there exists an ε-computationally indistinguishable membership function $\widehat{\mu}_\delta$, which is in $[0,1]_n^X$.

Suppose $f : [0,1] \to [0,1]$ is a continuous function, that is for any $\varepsilon > 0$ and any $b \in [0,1]$ there exists $\delta > 0$ such that if $|b - a| < \delta$ then $|f(b) - f(a)| < \varepsilon$.

Provided $\varepsilon > 0$ is the computational precision threshold in $[0,1]$ and since f is, in particular, continuous in the point $\mu(x) \in [0,1]$, there exists $\delta > 0$ such that if $a \in [0,1]$ verifies $|\mu(x) - a| < \delta$ then it is $|f(\mu(x)) - f(a)| < \varepsilon$. Hence, there can be found the membership function $\widehat{\mu}_\delta$ (that is, verifying $|\mu(x) - \widehat{\mu}_\delta(x)| < \delta$ for all $x \in X$) such that $f(\widehat{\mu}_\delta(x))$ is ε-computationally indistinguishable from $f(\mu(x))$ for all $x \in X$. *Thus, if μ is composed with a continuous function f, we can find $\widehat{\mu}_\delta$ in $[0,1]_n^X$, that makes $f \circ \mu$ and $f \circ \widehat{\mu}_\delta$ ε-computationally indistinguishable.*

In particular, if $N : [0,1] \to [0,1]$ is a strong-negation its continuity implies that $\mu\prime = N \circ \mu$ and the corresponding $\widehat{\mu}' = N \circ \widehat{\mu_\delta}$ are ε-computationally indistinguishable.

Theorem 4. *Given $\varepsilon > 0$, a membership function μ and a finite family of continuous functions $\mathfrak{F} = \{f_1, \ldots, f_n\}$, there can be found $\widehat{\mu}_\delta$ that makes $f_i \circ \mu$ and $f_i \circ \widehat{\mu}_\delta$ ε-computationally indistinguishable for all $i \in \{1, \ldots, n\}$.*

Proof. For each f_i ($i \in \{1, \ldots, n\}$), as it has just been shown, there exists $\delta_i > 0$, such that $f_i \circ \mu$ and $f_i \circ \widehat{\mu}_{\delta_i}$ are ε-computationally indistinguishable. Thus with $\delta = \min(\delta_1, \ldots, \delta_n)$, $\widehat{\mu}_\delta$ verifies $|(f_i \circ \mu)(x) - (f_i \circ \widehat{\mu}_\delta)(x)| < \varepsilon$ for all $x \in X$ and for all $i \in \{1, \ldots, n\}$.

Analogously, it can be proven that if $F : [0,1] \times [0,1] \to [0,1]$ is a continuous function in both variables, $F \circ (\mu \times \sigma)$ is ε-computationally indistinguishable from some $F \circ (\widehat{\mu}_\delta \times \widehat{\sigma}_\delta)$, with $\widehat{\mu}_\delta, \widehat{\sigma}_\delta \in [0,1]_n^X$. Obviously, if $(F \circ (\widehat{\mu}_\delta \times \widehat{\sigma}_\delta))(x) \neq n$ $\forall\, x \in X$, we get $F \circ (\widehat{\mu}_\delta \times \widehat{\sigma}_\delta) \in [0,1]_n^X$.

In particular, if $F = T$ is a continuous t-norm (or $F = S$ is a continuous t-conorm), there can be found two membership functions $\widehat{\mu}_\delta, \widehat{\sigma}_\delta \in [0,1]_n^X$ such that $T \circ (\mu \times \sigma)$ is ε-computationally indistinguishable from $T \circ (\widehat{\mu}_\delta \times \widehat{\sigma}_\delta)$ (or $S \circ (\mu \times \sigma)$ from $S \circ (\widehat{\mu}_\delta \times \widehat{\sigma}_\delta)$). $\square$

Remark 4. In addition, $\mu\prime = 1 - \mu, (\mu \cdot \sigma)(x) = \min(\mu(x), \sigma(x))$, and $(\mu + \sigma)(x) = \max(\mu(x), \sigma(x))$ are ε-computational indistinguishable from $\widehat{\mu}'_\varepsilon = 1 - \widehat{\mu}_\varepsilon, (\widehat{\mu}_\varepsilon \cdot \widehat{\sigma}_\varepsilon)(x) = \min(\widehat{\mu}_\varepsilon(x), \widehat{\sigma}_\varepsilon(x))$, and $(\widehat{\mu}_\varepsilon + \widehat{\sigma}_\varepsilon)(x) = \max(\widehat{\mu}_\varepsilon(x), \widehat{\sigma}_\varepsilon(x))$, respectively.

6. Crisp decisions with fuzzy sets

In several decision processes the final result has to be stated as a precise one, even though it is usually based on imprecise information [2]. In order to transform an imprecise conclusion into a precise one, we must reduce fuzzy sets to crisp sets, and sometimes to crisp singletons.

Using the mapping α we obtain the sets

$$\alpha(\mu_P) = \{x \in X; \varphi^{-1}(1/2) < \mu_P(x)\} = \{x \in X; N(\mu_P(x)) < \mu_P(x)\} = \{x \in X; \mu'_P(x) < \mu_P(x)\},$$

which consist of the elements in X that are 'more P, than not P'. Analogously,

$$\alpha(\mu'_P) = \{x \in X; \varphi^{-1}(1/2) < \mu'_P(x)\} = \{x \in X; \mu_P(x) < \varphi^{-1}(1/2)\} = \{x \in X; \mu_P(x) < N(\mu_P(x))\} = \{x \in X; \mu_P(x) < \mu'_P(x)\},$$

which are the elements in X that are 'more not P, than P'.

Example. If $X = [0,10] \subset \mathbb{R}, P = Big$ with the fuzzy set representation $\mu_P(x) = \frac{x}{10}$, and considering the usual strong negation $N = 1 - id$, then

$$\mu_{Big}(x) > \mu'_{Big}(x) \iff \frac{x}{10} > 1 - \frac{x}{10} \iff x > 5,$$

and the set $\alpha(\mu_{Big}) = \{x \in X; x > 5\}$ contains the elements that are more *Big* than *not-Big*, and the set $\alpha(\mu'_{Big}) = \{x \in X; x < 5\}$ the elements that are more *not-Big* than *Big*.

In this example the point $5 = \varphi^{-1}(1/2)$ is allocated neither to $\alpha(\mu_P)$, nor to $\alpha(\mu'_P)$, so it is an undecidable point.

If we transform this fuzzy set to its ε-computationally indistinguishable $\widehat{\mu}_{Big,\delta}$ given in Section 5, this point will be allocated to $\alpha(\widehat{\mu}'_{P,\delta})$. But considering this other valid definition of $\widehat{\mu}_{P,\delta}$ the point 5 will be assigned to $\alpha(\widehat{\mu}_{P,\delta})$:

$$\widehat{\mu}_{P,\delta}(x) = \begin{cases} \mu_P(x), & \text{if } x \notin \mu_P^{-1}(n), \\ \mu_P(x) + \frac{\delta}{2}, & \text{if } x \in \mu_P^{-1}(n). \end{cases}$$

Therefore this point plays a pivotal role in the decision and can be consider as the separation point between the two sets.

6.1. Threshold of separation of a predicate

The threshold of separation of an imprecise predicate P on X from its negation $\neg P$ = not P, once both are represented by well designed membership functions μ_P and $\mu_{\neg P} = \mu'_P$, respectively, is obtained (see [11]) through the analysis of the inequalities $\mu'_P(x) < \mu_P(x)$ and $\mu_P(x) < \mu'_P(x)$.

Based on these inequalities we define for all $\varepsilon > 0$ the *set of separation points* as follows,

$$\langle \mu_P \rangle = \bigcap_{\varepsilon > 0} \{ \{ x \in X; x - \varepsilon \in \alpha(\mu'_P) \ \& \ x + \varepsilon \in \alpha(\mu_P) \} \cup \{ x \in X; x + \varepsilon \in \alpha(\mu'_P) \ \& \ x - \varepsilon \in \alpha(\mu_P) \} \}.$$

Examples

In the previous example the *threshold of separation* of *Big* will be

$$\langle \mu_{Big} \rangle = \{5\}$$

But if we change the negation to $N(x) = \frac{1-x}{1+x}$ then

$$\mu'_{Big} < \mu_{Big} \iff \frac{10-x}{10+x} < \frac{x}{10} \iff 0 < x^2 + 20x - 100 \iff 200 < (x+10)^2,$$

$\alpha(\mu_{Big}) = \{x \in [0,10]; 200 < (x+10)^2\} = \{x \in [0,10]; 10\sqrt{2} - 10 < x\}, \alpha(\mu'_{Big}) = \{x \in [0,10]; 10\sqrt{2} - 10 > x\}$ and a different *threshold of separation* of *Big* will be obtained,

$$\langle \mu_{Big} \rangle = 10\sqrt{2} - 10 = 10(\sqrt{2} - 1) \approx 4.1$$

Using the same negation $N(x) = \frac{1-x}{1+x}$, and representing P = *small* by the decreasing function $\mu_P(x) = 1 - \frac{x}{10}$, we get that

$$\mu'_{Small} < \mu_{Small} \iff N\left(1 - \frac{x}{10}\right) = \frac{x}{20-x} < 1 - \frac{x}{10} \iff 0 < x^2 - 40x + 200 \iff x < 5.9$$

hence, $\alpha(\mu_{Small}) = \{x \in [0,10]; x < 5.9\}$ and $\alpha(\mu'_{Small}) = \{x \in [0,10]; 5.9 < x\}$ and the *threshold of separation* of *Small* will be

$$\langle \mu_{Small} \rangle = \{5.9\},$$

Hence, the study of the inequalities $\mu < \mu\prime$ and $\mu > \mu\prime$ is a way for obtaining a crisp decision from a fuzzy set.

6.2. Confidence on the crisp decisions

Although the threshold of separation of an imprecise predicate P on X can be found through the sets $\alpha(\mu_P)$ and $\alpha(\mu'_P)$, in most real-world cases it would be unrealistic to take that threshold as a crisp edge since a very small change in the values of μ_P or μ'_P could produce opposite decisions for the same elements.

Therefore, let us define *a confidence function* ($\gamma_P : X \to [0,1]$) on the decision of taking "x is P" when "x is more P than not-P" (see Fig. 2), and ($\gamma_{P'} : X \to [0,1]$) on the decision of taking "x is not P" when "x is more not-P than P" (see Fig. 4):

$$\gamma_P(x) = \begin{cases} \mu_P(x) - \mu'_P(x), & x \in \alpha(\mu_P) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_{P'}(x) = \begin{cases} \mu'_P(x) - \mu_P(x), & x \in \alpha(\mu'_P) \\ 0, & \text{otherwise} \end{cases}$$

For instance in the previous example of *Big* the confidence functions will be:

$$\gamma_{Big}(x) = \begin{cases} \frac{x}{10} - \left(1 - \frac{x}{10}\right) = \frac{x}{5} - 1, & x \in (5,10] \\ 0, & x \in [0,5] \end{cases}$$

$$\gamma_{not-Big}(x) = \begin{cases} \left(1 - \frac{x}{10}\right) - \frac{x}{10} = 1 - \frac{x}{5}, & x \in [0,5] \\ 0, & x \in (5,10] \end{cases}$$

Thus, γ_{Big} allows us to take "x = 10" as "Big" with a confidence degree of 1, since $\gamma_{Big}(10) = \frac{10}{5} - 1 = 1$, while taking "$x$ = 7.5" as "Big" can be done with a confidence degree of 0.5, since $\gamma_{Big}(7.5) = \frac{7.5}{5} - 1 = 0.5$ (see Fig. 3).

Function $\gamma_{\text{not-Big}}$ allows to take "x = 0", "x = 2.5" or "x = 5" as "not-big" with the confidence degrees given by $\gamma_{\text{not-Big}}(0) = 1 - \frac{0}{5} = 1$, $\gamma_{\text{not-Big}}(2.5) = 1 - \frac{2.5}{5} = 0.5$ and $\gamma_{\text{not-Big}}(5) = 1 - \frac{5}{5} = 0$, (see Fig. 4) respectively (see Fig. 5).

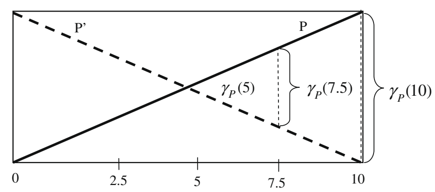

Fig. 2. "x is more Big than not-Big".

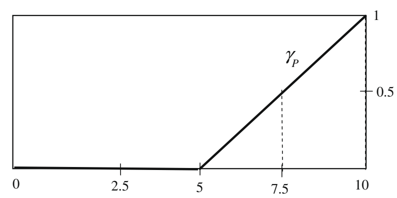

Fig. 3. Confidence function γ_{Big}.

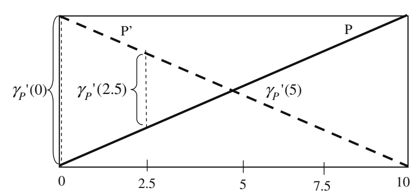

Fig. 4. "x is more not-Big than Big".

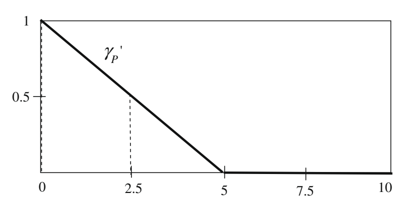

Fig. 5. Confidence function $\gamma_{\text{not-Big}}$.

Looking at functions γ_{Big} and $\gamma_{\text{not-Big}}$ can be seen that the confidence degrees of taking "$x = 5$" as "Big" or as "not-Big" are 0, as expected for an undecidable point.

Remark 5. If the predicate P is crisp in X, that is, $\mu_P \in \{0,1\}^X$, then $\alpha(\mu_P) = \{x \in X; \mu_P(x) > 1 - \mu_P(x)\} = \{x \in X; \mu_P(x) > \frac{1}{2}\} = \mu_P^{-1}(1)$, since in this case it is $\mu'_P(x) = N(\mu_P(x)) = 1 - \mu_P(x)$ for all $x \in X$, and $\mu_P(x) > \frac{1}{2}$ is equivalent to $\mu_P(x) = 1$. Hence,

$$\gamma_P(x) = \begin{cases} \mu_P(x) - (1 - \mu_P(x)) = 2\mu_P(x) - 1 = 1, & x \in \mu_P^{-1}(1) \\ 0, & x \notin \mu_P^{-1}(1) \end{cases}$$

that is, $\gamma_P(x) = \mu_P(x)$ for all $x \in X$. *In the limiting case when the predicate is crisp, the confidence function is nothing else than its membership function*, that is, the confidence is total for $x \in P$. Of course, in general, the closer γ_P is to μ_P in $\alpha(\mu_P)$, the more crisp is the set.

7. Conclusion

In this paper it has been proved that to obtain a Boolean algebra of classes of fuzzy sets isomorphic to $(\mathscr{P}(X), \cap, \cup, ^c)$ *some continuous functions should always be avoided*. The functions in $[0,1]^X$ to be avoided are those reaching the level given by the fix-point of the negation. However, in this paper it has also been proven that $[0,1]_n^X$ and $[0,1]^X$ are computationally indistinguishable.

To reach a crisp decision from imprecise information we have taken an approach based on the threshold of separation that could be stated as follows: we make a decision over a certain threshold and below it we make the opposite decision, while in the rare case of exactly matching the threshold we cannot make a decision with confidence. Also the confidence on the decision have been introduced, allowing to distinguish decisions with different degrees of confidence.

In this work we have answered two questions: how to find an appropriate threshold, and what is the meaning of this threshold. Although not exactly with the same aim, these points were previously discussed in [9] and in [11].

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References

[1] C. Alsina, M.J. Frank, B. Schweizer, Associative Functions. Triangular Norms and Copulas, World Scientific, Singapore, 2006.
[2] R.E. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Management Science 17 (4) (1970) B141–B164.
[3] C.W. Entemann, Fuzzy logic: misconceptions and clarifications, Artificial Intelligence (17) (2002) 65–84.
[4] A.O. Esogbue, Qiang Song, On optimal defuzzification and learning algorithms: theory and applications, Fuzzy Optimization and Decision Making 2 (4) (2003) 283–296.
[5] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic, Prentice Hall PTR, Upper Saddle River, NJ, 1995.
[6] A. Pradera, E. Trillas, E. Renedo, An overview on the construction of fuzzy set theories, New Mathematics and Natural Computation 1 (3) (2005) 329–358.
[7] E. Trillas, Sobre funciones de negación en la teoría de conjuntos difusos, Stochastica 3 (1) (1979) 47–59. Reprinted (English version), in: S. Barro et al. (Eds.), Advances of Fuzzy Logic, Press of the Universidad de Santiago de Compostela, 1998, pp. 31–43.
[8] E. Trillas, C. Alsina, A. Pradera, Searching for the roots of non-contradiction and excluded-middle, International Journal of General Systems 5 (31) (2002) 499–513.
[9] E. Trillas, A.R. de Soto, On the threshold of bounded pseudo-distances, Mathware and Soft Computing 15 (2) (2008) 189–200.
[10] E. Trillas, E. Renedo, S. Guadarrama, On a new theory of fuzzy sets with just one self-contradiction, in: Proceedings of the 10th IEEE International Conference on Fuzzy Systems, Melbourne, Australia, vol. 2, 2001, pp. 105–108.
[11] E. Trillas, L.A. Urtubey, Towards the dissolution of the Sorites paradox, Applied Soft Computing, in press, doi: 10.1016/j.physletb.2003.10.071.
[12] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Evaluating premises, partial consequences and partial hypotheses *

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Abstract— *To evaluate premises, consequences and hypotheses, on this paper relevance and support ratios are defined for each of them. This allows to distinguish consequences based on the number of premises that support them, and also to reduce the set of premises while maintaining the same consequences. Since the relation between premises and hypotheses is, in some sense, similar to the relation between consequences and premises, analogous ratios are defined for hypotheses and premises.*

Keywords— Conjectures, Consequences, Hypotheses, Relevance, Support.

1 Introduction

The aim of most problems is to make choice between possible solutions, a clear example is a medical diagnosis problem. In this paper we allocate degrees for the elements in the set of consequences, hypotheses or premises in order to choose the consequence, hypothesis or premise with the biggest degree. Papers [2] and [3] had dealt with that idea of graded consequences.

The following section will show Conjectures, Hypotheses, and Consequences (CHC) models introduced in [7], which was suggested, in part by Watanabe in [10], and takes into account the particular case of partial consequence's operator [8] [9]. Partial operators of consequences are that allow to get consequences of each premise, or subset of premises, and obtaining the final set of consequences as the union of all these partial consequences.

To define support for each consequence (section 3) we consider that consequences with bigger support, are those that are supported by more premises or subsets of premises. By the way, different degrees are allocated for consequences. So, for example, following with a medical diagnosis problem in which the premises are diseases and the consequences are symptoms, we can choose between consequences, and select as the stronger, the one with biggest support.

In section 4, we deal with a measure of relevance for premises that is useful for knowing which premises have more importance, in the sense of how many consequences can be deduced from them. Thanks to that measure, the set of premises can be reduced to a smaller set with the same relevance. This reduced set of premises gets rid of superfluous premises and yet allows to work with less premises, while getting the same set of consequences. Till now all premises seemed to have the same importance.

Finally, in section 5, we also consider partial hypotheses, that is, hypotheses of one premise, and not hypotheses of all premises. And analogous measure of support for premises, as well as of relevance for partial hypotheses, are defined. This allows to evaluate subset of partial hypotheses by counting how many premises they give as consequences.

2 Basic concepts

2.1 CHC models

Reasoning can be understood as a process allowing to get conjectures from a set of premises, P. There are three basic types of reasoning: deduction, abduction and induction. A process that allows to get consequences is a deductive reasoning, a process that allows to get hypotheses is an abductive reasoning, and finally, if the process allows to get speculations, it is an speculative reasoning.

In this paper, CHC models are defined on a preordered set $(L, \leq)$, endowed with a negation, $'$. And when it is necessary, the preordered set will be endowed with an infimum, $\cdot$, and a supremum, $+$, operations $(L, \leq, ', \cdot, +)$, a preorder with infimum and supremum operations is a partial ordered set (poset)with these operations. The infimum of L is called first element and it is denoted by 0, the supremum of L is called the last element and it is denoted by 1. The paper only deals with finite algebraic structures, that is, with a finite set L.

CHC models can be based on consequences operators [8, 9],

Definition 2.1 *If L is a set, and $\mathfrak{F} \subset \mathbb{P}(L)$, it is said that $(L, \mathfrak{F}, C)$ is a* structure of consequences*, provided that C : $\mathfrak{F} \to \mathfrak{F}$ verifies,*

1. *$P \subset C(P)$, for all $P \in \mathfrak{F}$ (C is* extensive*)*
2. *If $P \subset Q$, then $C(P) \subset C(Q)$, for all $P, Q \in \mathfrak{F}$ (C is* monotonic*)*
3. *$C(C(P)) = C(P)$, or $C^2 = C$, for all $P \in \mathfrak{F}$ (C is a* clausure*)*

i.e. C is an operator of consequences *(in the sense of Tarski) for $\mathfrak{F}$ in L.*

For each $\{q\} \in \mathfrak{F}$, let us write $C(q) = C(\{q\})$.

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Definition 2.2 *A consequences' operator C is consistent in P, if for all $q \in C(P)$, $q' \notin C(P)$.*
A structure of consequences $(L, \mathfrak{F}, C)$ is consistent if C is consistent for all $P \in \mathfrak{F}$.

Let P be the set of premises, $P \neq \emptyset$, and $C(P)$ a set of consequences for P. Conjectures of P can be defined from a consistent consequences operator C, as those elements whose negation is not in $C(P)$, $Conj_C(P) = \{q \in L; q' \notin C(P)\}$. Hypothesis can be defined by $Hyp_C(P) = \{h \in Conj_C(P) - C(P); \{h\} \in \mathfrak{F},\ P \subset C(h)\}$.

Finally, speculations are those conjectures that are neither consequences, nor hypotheses. Hence, $Sp_C(P) = Conj_C(P) - [C(P) \cup Hyp_C(P)] = \{q \in L, q \notin C(P), q' \notin C(P), q \notin Hyp_C(P)\}$.

2.2

Definition 2.3 *A consequences' operation C is a partial consequences operator if $C(P) = \bigcup_{R \subset P, R \in \mathfrak{F}} C(R)$.*

Definition 2.4 *A decomposable consequences operator is a consequences' operator such that $C(P) = \bigcup_{p \in P} C(R)$.*

This paper considers *partial hypothesis*, elements that are hypotheses of a subset of the set of premises P. This idea comes from that of *partial consequences*.

Definition 2.5 *For each set P of premises the partial hypotheses set is,*

$$Hyp^*_C(P) = \{h \in \{L - 0\} - P; \{h\} \in \mathfrak{F}, \exists R \subset P,$$
$$R \subset C(h)\}.$$

Obviously, hypotheses are partial hypotheses, since $P \subset P$ and $P \subset C(h)$, provided h is a hypothesis.

Remark 2.6 *Although hypotheses are anti-monotonic ($P_1 \subset P_2$, implies $Hyp(P_2) \subset Hyp(P_1)$), partial hypotheses are (as it is easy to prove) monotonic ($P_1 \subset P_2$, implies $Hyp^*(P_1) \subset Hyp^*(P_2)$). That is why they can not be considered classical hypotheses.*

2.3

The paper deals with a general concept of measure [6], defined in a preordered set $(L, \leq)$. A measure is a mapping $m : L \to [0,1]$, such that:

- There exists a minimal $x_0 \in L$, such that $m(x_0) = 0$
- There exists a maximal $x_1 \in L$, such that $m(x_1) = 1$
- If $x \leq y$, then $m(x) \leq m(y)$.

3 Consequences support

This section introduces a ratio in order to distinguish which consequences are the more supported by a given set of premises. And proof in which cases is a measure

Let's recall that in this paper L is assumed to be a finite set.

Definition 3.1 *The support of $q \in L$ is the ratio of subsets of premises that allow getting q as a consequence, to all possible subsets of premises.*

$$Supp_{C,P}(q) = \frac{|\{R \in \mathbb{P}(P); q \in C(R)\}|}{2^{|P|} - 1} = \qquad (1)$$
$$\frac{|\{R \in \mathbb{P}(P); q \in C(R)\}|}{|\mathbb{P}(P) - \emptyset|}.$$

Since $P \neq \emptyset$, it is $|P| > 0$ and the quotient in the definition is possible.

The bigger support a consequence has, the more subsets of premises allow deducing it.
Notice that if $q \notin C(P), Supp_{C,P}(q) = 0$, since if there were $R \in \mathbb{P}(P)$ such that $q \in C(R)$, because of the monotonicity of the consequence operator, $C(R) \subset C(P)$ would imply $q \in C(P)$.

This ratio verifies the following properties,

- If $P \subset Q$, it is $Supp_{C,P}(q) \leq Supp_{C,Q}(q)$, for all $q \in L$.
- If $P = \{p\}, \forall q \in C(P)$, it is $Supp_{C,P}(q) = 1$.
- For all $q \in C(P)$, $Supp_{C,P}(q) > 0$.
- $Supp_{C,P}(q) = 1$ means that q is a consequence for all $R \in \mathbb{P}(P)$. Particularly, q is consequence of all $p \in P$.

The support defined by (1), is not a measure in general. For example, if $C(P) = P$, $\forall P \in \mathbb{P}(L)$, let P be a set with more than one element. If $q \in L$, it is either $q \in P$, or $Supp(q) = 0$. If $q \in P$, there exists $p \in P$ such that $p \neq q$, and $q \notin C(p)$ and $Supp(q) \neq 1$. Therefore, there is no $q \in L$ such that $Supp_{C,P}(q) = 1$.

Remark 3.2 *$Supp_{C,P}$ is monotonic with respect to the preorder given by C, $q_1 \leq_C q_2$ iff $q_2 \in C(q_1)$ [1],*

Proof. Since if $q_1 \leq_C q_2$, for each R such that $q_1 \in C(R)$, it is $C(q_1) \in C(C(R)) = C(R)$, and, as $q_2 \in C(q_1)$, it is also $q_2 \in C(R)$. So, $Supp_{C,P}(q_1) \leq Supp_{C,P}(q_2)$.
□

Since, given P, the relation defined between the pairs of elements in L with the same value of $Supp_{C,P}$, is an equivalence, the classes

$$[q] = \{v \in L;\ Supp_{C,P}(v) = Supp_{C,P}(q)\}$$

give a partition on L in a number of parts that is, at most, $2^{|P|}$.

3.1 The case of the operator $C_\bullet$

Let $(L, \leq)$ be now a preordered set in which is defined an infimum operation denoted by '$\cdot$'.

The partial consequences operator $C_\bullet$ gives consequences that are consequences for a subset of the set of premises P, it is $C_\bullet(P) = \{q \in L; \exists\{p_1, p_2, ..., p_n\} \subset P : p_1 \cdot p_2 \cdot ... \cdot p_n \leq q\}$. It is a partial consequences' operator and it obviously verifies $C_\bullet(P) = \bigcup_{R \subset P, R \in \mathfrak{F}} C_\bullet(R)$.

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Notice, that as L is finite, P is also finite and $C_\bullet(P) = \{q \in L; \inf P \leq q\}$, which equal to the infimum operator of consequences $C_\wedge$.

$$Supp_{C_\bullet,P}(q) = \frac{|\{R \in \mathbb{P}(P) \in P; \inf(R) \leq q\}|}{2^{|P|} - 1} \qquad (2)$$

Let's see what specific properties are verified by $Supp_{C_\bullet,P}$,

- If L has last element, it implies $1 \in L$, then $1 \in C_\bullet(P)$ and $Supp_{C_\bullet,P}(1) = 1$.
- If $q_1 \leq q_2$, then $Supp_{C_\bullet,P}(q_1) \leq Supp_{C_\bullet,P}(q_2)$. That is the function $Supp_{C_\bullet,P}$ is monotonic.
- $Supp_{C_\bullet,P}(\sup\{q_1,q_2\}) \geq \max\{Supp_{C_\bullet,P}(q_1), Supp_{C_\bullet,P}(q_2)\}$, provided $\sup\{q_1,q_2\}$ exists.
- $Supp_{C_\bullet,P}(\inf\{q_1,q_2\}) \leq \min\{Supp_{C_\bullet,P}(q_1), Supp_{C_\bullet,P}(q_2)\}$

Corollary 3.3 *Let $(L, \leq, ', \cdot, +)$ be a partial ordered set with infimum and supremum operations and first and last elements. If $P \neq \{0\}$, the function $Supp_{C_\bullet,P} : L \to [0,1]$ is a measure.*

Proof. It is monotonic, and it verifies the boundary conditions, since 0 is not a consequence $Supp_{C_\bullet,P}(0) = 0$ and $Supp_{C_\bullet,P}(1) = 1$. □

3.2 *The case of the operator $C_\leq$*

$C_\leq$ is the partial consequences operator that gives as consequences those elements that follow from at least one premise in P, formally, it is $C_\leq(P) = \{q \in L; \exists p \in P : p \leq q\}$, see [8]. Hence, it can be considered as a decomposable consequences' operator, since allows getting consequences that are not deduced from all premises. It is straightforward that $C_\leq(P) = \bigcup_{R \subset P, R \in \mathfrak{F}} C(R) = \bigcup_{p \in P} C_\leq(p)$.

In this case, a different definition of the support's ratio seems to be more convenient, since it deals only with consequences of each $p \in P$, nor with consequences of each subset of $\mathbb{P}(P)$.

Definition 3.4 *The support of $q \in C_\leq(P)$ is the ratio of premises that allow getting q as consequence to all premises.*

$$\widehat{Supp}_{C_\leq,P}(q) = \frac{|\{p \in P; p \leq q\}|}{|P|} \qquad (3)$$

Since $P \neq \emptyset$, it is $|P| > 0$ and the quotient in the definition is possible.

If $q \notin C_\leq$, then $\widehat{Supp}_{C_\leq,P}(q) = 0$.

So, the bigger Support an element has, the more premises allow to reach it.

$\widehat{Supp}_{C_\leq,P}$ verifies,

- If $P = \{p\}, \forall q \in C_\leq(p)$, it is $\widehat{Supp}_{C_\leq,P}(q) = 1$.
- If L has last element, 1, then $1 \in C_\leq(P)$ and $\widehat{Supp}_{C_\leq,P}(1) = 1$.
- For no $q \in C_\leq(P)$ is $\widehat{Supp}_P(q) = 0$. That is, for all $q \in C_\leq(P)$, $\widehat{Supp}_{C_\leq,P}(q) > 0$.
- $\widehat{Supp}_{C_\leq,P}(q) = 1$ means that q is a consequence for all $p \in P$.
- If $q_1 \leq q_2$, then $\widehat{Supp}_{C_\leq,P}(q_1) \leq \widehat{Supp}_{C_\leq,P}(q_2)$. That is the function $\widehat{Supp}_{C_\leq,P}$ is monotonic with respect to $\leq$.

Remark 3.5 *In order to know what happens if we calculate the support for the infimum or supremum, of two consequences, provided it exists and it is a consequence, weak boundaries are found,*

- $\widehat{Supp}_{C_\leq,P}(\sup\{q_1,q_2\}) \geq \max\{\widehat{Supp}_{C_\leq,P}(q_1), \widehat{Supp}_{C_\leq,P}(q_2)\}$
- $\widehat{Supp}_{C_\leq,P}(\inf\{q_1,q_2\}) \leq \min\{\widehat{Supp}_{C_\leq,P}(q_1), \widehat{Supp}_{C_\leq,P}(q_2)\}$

Obviously, if the operator is consistent, that is, if $q \in C_\leq(P)$, then $q' \notin C_\leq(P)$, it follows $\widehat{Supp}_{C_\leq,P}(q') = 0$.

Theorem 3.6 *Let $(L, \leq, ', \cdot, +)$ be a partial ordered set with first and last elements and $P \neq \{0\}$. Function $\widehat{Supp}_{C_\leq,P} : L \to [0,1]$ is a measure.*

Proof. It is monotonic as it is stated above. Since 0 is not a consequence $\widehat{Supp}_{C_\leq,P}(0) = 0$. Finally, it is obvious that $1 \in C_\leq(P)$ and $\widehat{Supp}_{C_\leq,P}(1) = 1$. □

From $\widehat{Supp}_{C_\leq,P}$ we can calculate $Supp_{C_\leq,P}$. If $\widehat{Supp}_{C_\leq,P}(q) = k$ and $|P| = n$, it is $|\{R \in \mathbb{P}(P); q \in C(R)\}| = k \cdot n$. Hence,

$$Supp_{C_\leq,P} = \frac{2^n - \sum_{i \in \{0,\ldots,n-k\cdot n\}} \frac{(n-k\cdot n)!}{i!(n-k\cdot n-i)!}}{2^n - 1}. \qquad (4)$$

The numerator in (4) is the number of subsets of premises that contain at least one of the premises supporting q.

Corollary 3.7 *Let $(L, \leq)$ be a preordered set with first and last elements and $P \neq \{0\}$. Function $Supp_{C_\leq,P} : L \to [0,1]$ is a measure.*

Proof. It is monotonic as it is proven at the beginning of this section, and it verifies boundary conditions, since 0 is not a consequence $Supp_{C_\leq,P}(0) = \frac{2^n - 2^n}{2^n - 1} = 0$ and $Supp_{C_\leq,P}(1) = \frac{2^n - 1}{2^n - 1} = 1$. □

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Example 3.8 *Figure 1 represents a preordered set of medical symptoms and deseases for patients. Let's calculate the support for the consequences for a patient with* $P = \{antibody, bacterium\}$. *To such an end, let us notice that,* $C_{\leq}(P) = \{antibody, bacterium, fever, eruption, 1\}$. *Then,*

- $\widehat{Supp}_{C_{\leq},P}(antibody) = \frac{1}{2}$, *and* $Supp_{C_{\leq},P}(antibody) = \frac{2^2-(1+1)}{2^2-1} = \frac{2}{3}$.
- $\widehat{Supp}_{C_{\leq},P}(bacterium) = \frac{1}{2}$, *and* $Supp_{C_{\leq},P}(bacterium) = \frac{2}{3}$.
- $\widehat{Supp}_{C_{\leq},P}(fever) = 1$, *and* $Supp_{C_{\leq},P}(fever) = \frac{3}{3} = 1$.
- $\widehat{Supp}_{C_{\leq},P}(eruption) = \frac{1}{2}$, *and* $Supp_{C_{\leq},P}(eruption) = \frac{2}{3}$.
- $\widehat{Supp}_{C_{\leq},P}(1) = Supp_{C_{\leq},P}(1) = 1$.

Hence, the consequence with greatest support is 'fever'.

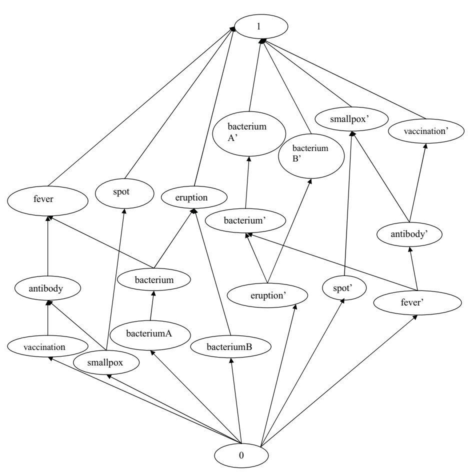

Figure 1: Preorder

4 Relevance for premises

This section introduces a measure to calculate the proportion of consequences that are gotten from a subset of premises, and what is more, it is shown how to reduce the set of premises using this ratio in order to give the same set of consequences.

Definition 4.1 *The relevance of a subset of premises* $R \in \mathbb{P}(P) - \{\emptyset\}$ *is the ratio of consequences deduced from a R, to all consequences.*

$$Rel_{C,P}(R) = \frac{|\{q \in L; q \in C(R)\}|}{|C(P)|} = \frac{|C(R)|}{|C(P)|}, \quad \textit{if } R \in \mathbb{P}(P) - \{\emptyset\}, \tag{5}$$

and, $Rel_{C,P}(\emptyset) = 0$.

Since $|P| > 0$ *and* $P \subset C(P)$, *it is* $|C(P)| > 0$, *the quotient does exist.*

If a subset of premises allows to deduce all consequences, the set of premises can be reduced to it, since both have the same set of consequences.

There are many properties that $Rel_{C,P}$ verifies,

- If there exists $R \in \mathbb{P}(P)$, such that $Rel_{C,P}(R) = 1$, it means that all consequences for P are consequences of R. So, $C(P) = C(R)$.
- If $R_1 \subset R_2$, then $Rel_{C,P}(R_1) \leq Rel_{C,P}(R_2)$. That is function $Rel_{C,P}$ is monotonic.
- It is $Rel_{C,P}(P) = 1$, and $Rel_{C,P}(\emptyset) = 0$.

Theorem 4.2 *Function* $Rel_{C,P} : \mathbb{P}(P) \to [0,1]$ *is a measure.*

Proof. Straightforward, by the last properties. □

Remark 4.3 *In this case, the concept of fuzzy measure is defined in the preordered set* $(\mathbb{P}(L), \subset)$, *since relevance is defined for all subsets of premises and not only for single premises. Remember that the support is defined for each element.*

The ratio of relevance applying for each premise allows to define a partition into the set of premises, in classes whose elements have the same relevance, $[q] = \{p \in L;\ Supp_{C,P}(p) = Supp_{C,P}(q)\}$. Analogously, it can be built a partition in the set $\mathbb{P}(P)$, defining each class as $[S] = \{R \in \mathbb{P}(P);\ Supp_{C,P}(R) = Supp_{C,P}(S)\}$. The maximum number of classes that can exist is $|C(P)| + 1$.

4.1 Using the operator $C_{\leq}$

For the operator $C_{\leq}$ it is useful to calculate the relevance for each $p \in P$, since it is sufficient to get consequences for each premise and then join them. So, in this case is enough to deals with

$$Rel_{C_{\leq},P} : P \to [0,1]$$

Definition 4.4 *The relevance for a premise* $p \in P$ *is the proportion of consequences deduced from* p.

$$Rel_{C_{\leq},P}(p) = \frac{|\{q \in L; p \leq q\}|}{|C_{\leq}(P)|} = \frac{|C(p)|}{|C(P)|}. \tag{6}$$

If a premise allows to deduce all consequences, the set of premises can be reduced to that premise, since both give the same set of consequences.

There are many properties that $Rel_{C_{\leq},P}(p)$ verifies,

- For all $p \in P$, $Rel_{C_{\leq},P}(p) > 0$, since $p \in \{q \in L; p \leq q\}$ implies $|\{q \in L; p \leq q\}| > 0$.
- If there exists $p \in P$ such that $Rel_{C_{\leq},P}(p) = 1$, it means that all consequence for P is consequence of p. So, $C_{\leq}(P) = C_{\leq}(p)$.

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- If $p_1 \leq p_2$, then $Rel_{\,C_{\leq},P}(p_2) \leq Rel_{\,C_{\leq},P}(p_1)$. That is the function $Rel_{\,C_{\leq},P}$ is anti-monotonic in this sense. Then, the function $1 - Rel_{\,C_{\leq}}$ is monotonic.
- Let L be endowed with an infimum operation. If $\inf P \in P$, as $Rel_{\,C_{\leq},P}(\inf P) = 1$, because $\forall q \in C_{\leq}(P)$ there exists $p \in P$ such that $p \leq q$, and $\inf P \leq p \leq q$. Then, $C_{\leq}(P) = C_{\leq}(\inf P)$.
  A common consequence's operator is $C_{\wedge}$, defined by $C_{\wedge}(P) = \{q \in L; \inf P \leq q\}$, that can be defined as $C_{\wedge}(P) = C_{\leq}(\inf P)$, so in that case $C_{\leq}(P) = C_{\wedge}(P)$.

Example 4.5 *Using the same preset in figure 1. Let's calculate the relevance for premises. Here, we have an example that allows us to quantify the relevance of deseases of a patient.*

If the patient has $P = \{antibody, smallpox\}$*, then,* $C_{\leq}(P) = \{antibody, smallpox, spot, fever, 1\}$*. Hence,*

- $Rel_{\,C_{\leq},P}(antibody) = \frac{|\{antibody, fever, 1\}|}{|C_{\leq}(P)|} = \frac{3}{5}$.
- $Rel_{\,C_{\leq},P}(smallpox) = \frac{|\{antibody, smallpox, spot, fever, 1\}|}{|C_{\leq}(P)|} = \frac{5}{5} = 1$
- *Obviously,* $Rel_{\,C_{\leq},P}(P) = 1$.

This example shows the case that a premise allows to deduce all consequences of P, since $C_{\leq}(P) = C_{\leq}(smallpox)$.

Theorem 4.6 *If* $(L, \leq,', \cdot, +)$ *is a partial ordered set and* $P \subset L$ *such that* $\inf P \neq 0$.
There exists $p \in P$ *such that* $Rel_{\,C_{\leq},P}(p) = 1$*, if and only if* $p = \inf P$.

Proof. If $Rel_{\,C_{\leq}}(p) = 1$, it is $p \leq q$ for all $q \in C_{\leq}(P)$. And since $C_{\leq}(P) \subset C_{\wedge}(P)$, it is $\inf P \leq q \ \forall q \in C_{\leq}(P)$. The infimum is the greatest lower bound of a set, then $p \leq \inf P$. It is also $\inf P \leq p$. Thus, $p = \inf P$, because L is a lattice, so verifies antisymmetric property and has an infimum for each subset.
On the other hand if $p = \inf P$, implies $\inf P \in P$, and as it is said, $Rel_{\,C_{\leq},P}(\inf P) = 1$. □

In the theorem and in the above example, it is shown that the set of premises can be reduced to an only premise with relevance one, but if there is no one premise with relevance one, it could be found a subset of premises that allows to obtain the same consequences as the initial set of premises. When models deal with not a very big number of premises, a simple program can be used in order to find a minimal set of premises by calculating all combination of premises.

This algorithm is exponential in the number of premises. So, others algorithms can be designed in order to deal with a big number of premises.

,

The algorithm is as follows.

First of all, we look for premises with greatest relevance, we put one of these premises (p_1) into the set of reduced premises, then we calculate a relative relevance

$$Rel_{\,C_{\leq},P-\{p_1\}}(p) = \frac{|\{q \in C_{\leq}(P) - C_{\leq}(p_1); p \leq q\}|}{|C_{\leq}(P) - C_{\leq}(p_1)|},$$

and we introduce a premise with the greatest relative relevance (p_2), and then we calculate other relative relevance,

$$Rel_{\,C_{\leq},P-\{p_1,p_2\}}(p) =$$

$$\frac{|\{q \in (C_{\leq}(P) - C_{\leq}(p_1)) - C_{\leq}(p_2); p \leq q\}|}{|(C_{\leq}(P) - C_{\leq}(p_1)) - C_{\leq}(p_2)|},$$

and this process is repeated till the lowest r that verifies $C_{\leq}(P) = \bigcup_{i \in \{1,\ldots,r\}} C_{\leq}(p_i)$. Then the reduced set searched in this way will be $\{p_1, \ldots, p_r\}$.

4.2 Using the operator $C_{\bullet}$

In this case we can particularize the definition of relevance for each subset of premises.

The relevance for a subsets of premises $R \subset \mathbb{P}(P) - \{\emptyset\}$ is the ratio of consequences deduced from R, to consequences deduced from P.

$$Rel_{\,C_{\bullet},P}(R) = \frac{|\{q \in L; q \in C_{\bullet}(R)\}|}{|C_{\bullet}(P)|} =$$

$$\frac{|\{q \in L; \exists \tilde{R} \subset \mathbb{P}(R), \inf \tilde{R} \leq q\}|}{|C_{\bullet}(P)|}$$

Example 4.7 *Let* P *be* $\{c, d, a'\}$ *defined in the preorder in figure 2. So,* $C_{\bullet}(P) = \{c, d, f, g, b, e, d', a', 1\}$*. Relevance for all subset of premises are,*

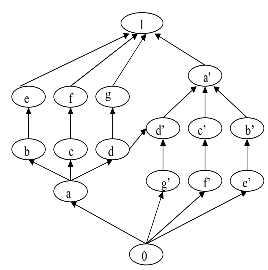

Figure 2: Preorder

- $Rel_{\,C_{\bullet},P}(\{c\}) = \frac{3}{7}$, $Rel_{\,C_{\bullet},P}(\{d\}) = \frac{5}{7}$, $Rel_{\,C_{\bullet},P}(\{a'\}) = \frac{2}{7}$
- $Rel_{\,C_{\bullet},P}(\{c, d\}) = 1$, $Rel_{\,C_{\bullet},P}(\{c, a'\}) = 1$, $Rel_{\,C_{\bullet},P}(\{d, a'\}) = \frac{5}{7}$

This example gives two reduced sets of premises, $\{c, d\}$ *and* $\{c, a'\}$*. Obviously* $C_{\bullet}(P) = C_{\bullet}(\{c, d\}) = C_{\bullet}(\{c, a'\})$.

5 Validating premises and partial hypotheses

Each premise is supported, or explained by hypotheses, so in this section, a support for each premise is defined. Then, we define the relevance of each partial hypothesis. This is analogously to the above section, because premises are consequences of hypotheses.

Let $Hyp^{*}_{C}(P) \neq \emptyset$.

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Definition 5.1 *The support of $p \in P$ is the ratio of subsets of hypotheses that allow getting p as consequence, to all subsets of partial hypotheses.*

$$Supp_{C,Hyp^*_C(P)}(p) = \frac{|\{H \subset Hyp^*_C(P); p \in C(H)\}|}{2^{|Hyp^*_C(P)|} - 1} \quad (7)$$

Since $|Hyp^*_C(P)| > 0$ the quotient in the definition is possible.

The bigger Support a premise has, the more hypotheses allow to deduce it.

The $Supp_{C,Hyp^*_C}$ verifies,

- If $P = \{p\}$, it is $Supp_{C,Hyp^*_C(P)}(p) = 1$.
- If $1 \in P$, then $Supp_{C,Hyp^*_C(P)}(1) = 1$.
- $Supp_{C,Hyp^*_C(P)}(p) = 1$ means that p is explained by all $h \in Hyp^*_C(P)$, in particular for all $h \in Hyp(P)$.
- If $p_1 \leq p_2$, then $Supp_{C,Hyp^*_C(P)}(p_1) \leq Supp_{C,Hyp^*_C(P)}(p_2)$. That is, function Sup is monotonic.
- $Supp_{C,Hyp^*_C(P)}(\sup\{p_1, p_2\}) \geq \max\{Supp_{C,Hyp^*_C(P)}(p_1), Supp_{C,Hyp^*_C(P)}(p_2)\}$.
- $Supp_{C,Hyp^*_C(P)}(\inf\{p_1, p_2\}) \leq \min\{Supp_{C,Hyp^*_C(P)}(p_1), Supp_{C,Hyp^*_C(P)}(p_2)\}$.

$Supp_{C,Hyp^*_C(P)}$, allows to compare premises in relation to hypotheses and to allocate different degrees to each premise.

Definition 5.2 *The relevance for a subset of partial hypotheses $H \subset Hyp^*_C(P)$ is the proportion of premises deduced from H.*

$$Rel_{\ C,Hyp^*_C(P)}(H) = \frac{|\{p \in P; p \in C(H)\}|}{|P|}, \quad (8)$$

if $H \in \mathbb{P}(Hyp^*_C(P)) - \{\emptyset\}$,

and, $Rel_{\ C,Hyp^*_C(P)}(\emptyset) = 0$.

Since $|P| > 0$, the quotient is possible.

There are many properties that $Rel_{\ C,Hyp^*_C(P)}$ verifies,

- If $P = \{p\}$, it is $Rel_{\ C,Hyp^*_C(P)}(H) = 1$, for all $H \subset Hyp^*_C(P)$.
- For all $H \subset Hyp^*_C(P)$, $Rel_{\ C,Hyp^*_C(P)}(H) > 0$.
- If $H_1 \subset H_2$, then $Rel_{\ C,Hyp^*_C(P)}(H_1) \leq Rel_{\ C,Hyp^*_C(P)}(H_2)$. That is, function $Rel_{\ C,Hyp^*_C(P)}$ is monotonic.
- If there exists $h \in Hyp^*_C(P)$ such that $Rel_{\ C,Hyp^*_C(P)}(\{h\}) = 1$, it means that h is a partial hypothesis that can explain all premises, so it is hypothesis.

Theorem 5.3 *If $Hyp_C(P) \neq \emptyset$, then $Rel_{\ C,Hyp^*_C(P)}$ is a measure.*

Proof. It is monotonic as it is stated above, $Rel_{\ C,Hyp^*_C(P)}(\emptyset) = 0$, and since there exists $h \in Hyp_C(P)$ $Rel_{\ C,Hyp^*_C(P)}(\{h\}) = 1$.
□

The measure, $Rel_{\ C,Hyp^*_C(P)}$, allows to compare partial hypotheses, and to distinguish which partial hypotheses are hypotheses in the classical sense, that are those one with relevance one.

6 Conclusions

In this paper, it is built a measure in the set of consequences, premises and partial hypotheses. That can be useful in decision problems in order to choose the consequence, premise or hypotheses with the biggest measure.

It should be also pointed out that by using a relevance measure, we can get rid of premises that do not add information, and still get the same set of consequences.

It is also introduced the concept of set of partial hypotheses, that contains the classical hypotheses one. The measure built allocate value one to that partial hypotheses that are hypotheses in the classical sense.

As future work it can be proposed to apply these measures to practical problems, for examples medical diagnosis problems much more bigger than the one that appears in this paper.

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References

[1] J. L. Castro and E. Trillas. Sobre preordenes y operadores de consecuencias de Tarski. *Theoria*, 4(11):419–425, 1989. (In Spanish).

[2] M.K. Chakraborty. Use of fuzzy set theory in introducting graded consequence in multiple-valued logic. *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, (1):247–257, 1988.

[3] M.K. Chakraborty and S. Dutta. *Theory of graded consequence and fuzzy logics*. LATD, Italy, 2008.

[4] D. Qiu. A note on Trillas' CHC models. *Artificial Intelligence*, 171:239–254, 2007.

[5] A. R. de Soto, A. Álvarez, and E. Trillas. Short note: Counting conjectures. *Mathware and Soft Computing*, 14(2):165–170, 2007.

[6] E. Trillas and C. Alsina. A reflection on what is a membership function. *Mathware & Soft-Computing*, 6:201–215, 1999.

[7] E. Trillas, S. Cubillo, and E. Castiñeira. On conjectures in orthocomplemented lattices. *Artificial Intelligence*, 117:255275, 2000.

[8] E. Trillas, I. García-Honrado, and A. Pradera. Consequences and conjectures in peordered sets. *ECSC's Tecnical Report (available upon request)*.

[9] E. Trillas, M. Mas, and M. Monserrat. Conjecturing from consequences. *Forthcoming in Int. journal of General Systems*.

[10] Satosi Watanabe. *Knowing and guessing. A Quantitative Study of Inference and Information*. John Wiley and sons, New York, 1969.

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Notes on the Exclusive Disjunction *

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Abstract

This short paper just contains some reflections on the symmetric difference operator (Δ) translating into an algebraic framework the connective *exclusive disjunction*, the linguistic *either/or*. In particular, it tries to find an upper bound for the fuzzy operators generalizing the classical symmetric difference, that is, those to deal with imprecise statements. This search is made throughout the preservation of the inferential schemes

$$a\Delta b \;\;\&\;\; a' : b, \text{ and } \; a\Delta b \;\;\&\;\; a : b',$$

in fuzzy logic. The paper tries to stress the inferential interest of the symmetric difference.

1 Introduction

1.1

As it is well known, the exclusive disjunction *either/or* with precise statements is representable, in the framework of bounded lattices with negation [6], by the symmetric difference

$$a\Delta b = (a+b)\cdot(a\cdot b)',$$

that enjoys the properties: 1). $a\Delta b = b\Delta a$, 2). $a\Delta 0 = a$, 3). $a\Delta 1 = a'$, and 4). $a\Delta a = a\cdot a'$ (equal to 0 in ortholattices).

In the case of imprecise predicates representable by means of fuzzy sets in a given universe of discourse [12], [3], several models for the symmetric difference were presented in [2]. Among them, the following ones are remarkable,

$$\Delta_1(a,b) = \varphi^{-1}(|\varphi(a) - \varphi(b)|), \text{ and}$$

$$\Delta_2(a,b) = min(max(a,b), max(N(a), N(b))),$$

with an order-automorphism φ of the ordered unit interval $([0,1], \leq)$, and a strong negation N [7]. Notice that

- $\Delta_1(a,b) = \Delta_1(b,a)$, $\Delta_1(a,1) = N_\varphi(a)$,
- $\Delta_1(a,0) = \Delta_2(a,0) = a$, $\Delta_1(a,a) = 0$, like in the ortholattice case,
- $\Delta_2(a,a) = \min(a, N(a))$, that is equal to 0 if only if $a \in \{0,1\}$,

Remember that $N_\varphi = \varphi^{-1} \circ (1 - id) \circ \varphi$ is the strong negation generated by φ (see [7]).

2 Some properties of the symmetric difference in boolean algebras

2.1

It is obvious that in any lattice with negation is $a\Delta b \leq a + b$ and $a\Delta b \leq (a\cdot b)'$. In addition and in boolean algebras, from $a\cdot(a\Delta b) = a\cdot b'$ follows

$$a\Delta b = (a\Delta b) + a\cdot(a\Delta b) = (a\Delta b) + a\cdot b' \Rightarrow a\cdot b' \leq a\Delta b.$$

Hence, $a\cdot b' \leq a\Delta b \leq a + b$ and $a\cdot b' \leq a\Delta b \leq (a\cdot b)'$.

Analogously, from $a'\cdot(a\Delta b) = a'\cdot b$, follows $a'\cdot b \leq a\Delta b$, and $a'\cdot b \leq a\Delta b \leq a + b$ and $a'\cdot b \leq a\Delta b \leq (a\cdot b)'$.

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2.2

A curious, and interesting, property of the symmetric difference in boolean algebras is given by the following:

Proposition 2.1 *The solutions of the equation $a{\cdot}x = b{\cdot}x$, are $x \leq (a\Delta b)'$.*

`Proof.` If $a{\cdot}x = b{\cdot}x$ it follows $0 = a'{\cdot}b{\cdot}x$ and $0 = b'{\cdot}a{\cdot}x$, that is $0 = (a'{\cdot}b + a{\cdot}b')x = (a\Delta b){\cdot}x$, that is equivalent (in boolean algebras) to $x \leq (a\Delta b)'$. Reciprocally, if this last inequality holds, is $x \leq (a'{\cdot}b + a{\cdot}b')' = (a + b'){\cdot}(a' + b)$, and

- $a{\cdot}x \leq a{\cdot}(a + b')(a' + b) = a{\cdot}(a' + b) = a{\cdot}b \leq b \to a{\cdot}x \leq b{\cdot}x$
- $b{\cdot}x \leq b{\cdot}(a + b')(a' + b) = b{\cdot}(a + b') = b{\cdot}a \leq a \to b{\cdot}x \leq a{\cdot}x$.

Hence, $a{\cdot}x = b{\cdot}x$. □

Notice that in the Chinese Lantern orthomodular lattice (Figure 1),

Figure 1: Chinese Lantern

it is $b{\cdot}a = b'{\cdot}a$, and $(b\Delta b')' = (1{\cdot}0')' = 0$, but $a \neq 0$. Hence, in the framework of ortholattices proposition 2.1 does not hold in general.

In the case of De Morgan algebras, for instance, taking the De Morgan algebra $([0,1]^X, \min, \max, 1 - id)$, and dealing with the pointwise order, it is easy to find examples for which the property fails. With the discrete fuzzy sets $\mu = 0.8/1 + 0.5/2 + 0.5/3$, $\sigma = 0.5/1 + 0.5/2 + 0.8/3$ and $\delta = 0.3/1 + 0.9/2 + 0.3/3$, it is $\min(\mu, \delta) = \min(\sigma, \delta)$, but $\delta \not\leq (\mu\Delta\sigma)' = 0.5/1 + 0.5/2 + 0.5/3$.

Notwithstanding, in the linear De Morgan algebra $([0,1], \min, \max, 1 - id)$, since if $a \leq b$,

$$\min(a, x) = \min(b, x) \Leftrightarrow x \leq a,$$

and $(a\Delta b)' = (b{\cdot}a')' = a + b' = \max(a, 1 - b) \geq a$, it can be said that if $min(a, x) = min(b, x)$, then $x \leq (a\Delta b)'$, and an analogous result follows if $b \leq a$. But the reciprocal is not true: it is enough to take $a = 0.3$, $b = 0.4$, $x = 0.5$, since $x \leq (a\Delta b)' = 0.6$, but $0.3 = min(a, x) \neq min(b, x) = 0.4$.

3 Schemes of inference with Δ

In [3] there is an interesting comment for the boolean case, relative to a difference between the inclusive *or*, and the exclusive *either/or*, from an inferential point of view. It can be synthesized by: $a{\cdot}(a + b) = a$, but $a{\cdot}(a\Delta b) = a{\cdot}(a + b){\cdot}(a{\cdot}b)' = a{\cdot}(a' + b') = a{\cdot}b' \leq b'$, showing the different deductive schemes

$$\frac{\begin{array}{l} a \text{ or } b \\ a \end{array}}{a} \qquad \text{and} \qquad \frac{\begin{array}{c} \text{Either } a \text{ or } b \\ a \end{array}}{\text{not } b.}$$

Under the first scheme only one of the arguments can be deduced from itself, but under the second what can be deduced from an argument is the negation of the other. Such last scheme reflects a kind of forwards-backwards type of reasoning.

The inequality that follows:

$$a'{\cdot}(a\Delta b) = a'{\cdot}(a+b){\cdot}(a'+b') = a'{\cdot}(a+b) = a'{\cdot}b \leq b$$

reflects the scheme

$$\frac{\begin{array}{c} \text{Either } a \text{ or } b \\ \textit{not } a \end{array}}{b,}$$

that is a kind of backwards-forwards type of reasoning.

Remark 3.1 *Out of boolean algebras these schemes do not always hold. For instance, in the Chinese Lantern orthomodular lattice in figure 1, it is* $a'\cdot(a\Delta b) = a'\cdot(a+b)\cdot(a\cdot b)' = a'\cdot 1\cdot 0' = a' \nleq b$*, and* $a\cdot(a\Delta b) = a\cdot(a+b)\cdot(a\cdot b)' = a\cdot 1\cdot 0' = a \nleq b'$*. In the De Morgan algebra* $([0,1], \min, \max, 1-id)$*, is* $0.3\cdot(0.3\Delta 0.9) = 0.3 \nleq 0.1 = 0.9'$*, and* $0.5'\cdot(0.5\Delta 0.4) = 0.5\cdot 0.5 = 0.5 \nleq 0.4$.

4 On the three-valued logic case

A three valued logic [11] is a triplet (Ω, L, t), where Ω is a set of propositions closed by negation, disjunction, and conjunction. L is a set of three elements, in which two of them are $1, 0$ such that $0 \neq 1$ and $0' = 1$, and the third element will be denoted by $\frac{1}{2}$, L is endowed with $'$,$+$ and $\cdot$ operations given by tables. Finally $t : \Omega \to Ł$ is a function preserving the corresponding operations, that is $t(not\ \ p) = t(p)'$, $t(p\ \ and\ \ q) = t(p)\cdot t(q)$, $t(p\ or\ q) = t(p) + t(q)$.

Function t is supposedly translating the 'truth' of the propositions in Ω, for instance, 1 represents 'true', 0 represents 'false', and $\frac{1}{2}$ can represent 'undecided'.

As examples of three-valued logics it can be collected those of Łukasiewicz, Gödel, Kleene, Bochvar and Post, with the operations defined by the following tables ([4]):

- Łukasiewicz

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\cdot$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Gödel

	$'$
1	0
$\frac{1}{2}$	0
0	1

$\cdot$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Kleene

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\cdot$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Bochvar

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\cdot$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$+$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Post

	$'$
1	$\frac{1}{2}$
$\frac{1}{2}$	0
0	1

$\cdot$	1	$\frac{1}{2}$	0
1	0	0	$\frac{1}{2}$
$\frac{1}{2}$	0	1	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

So, it is possible to calculate the symmetric difference by the formula $a\Delta b = (a+b)\cdot(a\cdot b)'$, and only three different tables are obtained, one for Łukasiewicz, Kleene and Bochvar logics, other for Gödel's, and other for Post's.

- Łukasiewicz, Kleene and Bochvar

Δ	1	$\frac{1}{2}$	0
1	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Gödel

Δ	1	$\frac{1}{2}$	0
1	0	0	1
$\frac{1}{2}$	0	0	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

- Post

Δ	1	$\frac{1}{2}$	0
1	0	0	1
$\frac{1}{2}$	0	1	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Symmetric difference in the two first tables, verifies the properties in section 1.1. But, this is not the case for the Post three-valued logic, in which such properties are not verified. For instance, $\Delta(0,0) = \frac{1}{2} \neq 0$, $\Delta(0,1) = \frac{1}{2} \neq 0' = 1$, and $\Delta(\frac{1}{2}, \frac{1}{2}) = 1 \neq \frac{1}{2}\cdot(\frac{1}{2})' = \frac{1}{2}\cdot 0 = \frac{1}{2}$.

Regarding the verifications of the two schemes $a\cdot\Delta(a,b) \leq b'$ and $a'\cdot\Delta(a,b) \leq b$, the case of Gödel is the only in which they are verified. For instance, in Łukasiewicz three-valued logic with $a = \frac{1}{2}$ and $b = 1$, is $a\cdot\Delta(a,b) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \nleq 1' = 0$, and the first scheme fails. With $a = \frac{1}{2}$ and $b = 0$, $a'\cdot\Delta(a,b) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \nleq 0$, and the second scheme also fails.

Taking into account the three-valued logics of Łukasiewicz, Gödel, Kleene, and Bochvar it is also verified that $a'\cdot b \leq a\Delta b \leq a + b$, but it is not the case of the three-valued logic of Post which fails for the lower bound with the pairs ($a = 1$, $b = \frac{1}{2}$) and ($a = \frac{1}{2}$, $b = 1$).

5 The fuzzy case

5.1

In the case of imprecise predicates representable by fuzzy sets in a universe of discourse X, and once an algebra of fuzzy sets [8] $([0,1]^X, \cdot, +, ')$ has been selected, accordingly with the context and purposes of the current problem [10], the statements of the type

x is P *or* x is Q, and *Either* x is P *or* x is Q,

can be represented in the forms $(\mu_P + \mu_Q)(x)$, and $(\mu_P \Delta \mu_Q)(x)$, respectively. Although in fuzzy logic, $+$ is usually represented by a continuous t-conorm [1], the representation of Δ is not always done through a single type of numerical functions [2]. What is not yet clarified is which ones of these functions do verify the deductive scheme of disjunctive inference[1]

$$\frac{\text{Either } x \text{ is } P \text{ or } x \text{ is } Q \qquad x \text{ is } not\ P}{x \text{ is } Q,}$$

that is, $T_1(N(\mu_P(x)), \Delta(\mu_P(x), \mu_Q(x))) \leq \mu_Q(x)$, for some continuous t-norm T_1 [1], and a strong negation N. For instance, the models of the type (see [2]):

$$\Delta_T(a,b) = T(\max(a,b), \max(N(a), N(b))),$$

in general do not verify that scheme since with $T = \min$ and $N = 1 - id$, it is: $\Delta_{\min}(0.1, 0) = \min(0.1, 1) = 0.1$, and $T_1(N(a), \Delta_{\min}(a,b)) \leq \min(N(a), \Delta_{\min}(a,b))$, that is equal to $\min(0.9, 0.1) = 0.1 \nleq 0$.

Remark 5.1 *The numerical functions* $\Delta : [0,1] \times [0,1] \to [0,1]$ *allow to reach the functional expression* $(\mu\Delta\sigma)(x) = \Delta(\mu(x), \sigma(x))$ *for all* $x \in X$, *and* μ, σ *in* $[0,1]^X$. *With those* Δ, *the two deductive schemes whose validity is to be studied are* $\mu' \cdot (\mu\Delta\sigma) \leq \sigma$, *and* $\mu \cdot (\mu\Delta\sigma) \leq \sigma'$.

5.2

Within the framework of the standard algebras of fuzzy sets ([5]), the models of Δ verifying the first scheme are, consequently, submitted to satisfy the functional inequality

$$T_1(N(a), \Delta(a,b)) \leq b \qquad (1)$$

for all a, b in $[0,1]$ and some continuous t-norm T_1, with Δ such that, at least,

$$\Delta(a,b) = \Delta(b,a), \;\; \Delta(a,1) = N(a), \;\; \Delta(a,0) = a.$$

Lemma 5.2 *For the verification of the inequality (1) it is necessary*[2] *that* $T_1 = W_\varphi$, *and* $N \leq N_\varphi$ *for any order automorphism* φ *of the unit interval.*

`Proof.` Taking $b = 0$, (1) is $T_1(N(a), \Delta(a,0)) = T_1(N(a), a) = 0$, equivalent to $T_1 = W_\varphi$ and $N \leq N_\varphi$ [1]. $\square$

Theorem 5.3 *If* $T_1 = W_\varphi$ *and* $N = N_\varphi$, *inequality 1 holds if and only if* $\Delta \leq W^*_\varphi$.

`Proof.`

1. If $T_1 = W_\varphi$ and $N = N_\varphi$, inequality 1 is $W_\varphi(N_\varphi(a), \Delta(a,b)) \leq b \Leftrightarrow W(1 - \varphi(a), \varphi(\Delta(a,b))) \leq \varphi(b)$, that is, $\max(0, \varphi(\Delta(a,b)) - \varphi(a)) \leq \varphi(b)$, that implies $\varphi(\Delta(a,b)) \leq \varphi(b) + \varphi(a)$.

   Since $\varphi(\Delta(a,b)) \leq 1$, it follows $\Delta(a,b) \leq \varphi^{-1}(\min(1, \varphi(a) + \varphi(b))) = W^*_\varphi(a,b)$.

2. Provided $\Delta \leq W^*_\varphi$, it is : $W_\varphi(N_\varphi(a), \Delta(a,b)) \leq W_\varphi(N_\varphi(a), W^*_\varphi(a,b)) =$
   $\varphi^{-1} W(1 - \varphi(a), \varphi(W^*_\varphi(a,b))) =$
   $\varphi^{-1} \max(0, 1 - \varphi(a) + \varphi(W^*_\varphi(a,b)) - 1) =$
   $\varphi^{-1} \max(0, \min(1, \varphi(a) + \varphi(b)) - \varphi(a)) =$
   $\varphi^{-1} \max(0, \min(1 - \varphi(a), \varphi(b))) =$
   $\varphi^{-1} \min(1 - \varphi(a), \varphi(b)) =$
   $\min(N_\varphi(a), b) \leq b$.

$\square$

Notice that $\Delta(a,0) = \Delta(a,0) \leq W^*_\varphi(a,0) = a$, $\Delta(a,1) \leq W^*_\varphi(a,1) = 1$, and $\Delta(a,a) \leq W^*_\varphi(a,a) = \varphi^{-1}(\min(1, 2\varphi(a)))$.

[1] See [9] for the typical disjunctive reasoning

[2] Being φ an order automorphism, the t-norms of Łukasiewicz are $W_\varphi(a,b) = \varphi^{-1}\max(0, \varphi(a) + \varphi(b) - 1)$, and the t-conorms $W^*_\varphi(a,b) = \varphi^{-1}\min(1, \varphi(a) + \varphi(b))$

Remarks 5.4

- *Last theorem gives an upper bound for the operators Δ such that $\Delta(a,0)=a$.*
- *Since $W^*_\varphi(a,1) = 1 \neq N(a)$, it is clear that the upper bound W^*_φ is not an operator Δ.*
- *$\Delta \leq W^*_\varphi$ is a translation of $a\Delta b \leq a+b$ into the fuzzy case.*

5.3

The bounds for Δ verifying the second scheme $\mu\cdot(\mu\Delta\sigma) \leq \sigma'$, once translated into

$$T_1(a, \Delta(a,b)) \leq N(b), \tag{2}$$

for all a, b in $[0,1]$, are different from those verifying (1).

Lemma 5.5 *For the verification of the inequality (2) it is necessary that $T_1 = W_\varphi$, and $N \leq N_\varphi$ for any order automorphism φ of the unit interval.*

`Proof.` Taking $b=1$, in (2): $T_1(a,\Delta(a,1)) = T_1(a, N(a)) = 0$, equivalent to $T_1 = W_\varphi$ and $N \leq N_\varphi$. □

Theorem 5.6 *If $T_1 = W_\varphi$ and $N = N_\varphi$, inequality 2 holds if and only if $\Delta \leq W^*_\varphi \circ (N_\varphi \times N_\varphi)$.*

`Proof.`

1. If $T_1 = W_\varphi$, and $N = N_\varphi$, from 2 follows $W_\varphi(a,\Delta(a,b)) \leq N_\varphi(b)$. That is,

   $$W(\varphi(a), \varphi(\Delta(a,b))) \leq 1-\varphi(b), \text{ or}$$

   $$\max(0, \varphi(a) + \varphi(\Delta(a,b)) - 1) \leq 1-\varphi(b),$$

   that implies

   $$\varphi(a) + \varphi(\Delta(a,b)) - 1 \leq 1 - \varphi(b), \text{ or}$$

   $$\varphi(\Delta(a,b)) \leq 2 - \varphi(a) - \varphi(b).$$

   Hence,

   $$\varphi(\Delta(a,b)) \leq \min(1, 2-\varphi(a)-\varphi(b)), \text{ or}$$

   $$\Delta(a,b) \leq \varphi^{-1}(\min(1, 2-\varphi(a)-\varphi(b))$$

   or $\Delta(a,b) \leq \varphi^{-1}(\min(1, 1-\varphi(a)+1-\varphi(b))) = \varphi^{-1}W^*(\varphi(N_\varphi(a)), \varphi(N_\varphi(b))) = W^*_\varphi \circ (N_\varphi \times N_\varphi)(a,b)$.

2. If $\Delta \leq W^*_\varphi \circ (N_\varphi \times N_\varphi)$, it follows $\Delta(a,b) \leq \varphi^{-1}(\min(1, 1-\varphi(a)+1-\varphi(b)))$, or $\varphi(\Delta(a,b)) \leq \min(1, 1-\varphi(a)+1-\varphi(b)) \leq 1-\varphi(a)+1-\varphi(b)$, that implies $\varphi(a) + \varphi(\Delta(a,b)) - 1 \leq 1 - \varphi(b)$ or $\max(0, \varphi(a) + \varphi(\Delta(a,b)) - 1) \leq 1-\varphi(b)$. Finally, $\varphi^{-1}(\max(0, \varphi(a) + \varphi(\Delta(a,b)) - 1)) = W_\varphi(a, \Delta(a,b)) \leq \varphi^{-1}(1-\varphi(b)) \leq N_\varphi(b)$.

□

It should be noticed that the upper bound $W^*_\varphi \circ (N_\varphi \times N_\varphi)$ is not a Δ operator, since

- $W^*_\varphi \circ (N_\varphi \times N_\varphi)(a,0) = W^*_\varphi(N_\varphi(a), 1) = 1 \neq a$
- $W^*_\varphi \circ (N_\varphi \times N_\varphi)(a,a) = \varphi^{-1}(\min(1, 2-2\cdot\varphi(a))) = 1$. So, it is not $N_\varphi(a, N_\varphi(a)) = 0$.

Theorem 5.6 gives an upper bound for the operators Δ such that $\Delta(a,1) = N(a)$.

6 Last comment

It is well known the importance that in fuzzy logic has the inference involving fuzzy and crisp sets, as it is the case, for instance and in fuzzy control, of the Takagi-Sugeno model, in which the antecedents of the values are fuzzy, but the consequents are crisp.

For what concerns the fuzzy models Δ_1 and Δ_2 in section 1, and with a crisp set A, it is:

- $\Delta_2(\mu_A, \sigma)(x) = \Delta_2(\mu_A(x), \sigma(x)) = T(\max(\mu_A(x), \sigma(x)), \max(N(\mu_A(x)), N(\sigma(x)))) = \begin{cases} T(1, N(\sigma(x))) = N(\sigma(x)), & \text{if } x \in A \\ T(\sigma(x), 1) = \sigma(x), & \text{if } x \notin A \end{cases}$
- $\Delta_1(\mu_A, \sigma)(x) = \Delta_1(\mu_A(x), \sigma(x)) = \varphi^{-1}(|\varphi(\mu_A(x)) - \varphi(\sigma)|) = \begin{cases} \varphi^{-1}(1-\varphi(\sigma(x))) = N_\varphi(\sigma(x)), & \text{if } x \in A \\ \sigma(x), & \text{if } x \notin A. \end{cases}$

Hence,

- The scheme "$\mu'_A(x), \Delta(\mu_A(x), \sigma(x)) : \sigma(x)$", always holds since, $T(N(\mu_A(x), \Delta(\mu_A(x), \sigma(x)))) = \left.\begin{cases} 0, & \text{if } x \in A \\ \sigma(x), & \text{if } x \notin A \end{cases}\right\} \leq \sigma(x)$.

- The scheme "$\mu_A(x), \Delta(\mu_A(x), \sigma(x)) : \sigma'(x)$", also holds always since,
$T(\mu_A(x), \Delta(\mu_A(x), \sigma(x))) =$
$\left\{ \begin{array}{ll} \Delta(1, \sigma(x)) = N(\sigma(x)), & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{array} \right\} \leq$
$\leq N(\sigma(x))$.

7 Conclusion

Apart of showing some few properties of the symmetric difference in boolean algebras, and their not validity in ortholattices and De Morgan algebras, this paper tries to continue with a comment made by Bernard Bosanquet [3], concerning a scheme of boolean deduction based on the exclusive disjunction *either/or*. It also tries to stress the inferential interest of the symmetric difference, an operator that is yet scarcely studied in multiple-valued and fuzzy logic.

The two considered schemes, like the classicals $a' \cdot (a + b) \leq b$, and $b \cdot (a \cdot b)' = b \cdot (a' + b') = b \cdot a' \leq a'$, are of a backwards-forwards and forwards-backwards reasoning type, respectively. With this idea in mind, upper bounds for the operators of symmetric difference Δ, are obtained for the two fuzzy schemes given by $\mu' \cdot (\mu \Delta \sigma) \leq \sigma$, and $\mu \cdot (\mu \Delta \sigma) \leq \sigma'$, from the corresponding functional inequalities

$$T_1(N(a), \Delta(a, b)) \leq b, \text{ and } T_1(a, \Delta(a, b)) \leq N(b).$$

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References

[1] C. Alsina, M. J. Frank, and B. Schweizer. *Associative Functions. Triangular Norms and Copulas*. World Scientific, Singapore, 2006.

[2] C. Alsina and E. Trillas. On the symmetric difference of fuzzy sets. *Fuzzy Sets and Systems*, 153:181–194, 2005.

[3] B.Bosanquet. *Logic Or The Morphology of Knowledge (vol. I)*. Clarendom Press, Oxford, 1911.

[4] L. Bolc and P. Borowik. *Many-valued Logics*. Springer-Verlag, Berlin, 1992.

[5] H.T. Nguyen and E.A. Walker. *A First Course in Fuzzy Logic*. Chapman & Hall/CRC, 2000.

[6] E. Renedo, E. Trillas, and C. Alsina. A note on the symmetric difference in lattices. *Mathware & Soft Computing*, 12(1):75–81, 2005.

[7] E. Trillas. Sobre funciones de negación en la teoría de conjuntos difusos. *Stochastica*, 3(1):47–59, 1979. (In Spanish). Reprinted (English version) in *Advances of Fuzzy Logic* (S. Barro et al. (eds.)):31-43. Press of the Universidad de Santiago de Compostela, 1998.

[8] E. Trillas, C. Alsina, and A. Pradera. On a class of Fuzzy Set Theories. In *Proc. FUZZ-IEEE 2007*, pages 1–5, London, 2007.

[9] E. Trillas, C. Alsina, and E. Renedo. On some classical schemes of reasoning in fuzzy logic. Technical Report FSC-2009-6, European Centre for Soft Computing, 2009. Forthcoming in New Mathematics and Natural Computation (Available upon request).

[10] E. Trillas and S. Guadarrama. Fuzzy representations need a careful design. *International Journal of General Systems*, 39(3):329–346, 2010.

[11] E. Trillas, C. Moraga, and E. Renedo. On Aristotle's NC and EM principles in three-valued logic. In *IFSA-EUSFLAT World Congress*, pages 879–884, 2009. IFSA'09.

[12] H. J. Zimmermann. *Fuzzy Sets Decision Making and Expert Systems*. Kluwer Academic Publishers, United States, 1987.

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Some (Unended) Queries on Conjecturing

Itziar García-Honrado, Adolfo R. de Soto, and Enric Trillas

***Abstract*— This paper deals with some of the queries still waiting for a good answer in the new field of the so called CHC Models related to approach ordinary reasoning through the conjectures concept. Namely, with two subjects of which the first is of a practical interest, and the second is of a theoretical one: The growing in the number of conjectures once new premises of a different character are added, and the relationships between Galois' Connections and Conjecture Operators. Since 'conjecturing' mainly refers to ordinary reasoning, CHC Models can be included in the new field of Computing With Words and Perceptions.**

Keywords: Conjectures, Galois' Connections, Crisp+Fuzzy Information.

I. Introduction

Ordinary, everyday, or commonsense reasoning basically consists in processes that, starting with a set P of premises, conduct to a conclusion q such that either

1) q follows *necessarily* from P, that is, q is a 'safe' conclusion of P, or
2) q is a *contingent explanation* of the premises in P, in the sense that every p in P is a 'safe' conclusion of the singleton q, or
3) q is a *contingent conclusion* that is neither a 'safe' one, nor a contingent explanation of P.

Those processes are respectively called: Deductive, or Formal, in case (1), Abductive, or Presumptive, in case (2), and Speculative, or Tentative, in case (3).

Although deduction is formalized from time ago by the consequence operators defined by Alfred Tarski, abduction and speculation only recently were included in a unified formal framework separately containing the above three types of conclusions. Following William Whewell ([15]), the elements in this unifying framework are called 'conjectures', and respectively denoted by the linguistic terms 'logical consequences', 'hypotheses', and 'speculations'.

Let us remember that the term 'abduction' was introduced by Charles S. Peirce ([8]), and that *a logic* in a universe of discourse X is commonly understood as a pair (X, C), where C is a Tarski's operator of consequences defined in some subset of $\mathbb{P}(X)$, the family of subsets of X. By adding to these three processes those known as *reasoning by similitude* or *analogy* ([1], [2]) that are not considered in the current paper, most of the ways with which humans conduct their reasonings are captured. Since the new field of Computing with Words and Perceptions (CW/P), introduced by Lotfi A. Zadeh ([16]), deals with ordinary reasoning in Natural Language, this paper although it is not in the current lines of research in CW/P, deserves to be included in it.

The formal study of conjecturing, under the so-called CHC Models (Consequences, Hypotheses and Conjectures (see [9])), is a young research subject actually born with the paper ([3]), published in the last year of the XX Century. Most of the several questions that immediately followed after this paper, were posed and solved or just partially solved, in the papers that subsequently appeared (see [12],[13],[11]). Anyway, there remain some neither posed, nor yet solved or approached questions; among them, for instance,

1) To find 'rules', like those presented by J.S. Mill ([7]), for obtaining hypotheses and speculations from the premises. These rules can conduct to find computer programs or algorithms actually able, when possible, to reach either hypotheses or speculations from the premises.
2) To obtain suitable ways of numerically measuring the support a speculation or a hypothesis deserves from a given set of premises.
3) To clarify how, from a previously solved problem and by analogy or similitude, it is possible to obtain hypotheses or logical consequences, through 'similar' speculations.
4) To study what happens to the conjectures when new information, supplied by new premises added out of the initial formal framework, appear.
5) To find theoretical alternative ways of dealing with all kind of conjectures which are able to better explain their behavior in ordinary reasoning.

and etc.

One of the drawbacks shown by the current formalization of ordinary reasoning by CHC Models, concerns its 'static' character relatively to the information contained in the premises that, actually, in a lot of cases and from several points of view [14], is in flux. The reasoning processes humans develop are not static, but dynamic. Although the subject is actually a wide one, this paper just tries to present a first case-example that can be considered typical of fuzzy logic.

So that, this paper tries to begin with point (4) in the case the old information is crisp and the new is fuzzy, and with point (5) with the help of Galois connections. What it contains should be just considered as a first attempt to deal

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with such topics.

II. Basic Concepts

A possible model for commonsense reasoning can be built (through some information) by the concept of a conjecture, which includes those of consequence, hypothesis and speculation (see [5], [9], [11]).

This model was initially defined in a given universe of discourse, in which there are both the available information and all conjectures. But it could be said that such definition is 'static', and that it lacks to afford the actually real case in which information is in flux. Along this paper we try to consider what happens when new information from even other universe of discourse is added to the previously given one, or the order in the universe of discourse is changed.

In the model the order in the universe of discourse is relevant, because changing it the concept of deduction also changes. That is so because the concept of deduction from some information P is accommodated in this model by means of a consequence operator in the sense of Tarski, and there is an intimate relationship between these operators and preorders ([11]). Let us suppose that $(L, \leq)$ is, at least, a preordered set.

Definition 2.1 (Structure of consequences): Let L be the universe of discourse and $\mathfrak{F} \subseteq \mathbb{P}(L)$. It is said that $(L, \mathfrak{F}, C)$ is a *structure of consequences*, or, alternatively, that C is an *operator of consequences* (in the sense of Tarski) for $\mathfrak{F}$ in L, provided $C : \mathfrak{F} \to \mathfrak{F}$ verifies the four following properties:

1) $P \subseteq C(P)$, for all $P \in \mathfrak{F}$ (C is *extensive*)
2) If $P \subseteq Q$, then $C(P) \subseteq C(Q)$, for all $P, Q \in \mathfrak{F}$ (C is *monotonic*)
3) $C(C(P)) = C(P)$ for all $P \in \mathfrak{F}$, or $C^2 = C$ (C is a *clausure*)
4) If $q \in C(P)$, then $q' \notin C(P)$ (C is *consistent*, and q' =*not* q).

The operators $C_\wedge(P) = \{q \in L;\ \inf P \ \leq q\}$ and $C_\leq(P) = \{q \in L;\ \exists p \in P,\ p \leq q\}$ defined for all $P \in \mathfrak{F} = \{P \in \mathbb{P}(L); \inf P \neq 0\}$ form an structure of consequences provided all non-empty subset in L has infimum, that is, L is infimum complete.

Conjectures are those elements that are non-contradictory with the available information. They can be defined from an operator of consequences, in the following way

$$Conj_C(P) = \{q \in L; q' \notin C(P)\}, \tag{1}$$

understanding that an element q is not contradictory with the premises when its negation q' is not deducible under C from the premises. Obviously, $P \subset C(P) \subset Conj_C(P)$.

It should be noticed that the term non-contradictory can be understood in different ways, for instance in a infimum complete ordered structure $(L, \leq)$, the operators ($\cdot$ = *and*):

- $Conj_1(P) = \{q \in L; \inf(P) \cdot q \neq 0\}$,
- $Conj_2(P) = \{q \in L; \inf(P) \cdot q \nleq (\inf(P) \cdot q)'\}$,

are sets of conjectures that can not be written in the form (1) with C a consequence operator. Anyway, it is always $P \subset C_\wedge(P) \subset Conj_i(P)$ and $C_\leq(P) \subset Conj_i(P)$, for $i = 1, 2$.

Hypotheses are those elements that 'explain' the information supplied by P. They can be also built by a consequences operator in the sense of Tarski by $Hyp_C(P) = \{q \in Conj(P) - C(P); \{q\} \in \mathfrak{F}$ and $P \subseteq C(q)\}$. In the cases of $C_\wedge$ and $C_\leq$ this set reduces to $Hyp(P) = \{q \in Conj(P); \{q\} \in \mathfrak{F}$ and $0 < q \leq \inf P\}$.

Finally, speculations are those elements in the set of conjectures that are neither consequences, nor hypotheses: $Sp(P) = Conj(P) - [C(P) \cup Hyp(P)]$. With all that, it is the partition $L = Ref(P) \cup Conj(P) = Ref(P) \cup C(P) \cup Hyp(P) \cup Sp(P)$, with 'refutations' of P, $Ref(P) = \{q \in L; q \notin Conj(P)\}$.

III. Initial Points

3.1 An interesting, and not yet studied, problem in CHC models is to analyze the change in the sets of conjectures, hypotheses and consequences through a mapping between two lattices. Fuzzy Set Theory could offer an interesting example of this problem.

Let L_1, L_2 be the lattices

$$\begin{aligned} L_1 &= (\{0,1\}^X, \min, \max, 1 - \mathrm{id}) \approx (\mathbb{P}(X), \wedge, \vee, '), \\ L_2 &= ([0,1]^X, \min, \max, 1 - \mathrm{id}). \end{aligned}$$

of crisp sets, $\mathbb{P}(X)$, and fuzzy sets, $\mathscr{F}(X)$, on the same universe X, respectively. L_1 is a boolean algebra, L_2 is a de Morgan-Kleene algebra and L_1 is a sublattice of L_2. Obviously, there exists a injective morphism between L_1 and L_2.

Let us consider the case in which the conjectures of a set of premises P come from logical consequences given by the operator $C_\wedge$. To simplify the problem, let's take $P = \{p_1, p_2\} \subset L_1$, and suppose that a new premise $p_3^* \in \mathscr{F}(X) - \mathbb{P}(X)$ is considered and so $P^* = \{p_1, p_2, p_3^*\} \subset L_2$ is a set of two crisp, and one fuzzy premises.

Of course, $\mathrm{Conj}(P) = \{p \in \mathbb{P}(X) : p_1 \cdot p_2 = \min(p_1, p_2) \nleq p'\}$ is in L_1, but $\mathrm{Conj}^*(P) = \{\mu \in \mathscr{F}(X) : p_1 \cdot p_2 \cdot p_3^* = \min(p_1, p_2, p_3^*) \nleq \mu' = 1 - \mu\}$ is in L_2 and, since $p_1 \cdot p_2 \cdot p_3^* \leq p_1 \cdot p_2$, if $\mu \in C_\wedge(P)$ from $p_1 \cdot p_2 \leq \mu$ it follows $p_1 \cdot p_2 \cdot p_3^* \leq \mu$, and $C_\wedge(P) \subset C_\wedge^*(P^*)$. From the same type of argument it follows that

$$\mathrm{Conj}^*(P^*) \subset \mathrm{Conj}^*(P),$$

being also obviously that:

$$\mathrm{Conj}(P) \subset \mathrm{Conj}^*(P).$$

That is,

1) P^* has less conjectures than P in L_2.
2) P has more conjectures in L_2 than in L_1,

and the question arises when considering that our interest lies in the new conjectures in $\text{Conj}^*(P^*) - \text{Conj}(P)$.

Remark 3.1: What has been done can be identically repeated if L_1 is any boolean algebra B since, by the Stone's Representation Theorem, B is isomorphic with a boolean algebra of subsets of some universe. Provided B is isomorphic with a boolean algebra L_1 included in $\mathbb{P}(X)$, it is enough to take the isomorphism $(\mathbb{P}(X), \cap, \cup, ^c) \approx (\{0,1\}^X, \min, \max, 1 - id)$ to have the boolean algebra of B by the isomorphism in L_2.

Remark 3.2: Given a De Morgan algebra $\mathcal{L} = (L, \cdot, +, '; 0, 1)$, the set $L_0 = \{a \in L; a \cdot a' = 0\}$ of its 'boolean elements' is a boolean algebra $\mathcal{L}_0$ with the restriction of the operations in L. Hence, what has bean done can be also repeated in these cases.

3.2 The last can be seen as an example of a Galois Connection.

Definition 3.3 (Galois Connection): Given a couple of posets $(R, \leq)$ and $(U, \preceq)$, a Galois Connection $G =< R, U, \alpha, \gamma >$ is a couple of mappings $\alpha : R \longrightarrow U$ and $\gamma : U \longrightarrow R$ such as

$$\alpha(r) \preceq u \Longleftrightarrow r \leq \gamma(u) \quad \forall r \in R, u \in U.$$

Usually, the mapping α is called the lower adjoint, or *coadjoint*, while the mapping γ is called the upper adjoint, or *adjoint*.

Returning to the above case, it can be written as a Galois connection $G_1 =< L_1, L_2, \alpha_1, \gamma_1 >$ by defining two mappings between the lattices L_1 and L_2:

$$\alpha_1 : L_1 \longrightarrow L_2, \quad \alpha_1(p) = p, \forall p \in \mathbb{P}(X), \tag{2}$$

$$\gamma_1 : L_2 \longrightarrow L_1, \quad \gamma_1(\mu) = \lfloor \mu \rfloor, \forall \mu \in \mathcal{F}(X) \tag{3}$$

where

$$\lfloor \mu \rfloor(x) = \left\{ \begin{array}{ll} 1 & \text{if } \mu(x) = 1 \\ 0 & \text{otherwise} \end{array} \right\} = \sup\{p \in \mathbb{P}(X) \,:\, p \leq \mu\}.$$

Both mappings constitute a Galois connection because, with the pointwise ordering, it is evident that

$$\alpha_1(p) = p \leq \mu \Longleftrightarrow p \leq \gamma_1(\mu) = \lfloor \mu \rfloor.$$

Remark 3.4: Note that $\gamma_1 \circ \alpha_1 = id_{L_1}$, but it is only $\alpha_1 \circ \gamma_1 \leq id_{L_2}$.

3.3 The Galois connection defined in the previous paragraph can not be reversed. In other words, if the coadjoint member is taken as the adjoint member and vice versa, it is not a Galois connection. Notwithstanding, a Galois connection $G_2 =< L_2, L_1, \alpha_2, \gamma_2 >$ between L_2 and L_1 can be obtained with the mappings:

$$\alpha_2 : L_2 \longrightarrow L_1, \quad \alpha_2(\mu) =, \lceil \mu \rceil, \forall \mu \in \mathcal{F}(X) \tag{4}$$

$$\gamma_2 : L_1 \longrightarrow L_2, \quad \gamma_2(p) = p, \forall p \in \mathbb{P}(X), \tag{5}$$

where

$$\lceil \mu \rceil(x) = \left\{ \begin{array}{ll} 1 & \text{if } \mu(x) > 0 \\ 0 & \text{otherwise} \end{array} \right\} = \inf\{p \in \mathbb{P}(X) \,:\, \mu \leq p\}.$$

Again, the proof is easy because to prove that

$$\alpha(\mu) = \lceil \mu \rceil \leq p \Longleftrightarrow \mu \leq \gamma(p) = p,$$

is enough to take account of $\mu \leq \lceil \mu \rceil$, and also if $\mu \leq p$ then μ is 0 when p is 0 and so, in this case, $\lceil \mu \rceil \leq p$. This Galois connection will be named G_2.

IV. CHC models through Galois connection

Let $G =< R, U, \alpha, \gamma >$ be a Galois connection. Let us suppose that the conjectures in R and U are given by the consequences operator $C_\wedge$ and let C_R, C_U be that operator defined on R and U, respectively.

A. Consequences

The following properties are satisfied.

Proposition 4.1: $\alpha(C_R(r)) \subseteq C_U(\alpha(r))$.
Proof. By monotonicity of the coadjoint α. □

Proposition 4.2: $\gamma^{-1}(C_R(r)) = C_U(\alpha(r))$.
Proof.

$$\begin{array}{rcl} u \in \gamma^{-1}(C_R(r)) & \Leftrightarrow & \exists s \in R \,:\, u = \gamma^{-1}(s), s \in C_R(r) \\ & \Leftrightarrow & r \leq s = \gamma(u) \\ & \Leftrightarrow & \alpha(r) \preceq u \\ & \Leftrightarrow & u \in C_U(\alpha(r)). \end{array}$$

□

Corollary 4.3: $\gamma(C_U(\alpha(r))) \subseteq C_R(r)$.
Corollary 4.4: If γ is a surjective mapping, then $\gamma(C_U(\alpha(r))) = C_R(r)$.

The Galois Connection G_1 has a surjective adjoint and then last corollary applies. So in this case is $\gamma_1(C_\mathcal{F}(p)) = C_\mathbb{P}(p)$, but it only says that the projection of the fuzzy consequences of a classical predicate is the set of its classical consequences. It is known that when the adjoint of a Galois connection is a surjective mapping, the coadjoint mapping is one-to-one, and, of course, this is the case of G_1. These kind of Galois connections are called *embeddings*.

In the case of the Galois connection G_2 the adjoint mapping is not surjective, so it is only valid that $C_\mathbb{P}(\alpha_2(\mu)) \subset C_\mathcal{F}(\mu)$. This expression says that the set of classical consequences of the projection of a fuzzy predicate is a subset of the set of fuzzy consequences of the fuzzy predicate. Remember that in this case the projection of a fuzzy predicate is given by the adjoint γ_2. This relation can be used to invalidate some consequences because if a classical predicate is not a consequence of the projection of a fuzzy predicate, then it can not be a fuzzy consequence of that fuzzy predicate.

Galois connection G_2 is an example showing that the condition on the adjoint mapping of being a surjection is a necessary condition to obtain the result of the corollary 4.4. In this case, a fuzzy set μ is an image by the adjoint γ_2 if and only if it is a classical set. If the corollary 4.4 is reduced to classical sets, then it is also true for G_2, so any fuzzy consequence of μ can be calculated by means of $C_\mathbb{P}(\alpha_2(\mu)) = C_\mathbb{P}(\lceil \mu \rceil)$ because any fuzzy consequence of μ

is also a classical consequence of $\lceil \mu \rceil$ because in this case it is just μ.

B. Hypotheses

Usually, in CHC models the definition of the set of hypotheses of a predicate does not include the predicate itself, but in this case we consider the definition of hypotheses as

$$Hyp(p) = \{q \,:\, 0 < q \leq p\}.$$

Proposition 4.5: $\alpha(Hyp_{\leq}(r)) \subseteq Hyp_{\preceq}(\alpha(r))$.
Proof. By monotonicity of the coadjoint α. □

Proposition 4.6: $\alpha^{-1}(Hyp_{\preceq}(u)) = Hyp_{\leq}(\gamma(u))$.
Proof.

$$\begin{aligned} r \in \alpha^{-1}(Hyp_{\preceq}(u)) &\Leftrightarrow \exists v \in U \,:\, r = \alpha^{-1}(v), \\ & \quad\; v \in Hyp_{\preceq}(u) \\ &\Leftrightarrow \alpha(r) = v \preceq u \\ &\Leftrightarrow r \leq \gamma(u) \\ &\Leftrightarrow r \in Hyp_{\leq}(\gamma(u)). \end{aligned}$$

□

Corollary 4.7: $Hyp_{\preceq}(u) \supseteq \alpha(Hyp_{\leq}(\gamma(u)))$.
Corollary 4.8: If α is a surjective mapping, then $Hyp_{\preceq}(u) = \alpha(Hyp_{\leq}(\gamma(u)))$.

The relation between hypotheses and premises is just the opposite with respect to the relation between consequences and premises, this fact explains that in the case of the Galois connection G_1, since the coadjoint is not surjective, it is $Hyp_{\preceq}(u) \supseteq \alpha_1(Hyp_{\leq}(\gamma_1(u)))$, therefore the projection of all classical hypotheses does not cover, the set of all hypotheses. Notwithstanding, the coadjoint of G_2 is surjective, so $Hyp_{\preceq}(u) = \alpha_2(Hyp_{\leq}(\gamma_2(u)))$, and that implies that classical hypotheses can be calculated as the projections of fuzzy hypotheses, once translating the crisp set into a fuzzy set though the identity γ_2.

C. The general case of conjectures

Taking into account the definition of conjectures from a consequence operator, that is $Conj_C(P) = \{q \in L; N(q) \notin C(P)\}$, and translating the results for consequences in section IV-A, for any Galois connection $G =< R, U, \alpha, \gamma >$ such that the adjoint is surjective and verifies $N \circ \gamma = \gamma \circ N$, it can be said that the following chain is verified

$Conj_C(\alpha(p)) = \{q \in U; N(q) \not\leq \gamma^{-1}(C(p))\} = \{q \in U; \gamma(N(q)) \not\leq C(p)\} = \{q \in U; N(\gamma(q)) \not\leq C(p)\} = \gamma(Conj(\alpha(p)))$.

Anyway neither G_1 nor G_2 verify this conditions, since γ_1 is surjective but it does not verify $N \circ \gamma = \gamma \circ N$, and γ_2 is not surjective. That is the reason why conjectures in one universe cannot be determined by conjectures in the other.

In the particular case of G_1, the set of conjectures in L_1 can be translated into the set of conjectures in L_2 by means of the mapping α_1 in G_1. It is $\alpha_1(Conj(p)) \subset Conj(\alpha_1(p))$. Since if $q \in \alpha_1(Conj(p))$, it is $p \not\leq N(q)$, that is, $\exists x \in X$, such that $p(x) > N(q(x))$, and that is $p(x) = 1$ and $q(x) = 1$, and for that $x \in X$, it is also $\alpha_1(p)(x) > N(\alpha_1(q)(x))$, so $\alpha_1(q) \in Conj(\alpha_1(p))$. The relation of inclusion is strict: it could exists $\sigma \in Conj(\alpha_1(p))$, and $\sigma \neq \alpha_1(q)$, for all $q \in Conj(p)$. Notice that all elements in $Conj(\alpha(p)) - \alpha(Conj_C(p))$ are new conjectures.

Regarding the set of refutations, when the lattices in a Galois connection are orthocomplemented it is easy to prove the next result.

Proposition 4.9: $\alpha^{-1}(Ref_{\preceq}(u)) = Hyp_{\leq}(\gamma(u'))$.
Proof.

$$\begin{aligned} r \in \alpha^{-1}(Ref_{\preceq}(u)) &\Leftrightarrow \exists v \in U \,:\, r = \alpha^{-1}(v), \\ & \quad\; v \in Ref_{\preceq}(u) \\ &\Leftrightarrow \alpha(r) = v \preceq u' \\ &\Leftrightarrow r \leq \gamma(u') \\ &\Leftrightarrow r \in Hyp_{\leq}(\gamma(u')). \end{aligned}$$

□

Corollary 4.10: $Ref_{\preceq}(u) \supseteq \alpha(Hyp_{\leq}(\gamma(u')))$.
Corollary 4.11: If α is a surjective mapping, then $Ref_{\preceq}(u) = \alpha(Hyp_{\leq}(\gamma(u')))$.

So, in the general case, it is possible to know that an element of R is not a refutation (so it is a conjecture) if it does not belong to the $\alpha-$range of the set of hypotheses of the image of its negation by γ. When α is a surjective mapping, that condition is also a necessary one.

V. Changing the order in the universe of discourse

Conjecturing between two frameworks, can be understood as conjecturing in two different universes of discourse connected by a Galois connection, or as conjecturing in the same universe of discourse but ordered in different ways, this is what it will be developed in this section.

Taking the universe of discourse of fuzzy sets, $[0,1]^X$, its usual order is the pointwise order. Given $\mu, \sigma \in [0,1]^X$ it is said that $\mu \leq \sigma$ if and only if $\mu(x) \leq \sigma(x)$ for all $x \in X$, but it can be also used the sharpened order, being φ an order automorphism,

$$\mu \leq_{\varphi} \sigma \Leftrightarrow \begin{cases} \mu(x) \leq \sigma(x), & \text{if } \sigma(x) \leq \varphi^{-1}(\frac{1}{2}) \\ \sigma(x) \leq \mu(x), & \text{if } \sigma(x) > \varphi^{-1}(\frac{1}{2}) \end{cases}$$

introduced to study the behavior of the predicate $P = fuzzy$ in $[0,1]^X$, (see [10], [6], [4]).

From a body of information $P = \{\mu_1, \mu_2, ..., \mu_n\} \neq \emptyset$ and a preorder $\leq$, it can be built up the consequences operator $C_{\leq}(P) = \{\sigma \in [0,1]^X; \exists \mu_i \in P; \mu_i \leq \sigma\}$, that verifies $C_{\leq}(P) = \bigcup_{1 \leq i \leq n} C_{\leq}(\{\mu_i\})$ ([11]). So, we can reduce the study to that of CHC models for only one single premise.[1]

[1]Notice that in the case of $C_{\wedge}$, consequences are obtained through one element $InfP$, that is not necessary a premise.

From a premise $\mu \in [0,1]^X$, conjectures will be those elements that are non-contradictory with it: $Conj_{C_\leq}(\mu) = \{\sigma \in [0,1]^X; \mu \nleq N(\sigma)\}$, being N some strong negation, and it is $Conj_{C_\leq}(P) = \bigcap_{1\leq i\leq n} Conj_{C_\leq}(\mu_i)$. Hypotheses are those elements that explain the premise, that is, those whose consequences include the premise $Hyp_{C_\leq}(\mu) = \{\sigma \in Conj_{C_\leq}(\mu);\ 0 < \sigma \leq \mu\}$, and speculations are those conjectures that neither explain the premise nor are deduced form it, $Sp_{C_\leq}(\mu) = Conj_{C_\leq}(\mu) - [C_\leq(\mu) \cap Hyp_{C_\leq}(\mu)]$.

The consequences operator $C_\leq$ is determined by the order in the universe of discourse, therefore, changing such order, the obtained consequences also change. That is why the concept of deduction (inherent to the concept of consequence operator) depends on the order of the universe, and the reason why consequences obtained using one order can became speculation in the other order. In fact, that is what happens with some consequences obtained by the sharpened order that became speculations using the pointwise order.

First of all, let us check that $C_{\leq_\varphi} \subset Conj_{C_\leq}$,

Proposition 5.1: If $\mu \neq \mu_0$, every consequence obtained with the sharpened order and having prototypes ($x_1 \in X$, such that $\sigma(x_1) = 1$) is a conjecture obtained with the pointwise order.

Proof. If $\sigma \in C_{\leq_\varphi}(\mu)$, it is $\mu \leq_\varphi \sigma$ or $\begin{cases} \mu(x) \leq \sigma(x), & \text{if } \sigma(x) \leq \varphi^{-1}(\frac{1}{2}) \\ \sigma(x) \leq \mu(x), & \text{if } \sigma(x) > \varphi^{-1}(\frac{1}{2}) \end{cases}$.

If σ has prototypes, $\exists x_1 \in X$, such that $\sigma(x_1) = 1 > \varphi^{-1}(\frac{1}{2})$, so $N(\varphi^{-1}(\frac{1}{2})) \geq N(\sigma(x_1))$ or $\varphi^{-1}(\frac{1}{2}) \geq N(\sigma(x_1))$.

Since $\sigma \in C_{\leq_\varphi}(\mu)$, it is $\mu \leq_\varphi \sigma$, therefore $\mu(x_1) = 1$.

Then, $N(\sigma(x_1)) \leq \varphi^{-1}(\frac{1}{2}) < \sigma(x_1) = \mu(x_1)$, that is $\mu \nleq N(\sigma)$, or $\sigma \in Conj_{C_\leq}(\mu)$. □

Then, let us clarify which kind of conjecture with respect to the pointwise order are some consequences obtained thought the sharpened order,

Proposition 5.2: Whenever μ, σ has prototypes and anti-prototypes ($x_0 \in X$, such that $\sigma(x_0) = 0$) and the sets of prototypes (and, respectively, anti-prototypes) of μ strictly includes the sets of prototypes (and, respectively, anti-prototypes) of μ. If $\mu \neq \mu_0$, every consequence of μ obtained through the sharpened order, denoted by σ, in the previous conditions, is a speculation obtained through the pointwise order.

Proof. By proposition 5.1, if $\sigma \in C_{\leq_\varphi}(\mu)$ has prototypes, then $\sigma \in Conj_{C_\leq}(\mu) = C_\leq(\mu) \cup Hyp_{C_\leq}(\mu) \cup Sp_{C_\leq}(\mu)$. So, it will be enough to check that $\sigma \notin C_\leq(\mu)$ and $\sigma \notin Hyp_{C_\leq}(\mu)$.

- If μ has prototypes not coincidental with the prototypes of σ, $\exists x_c \in X$ such that $\mu(x_c) = 1$, and $\sigma(x_c) < 1$, so $\sigma(x_c) < \mu(x_c)$ and $\sigma \notin C_\leq(\mu)$.
- If μ has anti-prototypes not coincidental with the anti-prototypes of σ, $\exists x_h \in X$ such that $\mu(x_h) = 0$ and $\sigma(x_h) > 0$, so $\mu(x_h) < \sigma(x_h)$, and $\sigma \notin Hyp_{C_\leq}(\mu)$.

□

So, it is interesting to note that the consequences actually vary when the order is changed.

Remark 5.3: It can not be built a Galois connection between $([0,1]^X, \leq)$ and $([0,1]^X, \leq_\varphi)$ with the adjoint $\gamma(\mu) = \begin{cases} 1, & \text{if } \mu(x) \geq \varphi^{-1}(\frac{1}{2}) \\ 0, & \text{if otherwise.} \end{cases}$ and the coadjoin $\alpha(p) = p$. Since the implication $p \leq \gamma(\mu) \Rightarrow \alpha(p) \leq_\varphi \mu$ is not verified. It is enough to take the following $\mu, \sigma \in [0,1]^{[0,6]}$: $\mu = \begin{cases} \frac{x}{3}, & \text{if } x \in [0,3] \\ \frac{6-x}{3}, & \text{if } x \notin [0,3]. \end{cases}$ and $p = \mu_{[2,2.5]} = \begin{cases} 1, & \text{if } x \in [2,2.5] \\ 0, & \text{otherwise,} \end{cases}$ and $N = 1 - id$ (that is $\varphi^{-1}(x) = \frac{1}{2}$), since $p \leq \gamma(\mu) = \mu_{[1.5,4.5]}$ but $\alpha(p) \nleq_\varphi \mu$.

Reciprocally, it is $\alpha(p) \leq_\varphi \mu \Rightarrow p \leq \gamma(\mu)$, since if it is supposed that $\alpha(p) \leq_\varphi \mu$ and $p \nleq \gamma(\mu)$, it exists $x_0 \in X$, such that $p(x_0) = 1$ and $\mu(x_0) \leq \varphi^{-1}(\frac{1}{2})$. So, $p(x_0) \leq \mu(x_0) \leq \varphi^{-1}(\frac{1}{2})$, then $p(x_0) = 0$, and an absurd is reached.

Following the same scheme of proof, if $\mu, \sigma \in \{0,1\}^X$ such that $\mu \leq_\varphi \sigma$, it is $\{x \in X; \mu(x) = 1\} \subset \{x \in X; \sigma(x) = 1\}$. Since, from $\sigma(x) = 0$, it obviously follows $\mu(x) = 0$, that is the same to from $\mu(x) = 1$, it follows $\sigma(x) = 1$.

VI. Example

A. Consequences and conjectures in two different universes of discourse

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It will be deal with the universes of crisp (L_1), and fuzzy sets (L_2).

Let it be the premise "*be between* 0.4 *and* 0.6", represented by $p = \mu_{[0.4,0.6]}$ in L_1,

$$\mu_{[0.4,0.6]}(x) = \begin{cases} 1, & \text{if } x \in [0.4, 0.6] \\ 0, & \text{if } x \notin [0.4, 0.6] \end{cases}$$

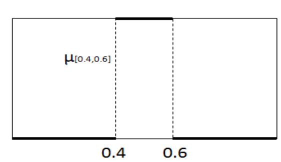

Fig. 1.

Consequences for p in L_1 with the consequences operators $C = C_{\leq}$ or $C_{\wedge}$, are those elements in $C(p) = \{q \in L_1; p \leq q\} = \{q \in L_1; \forall x \in [0.4, 0.6], q(x) = 1\}$, and in L_2 consequences are $C(\alpha(p)) = \{q \in L_2; p \leq q\} = \{\sigma \in L_2; \forall x \in [0.4, 0.6], \sigma(x) = 1\}$. In L_1, consequences are elements are all sets containing $[0.4, 0.6]$, but in L_2 are those fuzzy sets containing the crisp set $[0.4, 0.6]$, so they collect uncertainty that allow us to consider a predicate like "*being more or less between* 0.4 *and* 0.6".

For this example, all results obtained along this paper using a Galois connection will be shown. For instance, taking G_1, it is obvious that $\alpha_1(C(p)) \subset C(\alpha_1(p))$, but it is not $\alpha_1(C(p)) = C(\alpha_1(p))$, since taking

$$\sigma = \begin{cases} \frac{x-0.2}{0.2}, & \text{if } x \in [0.2, 0.4] \\ 1, & \text{if } x \notin [0.4, 0.6] \\ \frac{0.8-x}{0.2}, & \text{if } x \in [0.6, 0.8] \\ 0, & \text{otherwise,} \end{cases}$$

drawn in the picture 2, it is σ in $C(\alpha(p)) - \alpha(C(p))$.

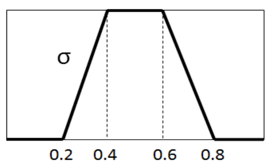

Fig. 2.

The equality $\gamma_1^{-1}(C(p)) = C(\alpha_1(p))$, is obviously verified, for instance the previous $\sigma \in C(\alpha_1(p))$, verifies that it exists $\mu_{[0.2,0.8]} \in C(p)$ such that $\mu_{[0.2,0.8]} \in \gamma_1^{-1}(\sigma)$.

Finally, as γ_1 is surjective $C(p) = \gamma_1(C(\alpha_1(p)))$. See for instance the case of $\mu_{[0.2,0.8]} = \gamma_1(\sigma)$.

Conjectures of p in L_1 are in the set, $Conj(p) = \{q \in L_1; p \nleq 1 - q\} = \{q \in L_1; \exists x \in [0.4, 0.6], q(x) = 1\}$, those sets that, at least, has a element in $[0.4, 0.6]$, linguistically it can be said that they are not in contradiction with p, and conjectures in L_2, are $Conj(\alpha(p)) = \{q \in L_1; p \nleq 1-q\} = \{q \in L_1; \exists x \in [0.4, 0.6], q(x) \neq 0\}$, those fuzzy sets that allocate a value different to 0 for at least one element in the set $[0.4, 0.6]$. It is obvious that $\alpha(Conj(p)) \subset Conj(\alpha(p))$.

B. What happens if adding a new premise?

Now, taking as set of premises two different premises, one crisp and the other fuzzy, $p =$"*be between* 0.4 *and* 0.6" and μ="*be near* 0.5", the representation will be $P = \{p, \mu\}$, with the previous $p \in L_1$ and $\mu \in L_2$, consequences and conjectures of P will be calculated.

First of all, consequences and conjectures for μ are computed with $C = C_{\leq}$ and $C_{\wedge}$ as, $C(\{\mu\}) = \{\sigma \in L_2; \mu \leq \sigma\} = \{\sigma \in L_2; \forall x \in [0, 1], \mu(x) \leq \sigma(x)\}$, and $Conj(\{\mu\}) = \{\sigma \in L_2; \mu \nleq 1 - \sigma\} = \{\sigma \in L_2; \exists x \in [0.4, 0.6], p(x) > 1 - q(x)\}$.

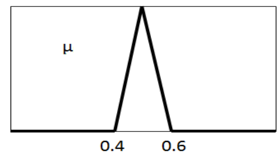

Fig. 3.

So, by the properties of the operator $C_{\leq}$, consequences and conjectures for P in L_2 can be calculated in the following way, $C_{\leq}(P) = C_{\leq}(\alpha(p)) \cup C_{\leq}(\mu) = C_{\leq}(\alpha(p))$ and $Conj_{\,C_{\leq}}(P) = Conj_{\,C_{\leq}}(\alpha(p)) \cap Conj_{\,C_{\leq}}(\mu) = Conj_{\,C_{\leq}}(\mu)$.

On the other hand, with the consequences operator $C_{\wedge}$, it is $InfP = \mu$, so $C_{\wedge}(P) = C_{\leq}(\mu)$ and $Conj_{\,C_{\wedge}}(P) = Conj_{\,C_{\wedge}}(\mu) = Conj_{\,C_{\leq}}(\mu)$.

See that adding a new premise it is $Conj(p) \subset Conj(\alpha(p))$, obtained by the antimonotonicity of the operator $Conj_C$.

C. What happens if using the sharpened order in L_2?

Till now, the universe of discourse $[0, 1]^X$ was ordered by the pointwise order, but changing it by sharpened order $\leq_{\varphi}$, different consequences and conjectures are obtained.

For instance, taking the premise representing "*being around* 0.5" by the fuzzy set $\delta = \begin{cases} \frac{x-0.43}{0.02}, & \text{if } x \in [0.43, 0.45] \\ 1, & \text{if } x \notin [0.45, 0.55] \\ \frac{0.57-x}{0.02}, & \text{if } x \in [0.55, 0.57] \\ 0, & \text{otherwise,} \end{cases}$ and $\varphi = id$:

$$C_{\leq_{\varphi}}(\{\delta\}) = \{\sigma \in [0, 1]^X; \mu \leq_{\varphi} \sigma\},$$

and for instance the representation of the predicate "*being around* 0.5" by μ

in figure 4 is in $C_{\leq_{\varphi}}(\{\delta\})$. So, it directly follows for the premise.

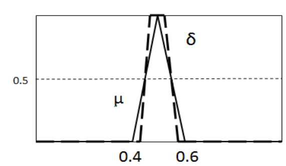

Fig. 4.

On the other hand, notice that dealing with the pointwise order, $\mu \in Conj(\{\delta\}) = \{\sigma \in L_2; \mu \nleq 1 - \sigma\} = \{\sigma \in L_2; \exists x \in [0.4, 0.6], p(x) > 1 - q(x)\}$. Since, taking $x = 0.5$, it is $p(x) = 1 > 1 - q(x) = 0$, that corroborate what is said in proposition 5.1. But, μ is an speculation, this is that it is neither in $C_{\leq}(\{\delta\})$ ($\delta \nleq \mu$), nor in $Hyp_{C_{\leq}}(\{\delta\})$ ($\mu \nless \delta$).

VII. Conclusions

This paper deals with two topics neither solved, nor clearly possed before, and it does not offer conclusive but only provisional results.

The main subject concerns the growing in the number of possible conjectures when, once some crisp information is known and from which either some logical consequences, or some conjectures followed, new but imprecise information is added. Such a realistic question, actually concerning Computing with Words, is posed by means of a particular example, and its general solution still remains an open problem. As a second topic, the paper tries to pose the relationships existing between consequence and conjecture operators with Galois' Connections.

If with the first of these topics the relationships are clear but the suitable theoretic methodology is not so, only some elementary and previous results are actually obtained with the second. Although more work at the respect is still to be done, it is to be remarked how the character of the beforehand conclusions (either consequences of hypotheses) can abruptly change by a change in the order of the universe.

This work is only an opening step for those topics, and the answers to the posed queries actually remain unended.

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References

[1] B. Bouchon-Meunier, C. Delechamp, C. Marsala, and M. Rifqi. Several forms of fuzzy analogical reasoning. In *FUZZ-IEEE'97*, pages 45–50, Barcelona, Spain, 1997.

[2] B. Bouchon-Meunier and L. Valverde. A fuzzy aproach to analogical resasoning. *Int. J. Soft Computing*, 3(3):141–147, 1999.

[3] J. L. Castro and E. Trillas. Sobre preordenes y operadores de consecuencias de Tarski. *Theoria*, 4(11):419–425, 1989. (In Spanish).

[4] A. DeLuca and S. Termini. Entropy of L-fuzzy sets. *Information and Control*, 24:55–73, 1974.

[5] E. Castiñeira E. Trillas, S. Cubillo. On conjectures in orthocomplemented lattices. *Artificial Intelligence*, 117:255–275, 2000.

[6] I. García-Honrado and E. Trillas. An essay on the linguistic roots of fuzzy sets. *Submitted to Information Sciences*, (Available upon request).

[7] J. Stuart Mill. *Sir William Hamilton's Philosophy and the principal philosophical questions discussed in his writings*. London Longmans, Green, and Co., New York, 1889.

[8] C. S. Peirce. Deduction, induction, and hypothesis. *Popular Science Monthly*, (13):470–482, 1878.

[9] D. Qiu. A note on Trillas' CHC models. *Artificial Intelligence*, 171:239–254, 2007.

[10] E. Trillas. On a model for the meaning of predicates. In Rudolf Seising, editor, *Fuzzy sets theory. Philosohpy and criticism*, volume 243 of *Studies in Fuzziness and Soft Computing*, chapter 9, pages 175–205. Springer Verlag, 2009.

[11] E. Trillas, I. García-Honrado, and A. Pradera. Consequences and conjectures in preordered sets. *Information Sciences*, 180(19):3573–3588, 2010.

[12] E Trillas and A. Pradera. An orthogonality-based classification of conjectures in ortholattices. *Mathware & soft computing*, 13(1):71–87, 2006.

[13] E. Trillas, A. Pradera, and A. Álvarez. On the reducibility of Hypotheses and Consequences. *Information Sciences*, 179(23):3957–3963, 2009.

[14] Satosi Watanabe. *Knowing and guessing. A Quantitative Study of Inference and Information*. John Wiley and sons, New York, 1969.

[15] W. Whewell. *Novum Organon Renovatum: Being The Second Part of The Philosophy of The Inductive Sciences*. John Parker and Son, London, 1858.

[16] L. A. Zadeh. Fuzzy Logic = Computing With Words. *IEEE Trans. Fuzzy Systems*, 4:103–111, 1996.